EXTENDING CONNECTIVITY FUNCTIONS ON $\mathbb{R}^n$

KRZYSZTOF CIESIELSKI, TOMASZ NATKANIEC, AND JERZY WOJCIECHOWSKI

ABSTRACT. A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *connectivity function* if for every connected subset C of $\mathbb{R}^n$ the graph of the restriction $f|C$ is a connected subset of $\mathbb{R}^{n+1}$, and f is an *extendable connectivity function* if f can be extended to a connectivity function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\mathbb{R}^n$ imbedded into $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \{0\}$. There exists a connectivity function $f : \mathbb{R} \to \mathbb{R}$ that is not extendable. We prove that for $n \geq 2$ every connectivity function $f : \mathbb{R}^n \to \mathbb{R}$ is extendable.

1. INTRODUCTION

Given functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}$, we say that g *extends* f if g extends the composition $f \circ \tau : \mathbb{R}^n \times \{0\} \to \mathbb{R}$, where $\tau : \mathbb{R}^n \times \{0\} \to \mathbb{R}^n$ and

$$\tau(\langle x_1, x_2, \ldots, x_n, 0\rangle) = \langle x_1, x_2, \ldots, x_n\rangle\,, \tag{1}$$

for every $\langle x_1, x_2, \ldots, x_n\rangle \in \mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *connectivity function* if for every connected subset C of $\mathbb{R}^n$ the graph of the restriction $f|C$ is a connected subset of $\mathbb{R}^{n+1}$, and f is an *extendable connectivity function* if there exists a connectivity function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ extending f.

It follows immediately from the definition that every extendable connectivity function is a connectivity function. Cornette [3] and Roberts [9] proved that there exists a connectivity function $f : \mathbb{R} \to \mathbb{R}$ that is not extendable. This result was surprising and sparked the interest in the family of extendable connectivity functions. Ciesielski and Wojciechowski [2] asked whether there exists a connectivity function $f : \mathbb{R}^n \to \mathbb{R}$,

---

1991 *Mathematics Subject Classification.* Primary 26B05; Secondary 54C30.

Key words and phrases. connectivity functions, extendable functions, Darboux-like functions.

This work was partially supported by NSF Cooperative Research Grant INT-9600548 with its Polish part being financed by Polish Academy of Science PAN.

with $n \geq 2$, that is not extendable. In this paper we will show that the answer to that question is negative.

Theorem 1. *If $n \geq 2$ then every connectivity function $f : \mathbb{R}^n \to \mathbb{R}$ is extendable.*

To prove Theorem 1 we will use ideas from Gibson and Roush [5] where is formulated a necessary and sufficient condition for a connectivity function $f : [0,1] \to [0,1]$ to be extendable to a connectivity function $f : [0,1]^2 \to [0,1]$ (if one considers $[0,1]$ to be embedded in $[0,1]^2$ as $[0,1] \times \{0\}$).

Our basic terminology and notation is standard. (See [1] or [4].) In particular, if A is a subset of a metric space X, then $\operatorname{bd} A$, $\operatorname{cl} A$ and $\operatorname{diam} A$ will denote the boundary, closure, and diameter of A in X respectively, and if f is a function and A is a subset of its domain, then $f[A]$ is the image of A under f.

The following additional terminology will be useful in our proof. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, a *peripheral pair* (*for* f) is an ordered pair $\langle A, I \rangle$ with I being a closed interval in $\mathbb{R}$ and A being an open bounded subset of $\mathbb{R}^n$ with $f[\operatorname{bd} A] \subseteq I$. Given $\varepsilon > 0$, an *ε-peripheral pair* is a peripheral pair $\langle A, I \rangle$ with $\operatorname{diam} A < \varepsilon$ and $\operatorname{diam} I < \varepsilon$. Given a point $x \in \mathbb{R}^n$, a peripheral pair for f *at* x is a peripheral pair $\langle A, I \rangle$ for f with $x \in A$ and $f(x) \in I$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *peripherally continuous* if for every $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there is an ε-peripheral pair for f at x.

The class of peripherally continuous functions $f : \mathbb{R} \to \mathbb{R}$ is strictly larger than the class of connectivity functions. However, the following result holds.

Theorem 2. *If $n \geq 2$ then a function $f : \mathbb{R}^n \to \mathbb{R}$ is peripherally continuous if and only if it is a connectivity function.*

The implication that a connectivity function is peripherally continuous in Theorem 2 was proved by Hamilton [7] and Stallings [10], and the opposite implication was proved by Hagan [6].

