# A DETERMINANT PROPERTY OF CATALAN NUMBERS

M. E. MAYS AND JERZY WOJCIECHOWSKI

Abstract. Catalan numbers arise in a family of persymmetric arrays with determinant 1. The demonstration involves a counting result for disjoint path systems in acyclic directed graphs.

## 1. Introduction

The *Catalan number* $c_n$ is the number of all sequences $\langle s_1, s_2, \ldots, s_{2n}\rangle$ such that $s_i \in \{-1, 1\}$, $\sum_{j=1}^{i} s_j \geq 0$ for every $i \in \{1, 2, \ldots, 2n-1\}$, and $\sum_{j=1}^{2n} s_j = 0$, in particular $c_0$ is the number of empty sequences so $c_0 = 1$. A *persymmetric* matrix is a square matrix with constant skew diagonals. In older literature, such matrices were called *orthosymmetric*.

Let $k, t$ be fixed integers with $k \geq 1$ and $t \geq 0$. Let $M_k^t = (m_{ij})_{i,j=1}^{k}$ be the persymmetric matrix with the sequence of consecutive Catalan numbers starting at $c_t$ being the first row of $M_k^t$. Explicitly, we have then $m_{ij} = c_{t+i+j-2}$, and

$$M_k^t = \begin{bmatrix} c_t & c_{t+1} & \cdots & c_{t+k-1} \\ c_{t+1} & c_{t+2} & \cdots & c_{t+k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{t+k-1} & c_{t+k} & \cdots & c_{t+2k-2} \end{bmatrix}.$$

Consider the infinite directed graph $G$ with $\mathbb{Z} \times \mathbb{Z}$ as the set of vertices and directed arcs from $(i, j)$ to $(i+1, j)$ and to $(i, j+1)$, for every $i, j \in \mathbb{Z}$. Let $d_i$ denote the vertex $(i, i)$ in $G$, $i \in \mathbb{Z}$. Note that the number of directed paths in $G$ from $d_i$ to $d_j$, with $j \geq i$, is equal to the Catalan number $c_{j-i}$. Let $Q_k^t$ be the family consisting of all sets of $k$ pairwise vertex disjoint directed paths $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ in $G$ such that $\gamma_i$ joins $d_{-i}$ with $d_{t+i}$, $i = 0, 1, \ldots, k-1$.

We are going to prove the following result.

**Theorem 1.** *The determinant* $\det M_k^t$ *is equal to the cardinality* $\left|Q_k^t\right|$.

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In particular, it is easy to see that $\left|Q_k^0\right| = \left|Q_k^1\right| = 1$, so we have the following corollary.

**Corollary 2.** *The determinants* $\det M_k^0$ *and* $\det M_k^1$ *are both equal to* 1.

It is also easy to see that

$$\det M_k^2 = \left|Q_k^2\right| = k + 1$$

and (see section 4)

$$\det M_k^3 = \left|Q_k^3\right| = \sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Our proof of Theorem 1 will be based on a result of Gronau et al. [1] on disjoint path systems in acyclic directed graphs. For the convenience of the reader we will include the proof of that result. For $t = 1$, our main result also follows from work of Shapiro [2], who noted an LU factorization of $M_k^1$ using an array he called a Catalan triangle.

## 2. Disjoint path systems

Let $G = (V, E)$ be an acyclic directed graph, where $V$ is a finite set of vertices and $E$ is a set of ordered pairs of vertices (directed edges of $G$). If $e = (v, w) \in E$, then $v$ is the *tail* of $e$ and $w$ is the *head* of $e$. The assumption that $G$ is acyclic means that there are no directed cycles in $G$, that is, there does not exist a sequence of vertices $v_0, v_1, \ldots, v_k$ with $(v_i, v_{i+1 \bmod k}) \in E$ for every $i \in \{0, 1, \ldots, k\}$, in particular, $(v, v) \notin E$ for every $v \in V$. A *source* (*sink*) in $G$ is a vertex of indegree 0 (outdegree 0), that is, a vertex that is not a head (tail) of any edge. A *path* in $G$ is a sequence $v_0, v_1, \ldots, v_k$ of distinct vertices such that $(v_i, v_{i+1}) \in E$ for every $i \in \{0, 1, \ldots, k-1\}$. We say that such a path *leads* from $v_0$ to $v_k$.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a certain fixed set of sources, and $B = \{b_1, b_2, \ldots, b_n\}$ be a fixed set of sinks in $G$. A *path system* in $(G, A, B)$ is a set $W = \{w_1, w_2, \ldots, w_n\}$ of paths in $G$ such that there exist a permutation $\sigma = \sigma(W) \in S_n$ so that $w_i$ leads from $a_i$ to $b_{\sigma(i)}$ for every $i \in \{1, 2, \ldots, n\}$. We say that $W$ is *disjoint* if for every $i$ and $