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and $\mathcal{P}$ be a family of peripheral pairs for f. We say that $\mathcal{P}$ *locally converges to* 0 if for every $\varepsilon > 0$ and every bounded set $X \subseteq \mathbb{R}^n$ the set

$$\{\langle A, I \rangle \in \mathcal{P} : A \cap X \neq \emptyset \text{ and } \operatorname{diam} A \geq \varepsilon\}$$

is finite, and that $\mathcal{P}$ has the *intersection property* provided $I \cap I' \neq \emptyset$ for any $\langle A, I\rangle, \langle A', I'\rangle \in \mathcal{P}$ such that each of the sets $A \cap A'$, $A \setminus A'$, and $A' \setminus A$ is nonempty. Given $X \subseteq \mathbb{R}^n$, we say that $\mathcal{P}$ is an *f-base for X* if for every $\varepsilon > 0$ and $x \in X$ there exists an ε-peripheral pair for f at x that belongs to $\mathcal{P}$. Note that a function $f : \mathbb{R}^n \to \mathbb{R}$ is peripherally continuous if and only if there exists an f-base for some set $X \subseteq \mathbb{R}^n$ that contains all points of discontinuity of f. A *peripheral family for* $f : \mathbb{R}^n \to \mathbb{R}$ is a countable family of peripheral pairs for f that locally converges to 0, has the intersection property, and is an f-base for $\mathbb{R}^n$.

Theorem 1 follows from Theorem 2 and the following two results.

Theorem 3. *If $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a peripherally continuous function, then there exists a peripheral family for f.*

If $\langle A, I\rangle$ is a peripheral pair (for some $f : \mathbb{R}^n \to \mathbb{R}$), then the *cylindrical extension* of $\langle A, I\rangle$ is a pair $\langle A', I\rangle$, where

$$A' = A \times (-\operatorname{diam} A, \operatorname{diam} A) \subseteq \mathbb{R}^{n+1}.$$

If $\mathcal{P}$ is a set of peripheral pairs, then the *cylindrical extension* of $\mathcal{P}$ is the set of cylindrical extensions of all the elements of $\mathcal{P}$.

The case $n = 1$ of the following theorem is a modification of a result of Gibson and Roush [5].

Theorem 4. *If $n \geq 1$ and $\mathcal{P}$ is a peripheral family for $f : \mathbb{R}^n \to \mathbb{R}$, then there exists a continuous function*

$$h : \mathbb{R}^{n+1} \setminus (\mathbb{R}^n \times \{0\}) \to \mathbb{R}$$

such that every element of the cylindrical extension of $\mathcal{P}$ is a peripheral pair for the function

$$g = h \cup (f \circ \tau) : \mathbb{R}^{n+1} \to \mathbb{R},$$

where $\tau : \mathbb{R}^n \times \{0\} \to \mathbb{R}^n$ is the bijection as in (1).

The proof of Theorem 3 is given in section 2, and the proof of Theorem 4 can be found in section 3. Now we shall give the proof of Theorem 1.

Proof of Theorem 1. Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a connectivity function. Since f is peripherally continuous, it follows from Theorem 3 that there exists a peripheral family $\mathcal{P}$ for f. Let $\mathcal{Q}$ be the cylindrical extension of $\mathcal{P}$. By Theorem 4 there exists a function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ such that g extends f, the restriction of g to $\mathbb{R}^{n+1} \setminus (\mathbb{R}^n \times \{0\})$ is continuous, and every element of $\mathcal{Q}$ is a peripheral pair for g. The proof will be complete when we show that $\mathcal{Q}$ is a g-base for $\mathbb{R}^n \times \{0\}$ since then it will follow that g is peripherally continuous and hence a connectivity function.

Let $\varepsilon > 0$ and $x = \langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$. Since $\mathcal{P}$ is an f-base for $\mathbb{R}^n$, there is $\langle A, I \rangle \in \mathcal{P}$ such that $\operatorname{diam} A < \varepsilon/\sqrt{5}$, $\operatorname{diam} I < \varepsilon$, $x \in A$, and $f(x) \in I$. Then the cylindrical extension $\langle A', I \rangle \in \mathcal{Q}$ of $\langle A, I \rangle$ is an ε-peripheral pair for g at $\bar{x} = \langle x_1, \ldots, x_n, 0 \rangle$ implying that $\mathcal{Q}$ is a g-base for $\mathbb{R}^n \times \{0\}$.

2. Peripheral families for connectivity functions

In this section we are going to prove Theorem 3. First, let us introduce some more terminology. Throughout this section we will assume that n is a fixed integer and that $n \geq 2$.

Given $X, Y \subseteq \mathbb{R}^n$, the *boundary* of $X \cap Y$ in X will be denoted by $\operatorname{bd}_X Y$. The *inductive dimension* $\operatorname{ind} X$ of a subset $X \subseteq \mathbb{R}^n$ is defined inductively as follows. (See for example Engelking [4].)

(i) $\operatorname{ind} X = -1$ if and only if $X = \emptyset$.
(ii) $\operatorname{ind} X \leq m$ if for any $p \in X$ and any open neighborhood W of p there exists an open neighborhood $U \subseteq W$ of p such that $\operatorname{ind} \operatorname{bd}_X U \leq m - 1$.
(iii) $\operatorname{ind} X = m$ if $\operatorname{ind} X \leq m$ and it is not true that $\operatorname{ind} X \leq m - 1$.

A fundamental result of dimension theory states that $\operatorname{ind} \mathbb{R}^n = n$.

Given a set $A \subseteq \mathbb{R}^n$ and an integer $m \geq 1$, we say that A is an *m-dimensional Cantor manifold* if A is compact, $\operatorname{ind} A = m$, and for every $X \subseteq A$ with $\operatorname{ind} X \leq m-2$, the set $A \setminus X$ is connected. (See [8].) Given a subset A of $\mathbb{R}^n$, we say that A is a *quasiball* if A is a bounded and connected open set, and $\operatorname{bd} A$ is an $(n-1)$-dimensional Cantor manifold. (See [2].) A peripheral pair $\langle A, I \rangle$ with A being a quasiball will be called a *nice* peripheral pair. Given $\varepsilon, \delta > 0$, an $\langle \varepsilon, \delta \rangle$*-peripheral pair* is a peripheral pair $\langle A, I\rangle$ with $\operatorname{diam} A < \varepsilon$ and $\operatorname{diam} I < \delta$. The following theorem follows immediately from Corollary 5.5 in [2].

Theorem 5. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a peripherally continuous function, then for any $\varepsilon, \delta > 0$ and $x \in \mathbb{R}^n$ there exists a nice $\langle \varepsilon, \delta\rangle$-peripheral pair for f at x.*

We say that quasiballs A and A' are *independent* if each of the sets $A \cap A'$, $A \setminus A'$, and $A' \setminus A$ is nonempty. The following lemma is a restatement of Lemma 5.6 in [2].

Lemma 6. *If A and A' are independent quasiballs in $\mathbb{R}^n$, then $\operatorname{bd} A \cap \operatorname{bd} A' \neq \emptyset$.*

The following lemma follows immediately from Lemma 6.

Lemma 7. *If $\mathcal{P}$ is a family of nice peripheral pairs, then $\mathcal{P}$ has the intersection property.*

For every positive integer $i \in \mathbb{N}$, let

$$D_i = \left\{ \frac{-4i^2}{4i}, \frac{-4i^2+1}{4i}, \ldots, \frac{4i^2}{4i} \right\}$$

and

$$\mathcal{J}_i = \left\{ J_{i,q} : q \in D_i \right\},$$

where $J_{i,q}$ is the open interval

$$J_{i,q} = \left(q - \frac{1}{4i}, q + \frac{1}{4i} \right),$$

for each $q \in D_i$.

Lemma 8. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and, for every $i \in \mathbb{N}$ and $q \in D_i$, let*

$$\mathcal{P}_{i,q} = \{ \langle A_\gamma, I_\gamma \rangle : \gamma \in \Gamma_{i,q} \}$$

be a family of $(1/i)$-peripheral pairs for f such that

$$f^{-1}(J_{i,q}) \subseteq \bigcup_{\gamma \in \Gamma_{i,q}} A_\gamma \qquad \text{and} \qquad J_{i,q} \subseteq \bigcap_{\gamma \in \Gamma_{i,q}} I_\gamma.$$

Then

$$\mathcal{P} = \bigcup_{i \in \mathbb{N}} \bigcup_{q \in D_i} \mathcal{P}_{i,q}$$

is an f-base for $\mathbb{R}^n$.

Proof. Let $\varepsilon > 0$ and $x \in \mathbb{R}^n$. Then there are $i \in \mathbb{N}$ and $q \in D_i$ with $1/i \leq \varepsilon$ and $f(x) \in J_{i,q}$. Since

$$f^{-1}(J_{i,q}) \subseteq \bigcup_{\gamma \in \Gamma_{i,q}} A_\gamma,$$

there is $\delta \in \Gamma_{i,q}$ such that $x \in A_\delta$. Since

$$J_{i,q} \subseteq \bigcap_{\gamma \in \Gamma_{i,q}} I_\gamma,$$

it follows that $\langle A_\delta, I_\delta \rangle$ is an ε-peripheral pair for f at x. ∎

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a peripherally continuous function. Fix $i \in \mathbb{N}$ and $q \in D_i$. By Theorem 5 for each $x \in f^{-1}(J_{i,q})$ there exists a nice $\langle 1/i, 1/4i \rangle$-peripheral pair $\langle A_{i,q,x}, I_{i,q,x} \rangle$ for f at x. Let

$$\mathcal{T}_{i,q} = \left\{ \langle A_{i,q,x}, \mathrm{cl}(I_{i,q,x} \cup J_{i,q}) \rangle : x \in f^{-1}(J_{i,q}) \right\}.$$

Note that since

$$f(x) \in I_{i,q,x} \cap J_{i,q} \neq \emptyset$$

for every $x \in f^{-1}(J_{i,q})$, the elements of $\mathcal{T}_{i,q}$ are $\langle 1/i, 3/4i \rangle$-peripheral pairs for f.

Let $j, k \in \mathbb{N}$ be any positive integers with $j > i$. Set

$$\mathcal{T}_{i,q}^{k} = \{ \langle A, I \rangle \in \mathcal{T}_{i,q} : A \cap B_k \neq \emptyset \text{ and } A \cap B_{k'} = \emptyset \text{ for every } k' < k \},$$

where B_k is the open ball of center $\langle 0, 0, \ldots, 0 \rangle$ and radius k, and

$$\mathcal{T}_{i,q}^{k,j} = \left\{ \langle A, I \rangle \in \mathcal{T}_{i,q}^{k} : \frac{1}{j} \leq \mathrm{diam}\, A < \frac{1}{j-1} \right\}.$$

Moreover, let

$$C_{i,q}^{k,j} = \mathrm{cl}\left(\bigcup_{\langle A, I \rangle \in \mathcal{T}_{i,q}^{k,j}} A \right),$$

and

$$E_{i,q}^{k,j} = C_{i,q}^{k,j} \setminus \bigcup_{\langle A, I \rangle \in \mathcal{T}_{i,q}^{k,j}} A.$$

Fix $y \in E_{i,q}^{k,j}$. Let $\left\langle A_y, I'_y \right\rangle$ be a nice $\langle 1/j, 1/4i \rangle$-peripheral pair for f at y. Since

$$E_{i,q}^{k,j} \subseteq \mathrm{cl}\left(\bigcup_{\langle A,I \rangle \in \mathcal{T}_{i,q}^{k,j}} \mathrm{bd}\, A \right),$$

there is $\langle A, I \rangle \in \mathcal{T}_{i,q}^{k,j}$ such that

$$A_y \cap \mathrm{bd}\, A \neq \emptyset.$$

Since $\mathrm{diam}\, A_y < \mathrm{diam}\, A$, it follows that the quasiballs A and A_y are independent and so Lemma 6 implies that $I \cap I'_y \neq \emptyset$. Let $I_y = I \cup I'_y$ for every $y \in E_{i,q}^{k,j}$ and

$$\mathcal{S}_{i,q}^{k,j} = \mathcal{T}_{i,q}^{k,j} \cup \left\{ \langle A_y, I_y \rangle : y \in E_{i,q}^{k,j} \right\}.$$

Note that $J_{i,q} \subseteq I$ for every $\langle A, I \rangle \in \mathcal{S}_{i,q}^{k,j}$. Since the set $C_{i,q}^{k,j}$ is compact and

$$C_{i,q}^{k,j} \subseteq \bigcup_{\langle A,I \rangle \in \mathcal{S}_{i,q}^{k,j}} A,$$

there is a finite subset $\mathcal{P}_{i,q}^{k,j}$ of $\mathcal{S}_{i,q}^{k,j}$ such that

$$C_{i,q}^{k,j} \subseteq \bigcup_{\langle A,I \rangle \in \mathcal{P}_{i,q}^{k,j}} A.$$

Let

$$\mathcal{P}_{i,q} = \bigcup_{k \in \mathbb{N}} \bigcup_{j > i} \mathcal{P}_{i,q}^{k,j} = \left\{ \langle A_\gamma, I_\gamma \rangle : \gamma \in \Gamma_{i,q} \right\}.$$

It is clear that the elements of $\mathcal{P}_{i,q}$ are $(1/i)$-peripheral pairs and

$$J_{i,q} \subseteq \bigcap_{\gamma \in \Gamma_{i,q}} I_\gamma.$$

Moreover,

$$f^{-1}(J_{i,q}) \subseteq \bigcup_{\langle A,I \rangle \in \mathcal{T}_{i,q}} A \subseteq \bigcup_{\langle A,I \rangle \in \mathcal{T}_{i,q}} \mathrm{cl}\, A \subseteq \bigcup_{k \in \mathbb{N}} \bigcup_{j > i} C_{i,q}^{k,j} \subseteq \bigcup_{\gamma \in \Gamma_{i,q}} A_\gamma,$$

implying, by Lemma 8, that

$$\mathcal{P} = \bigcup_{i \in \mathbb{N}} \bigcup_{q \in D_i} \mathcal{P}_{i,q}$$

is an f-base for $\mathbb{R}^n$. Of course $\mathcal{P}$ is countable and since all peripheral pairs in $\mathcal{P}$ are nice, it follows from Lemma 7 that $\mathcal{P}$ has the intersection property. It remains to prove the following claim.

Claim. The family $\mathcal{P}$ locally converges to 0.

We are going now to prove the claim. First note that if $\langle A, I\rangle \in \mathcal{T}_{i,q}^{k,j}$ and $k' < k$, then $A \cap B_{k'} = \emptyset$, implying that $y \notin B_{k'}$ (and hence $A_y \nsubseteq B_{k'}$) for any $y \in E_{i,q}^{k,j}$. Therefore

$$A \nsubseteq B_{k'} \text{ for any } \langle A, I\rangle \in \mathcal{S}_{i,q}^{k,j} \text{ and } k' < k. \tag{2}$$

Also note that

$$\operatorname{diam} A < \frac{1}{j'} \text{ for any } \langle A, I\rangle \in \mathcal{S}_{i,q}^{k,j} \text{ and } j' < j. \tag{3}$$

Now let $\varepsilon > 0$ and $X \subseteq \mathbb{R}^n$ be a bounded set. Then there are $j', k' \in \mathbb{N}$ such that $1/j' < \varepsilon$ and X is a subset of the ball $B_{k'-1}$. Let $\langle A, I\rangle \in \mathcal{P}$ be such that $A \cap X \neq \emptyset$ and $\operatorname{diam} A \geq \varepsilon$. Since $A \cap B_{k'-1} \neq \emptyset$ and $\operatorname{diam} A < 1$, it follows that $A \subseteq B_{k'}$. Therefore, since $\operatorname{diam} A \geq 1/j'$, it follows from (2) and (3) that if $\langle A, I\rangle \in \mathcal{P}_{i,q}^{k,j} \subseteq \mathcal{S}_{i,q}^{k,j}$, then $k \leq k'$ and $j \leq j'$. Thus

$$\langle A, I\rangle \in \mathcal{P}^{k',j'} = \bigcup_{k\leq k'} \bigcup_{j\leq j'} \bigcup_{i<j} \bigcup_{q\in D_i} \mathcal{P}_{i,q}^{k,j}.$$

Since the set $\mathcal{P}^{k',j'}$ is finite, the proof of the claim, and hence of the theorem is complete.

3. Connectivity functions are extendable

In this section we are going to prove Theorem 4.

A *partial order* on a set T is a binary relation $\preccurlyeq$ on T that is reflexive, transitive and antisymmetric (that is, $t \preccurlyeq s$ and $s \preccurlyeq t$ imply $t = s$ for every $s, t \in T$). We say that $\preccurlyeq$ has the *finite predecessor property* if for every $t \in T$ the set $\{s \in T : s \preccurlyeq t\}$ of $\preccurlyeq$-predecessors of t is finite. A partial order $\preccurlyeq^*$ on a set T is an ω-*order* if there is a bijection $f : \omega \to T$ (where $\omega = \{0, 1, \dots\}$) such that $f(t) \preccurlyeq^* f(s)$ if and only if

$t \leq s$. Given partial orders $\preccurlyeq$ and $\preccurlyeq^*$ on T, we say that $\preccurlyeq^*$ *extends* $\preccurlyeq$ if and only if $t \preccurlyeq s$ implies $t \preccurlyeq^* s$ for every $s, t \in T$.

Lemma 9. *If $\preccurlyeq$ is a partial order on an infinite countable set T with the finite predecessor property, then there is an ω-order $\preccurlyeq^*$ on T that extends $\preccurlyeq$.*

Proof. It is enough to show that there is a bijection $f : \omega \to T$ such that $f(i) \preccurlyeq f(j)$ implies $i \leq j$. Let $\lesssim$ be any fixed ω-order on T. We shall define the value $f(i)$ by induction on i. Let $i \in \omega$ and assume that $f(j)$ has been defined for every $j < i$. Let

$$T_i = T \setminus \{f(j) : j < i\},$$

and let T_i' consist of all $\preccurlyeq$-minimal elements in T_i. For every $t \in T_i$ the set of $\preccurlyeq$-predecessors of t is finite so there is $s \in T_i'$ with $s \preccurlyeq t$. In particular, T_i' is nonempty. Let $f(i)$ be the $\lesssim$-minimal element of T_i'.

It is obvious from the construction that f is injective and that $f(i) \preccurlyeq f(j)$ implies $i \leq j$ for every $i, j \in \omega$. To see that f is surjective note that for any $i \in \omega$ and $t \in T_i$ the set of $\preccurlyeq$-predecessors of t is finite, so one of them is in T_i'. This predecessor of t will eventually become a value of f since $\lesssim$ is an ω-order. Then the number of unassigned $\preccurlyeq$-predecessors of t becomes smaller and hence eventually t itself must become a value of f. ∎

A family $\mathcal{A}$ of subsets of a metric space X is *locally finite* if for every $x \in X$ some open neighborhood of x intersects only finitely many elements of $\mathcal{A}$. Let a *Tietze family* for a metric space X be a countable family

$$\mathcal{F} = \{\langle C_\gamma, I_\gamma \rangle : \gamma \in \Gamma\}$$

such that:

(1) $\mathcal{A} = \{C_\gamma : \gamma \in \Gamma\}$ is a locally finite closed cover of X with any C_γ intersecting only finitely many elements of $\mathcal{A}$;
(2) for every $\gamma \in \Gamma$, I_γ is either equal to $\mathbb{R}$ or is a closed interval in $\mathbb{R}$;
(3) for every $\Phi \subseteq \Gamma$

$$\text{if } \bigcap_{\gamma \in \Phi} C_\gamma \neq \emptyset \text{ then } \bigcap_{\gamma \in \Phi} I_\gamma \neq \emptyset.$$

The following result will be the key step in our proof of Theorem 4.

Theorem 10. *Let X be a metric space and $\mathcal{F} = \{\langle C_\gamma, I_\gamma\rangle : \gamma \in \Gamma\}$ be a Tietze family for X. Then there is a continuous function $h : X \to \mathbb{R}$ such that $h[C_\gamma] \subseteq I_\gamma$ for every $\gamma \in \Gamma$.*

Proof. Let $\mathcal{A} = \{C_\gamma : \gamma \in \Gamma\}$, and

$$T_{\mathcal{A}} = \left\{ \Phi \subseteq \Gamma : \bigcap_{\gamma\in\Phi} C_\gamma \neq \emptyset \right\}.$$

Let $\preccurlyeq_{\mathcal{A}}$ be the partial order of reversed inclusion on $T_{\mathcal{A}}$, that is, let $\Phi_1 \preccurlyeq_{\mathcal{A}} \Phi_2$ if and only if $\Phi_2 \subseteq \Phi_1$. Since every element of $\mathcal{A}$ intersects only finitely many elements of $\mathcal{A}$, it follows that the elements of $T_{\mathcal{A}}$ are finite sets and that $\preccurlyeq_{\mathcal{A}}$ has the finite predecessor property.

Let $\preccurlyeq^*_{\mathcal{A}}$ be an ω-order extending $\preccurlyeq_{\mathcal{A}}$ and for every $\Phi \in T_{\mathcal{A}}$ let

$$C_\Phi = \bigcap_{\gamma\in\Phi} C_\gamma \neq \emptyset.$$

Take the enumeration $\Phi_1, \Phi_2, \ldots$ of $T_{\mathcal{A}}$ with

$$\Phi_1 \preccurlyeq^*_{\mathcal{A}} \Phi_2 \preccurlyeq^*_{\mathcal{A}} \cdots$$

and for every $i = 1, 2, \ldots$ let

$$C_i = \bigcup_{j\leq i} C_{\Phi_j}, \qquad C'_i = C_i \cap C_{\Phi_{i+1}},$$

and

$$I_i = \bigcap_{\gamma\in\Phi_i} I_\gamma \neq \emptyset.$$

We are going to define a sequence $h_1, h_2, \ldots$ of continuous functions $h_i : C_i \to \mathbb{R}$ such that for every $i = 1, 2, \ldots$ the function h_{i+1} is an extension of h_i and

$$h_i[C_\gamma \cap C_i] \subseteq I_\gamma. \tag{4}$$

for every $\gamma \in \Gamma$. Having defined such a sequence of functions our proof will be complete since it is easy to see that the function

$$h = \bigcup_{i=1}^{\infty} h_i$$

satisfies the required conditions. Indeed, (4) implies that $h(C_\gamma) \subseteq I_\gamma$ for every $\gamma \in \Gamma$, and since $\mathcal{F}$ is a locally finite closed cover of X it follows that h is a continuous function on X.

Let $h_1 : C_1 \to I_1$ be any continuous function. Suppose that h_i has been defined in such a way that (4) is satisfied. Let h_i' be the restriction of h_i to C_i'. It follows from (4) that $h_i' : C_i' \to I_{i+1}$. Since C_i' is a closed subset of $C_{\Phi_{i+1}}$, it follows from Tietze Extension Theorem that h_i' can be extended to a continuous function $h_i'' : C_{\Phi_{i+1}} \to I_{i+1}$. Let $h_{i+1} = h_i \cup h_i''$. Since C_i and $C_{\Phi_{i+1}}$ are closed subsets of C_{i+1}, the function $h_{i+1} : C_{i+1} \to \mathbb{R}$ is continuous. It remains to show that (4) is satisfied for h_{i+1}.

Suppose that $\gamma \in \Gamma$ and $x \in C_\gamma \cap C_{i+1}$. If $x \in C_i$, then $h_{i+1}(x) = h_i(x) \in I_\gamma$ by the inductive hypothesis. Otherwise $x \in C_{\Phi_{i+1}}$ and so $h_{i+1}(x) = h_i''(x) \in I_{i+1}$. It suffices to show that $\gamma \in \Phi_{i+1}$.

Indeed, since $C_\gamma \cap C_{\Phi_{i+1}} \neq \emptyset$, it follows that $\Phi_{i+1} \cup \{\gamma\} \in T_{\mathcal{A}}$. Since

$$\Phi_{i+1} \cup \{\gamma\} \preccurlyeq_{\mathcal{A}} \Phi_{i+1}$$

and since $\preccurlyeq_{\mathcal{A}}^*$ extends $\preccurlyeq_{\mathcal{A}}$, it follows that there is $j \leq i+1$ with

$$\Phi_{i+1} \cup \{\gamma\} = \Phi_j.$$

Since $x \in C_\gamma \cap C_{\Phi_{i+1}} = C_{\Phi_j}$ and $x \notin C_i$, it follows that $j = i+1$. Thus $\gamma \in \Phi_{i+1}$ and so the proof is complete. ∎

Lemma 11. *Let $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{P}$ be a peripheral family for f, and $\mathcal{Q}$ be the cylindrical extension of $\mathcal{P}$. If $\{\langle A_j, I_j\rangle : 1 \leq j \leq k\} \subseteq \mathcal{Q}$ and $\operatorname{bd} A_i \cap \operatorname{bd} A_j \neq \emptyset$ for every $i, j \leq k$, then $\bigcap_{j=1}^k I_j \neq \emptyset$.*

Proof. First we shall prove the lemma for $k = 2$. Suppose, by way of contradiction, that there exist $\langle A_1, I_1\rangle, \langle A_2, I_2\rangle \in \mathcal{Q}$ with $\operatorname{bd} A_1 \cap \operatorname{bd} A_2 \neq \emptyset$ and $I_1 \cap I_2 = \emptyset$. Let

$\langle A_1', I_1\rangle, \langle A_2', I_2\rangle \in \mathcal{P}$ be such that

$$A_1 = A_1' \times (-a_1, a_1) \quad \text{and} \quad A_2 = A_2' \times (-a_2, a_2),$$

where $a_1 = \operatorname{diam} A_1'$ and $a_2 = \operatorname{diam} A_2'$.

Since $f[\operatorname{bd} A_1'] \subseteq I_1$ and $f[\operatorname{bd} A_2'] \subseteq I_2$, we have

$$\operatorname{bd} A_1' \cap \operatorname{bd} A_2' = \emptyset.$$

It follows that $A_1' \cap A_2' \neq \emptyset$ since otherwise we would have $\operatorname{cl} A_1' \cap \operatorname{cl} A_2' = \emptyset$ in contradiction with $\operatorname{bd} A_1 \cap \operatorname{bd} A_2 \neq \emptyset$. Since $\mathcal{P}$ has the intersection property, one of A_1', A_2' is a subset of the other.

Assume that $A_1' \subseteq A_2'$. Since $\operatorname{cl} A_1' \subseteq \operatorname{cl} A_2'$ and $\operatorname{bd} A_1' \cap \operatorname{bd} A_2' = \emptyset$, it follows that $\operatorname{cl} A_1' \subseteq A_2'$. Since the set $\operatorname{cl} A_1'$ is compact, there are $x_1, x_2 \in \operatorname{cl} A_1'$ with $\operatorname{diam} A_1'$ equal to the distance from x_1 to x_2. Since $x_1, x_2 \in A_2'$ and A_2' is open, it follows that

$$a_1 = \operatorname{diam} A_1' < \operatorname{diam} A_2' = a_2,$$

and so

$$\operatorname{bd} A_1 = \operatorname{bd} A_1' \times [-a_1, a_1] \cup A_1' \times \{-a_1, a_1\} \subseteq A_2' \times (-a_2, a_2) = A_2,$$

contradicting our assumption that $\operatorname{bd} A_1 \cap \operatorname{bd} A_2 \neq \emptyset$.

Now for $k > 2$ the assertion follows easily from the fact that if $\{I_j : 1 \leq j \leq k\}$ is a family of intervals in $\mathbb{R}$ and $I_j \cap I_m \neq \emptyset$ for every $j, m \leq k$, then $\bigcap_{j=1}^k I_j \neq \emptyset$. ∎

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, $\mathcal{P}$ be a peripheral family for f, $\mathcal{Q}$ be the cylindrical extension of $\mathcal{P}$, and

$$X = \mathbb{R}^{n+1} \setminus (\mathbb{R}^n \times \{0\}).$$

We need to construct a continuous function $h : X \to \mathbb{R}$ such that $h[\operatorname{bd} A] \subseteq I$ for every $\langle A, I\rangle \in \mathcal{Q}$. The existence of the function h will follow from Theorem 10 after we have constructed a Tietze family

$$\mathcal{F} = \{\langle C_\gamma, I_\gamma\rangle : \gamma \in \Gamma\}$$

for X such that for every $\langle A, I\rangle \in \mathcal{Q}$ there is $\Phi \subseteq \Gamma$ with

$$X \cap \operatorname{bd} A \subseteq \bigcup_{\gamma \in \Phi} C_\gamma \text{ and } I_\gamma = I \text{ for every } \gamma \in \Phi. \tag{5}$$

Let $\mathcal{K}$ consist of all closed intervals of the following forms: $[i, i+1]$, $[-i-1, -i]$, $[1/(i+1), 1/i]$, and $[-1/i, -1/(i+1)]$ for every $i = 1, 2, \ldots$. Set

$$\mathcal{A}_1 = \left\{ \left(\operatorname{cl} B_k^n \setminus B_{k-1}^n\right) \times [a,b] \subseteq \mathbb{R}^{n+1} : [a,b] \in \mathcal{K} \text{ and } k = 1, 2, \ldots \right\},$$

where $B_k^n \subseteq \mathbb{R}^n$ is the open ball with center $\langle 0, 0, \ldots, 0\rangle$ and radius k. Note that $\mathcal{A}_1$ is a locally finite closed cover of X.

Define

$$\mathcal{F}_1 = \{\langle C, \mathbb{R}\rangle : C \in \mathcal{A}_1\}$$

and

$$\mathcal{F}_2 = \{\langle \operatorname{bd} A \cap L, I\rangle : \langle A, I\rangle \in \mathcal{Q} \text{ and } L \in \mathcal{L}\},$$

where

$$\mathcal{L} = \{\mathbb{R}^n \times [a,b] : [a,b] \in \mathcal{K}\}.$$

Let Γ_1 and Γ_2 be disjoint sets of indices such that

$$\mathcal{F}_1 = \{\langle C_\gamma, I_\gamma\rangle : \gamma \in \Gamma_1\} \text{ and } \mathcal{F}_2 = \{\langle C_\gamma, I_\gamma\rangle : \gamma \in \Gamma_2\}.$$

Obviously, for every $\langle A, I\rangle \in \mathcal{Q}$ there is $\Phi \subseteq \Gamma_2$ such that (5) holds. Thus to complete the proof it remains to prove the following claim.

Claim. The family $\mathcal{F}_1 \cup \mathcal{F}_2$ is a Tietze family for X.

Let

$$\mathcal{A}_2 = \{C_\gamma : \gamma \in \Gamma_2\}.$$

Obviously, $\mathcal{A}_1 \cup \mathcal{A}_2$ is a closed cover of X. Since the family $\mathcal{P}$ is locally convergent to 0, every bounded subset of an element of $\mathcal{L}$ intersects only finitely many elements of $\mathcal{A}_2$. Since each point $x \in X$ has an open neighborhood contained in at most two elements of $\mathcal{L}$, it follows that $\mathcal{A}_2$ is locally finite, and hence $\mathcal{A}_1 \cup \mathcal{A}_2$ is locally finite.

Since every element C of $\mathcal{A}_1 \cup \mathcal{A}_2$ is a bounded subset of an element of $\mathcal{L}$, it follows that C intersects only finitely many elements in $\mathcal{A}_2$, and it is clear that C intersects only finitely many elements of $\mathcal{A}_1$. Thus every element of $\mathcal{A}_1 \cup \mathcal{A}_2$ intersects only finitely many elements in $\mathcal{A}_1 \cup \mathcal{A}_2$.

Now suppose that

$$\bigcap_{\gamma \in \Phi_1 \cup \Phi_2} C_\gamma \neq \emptyset$$

for some $\Phi_1 \subseteq \Gamma_1$ and $\Phi_2 \subseteq \Gamma_2$. Since $\bigcap_{\gamma \in \Phi_2} C_\gamma \neq \emptyset$, it follows from Lemma 11 that $\bigcap_{\gamma \in \Phi_2} I_\gamma \neq \emptyset$. Since $I_\gamma = \mathbb{R}$ for $\gamma \in \Phi_2$, we have

$$\bigcap_{\gamma \in \Phi_1 \cup \Phi_2} I_\gamma = \bigcap_{\gamma \in \Phi_2} I_\gamma \neq \emptyset.$$

Thus $\mathcal{F}_1 \cup \mathcal{F}_2$ is a Tietze family for X, and so the proof of the claim and hence of the theorem is complete.

References

[1] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Student Texts **39**, Cambridge Univ. Press 1997.

[2] K. Ciesielski and J. Wojciechowski, *Sums of connectivity functions on* $\mathbb{R}^n$, Proc. London Math. Soc. **76** (2) (1998), 406–426.

[3] J. L. Cornette, *Connectivity functions and images on Peano continua*, Fund. Math. **58** (1966), 183–192.

[4] R. Engelking, *General Topology, Revised and Completed Edition*, Sigma Series in Pure Mathematics, vol. **6**, Heldermann Verlag, Berlin 1989.

[5] R. G. Gibson, and F. Roush, *A characterization of extendable connectivity functions*, Real Anal. Exchange **13** (1987-88), 214–222.

[6] M. Hagan, *Equivalence of connectivity maps and peripherally continuous transformations*, Proc. Amer. Math. Soc. **17** (1966), 175–177.

[7] O. H. Hamilton, *Fixed points for certain noncontinuous transformations*, Proc. Amer. Math. Soc. **8** (1957), 750–756.

[8] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, 1948.

[9] J. H. Roberts, *Zero-dimensional sets blocking connectivity functions*, Fund. Math. **57** (1965), 173–179.

[10] J. R. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263.

(K. Ciesielski and J. Wojciechowski) Department of Mathematics, West Virginia University, PO Box 6310, Morgantown, WV 26506-6310, USA

E-mail address, K. Ciesielski: kcies@wvnvms.wvnet.edu

(T. Natkaniec) Department of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland

E-mail address, T. Natkaniec: mattn@ksinet.univ.gda.pl

E-mail address, J. Wojciechowski: jerzy@math.wvu.edu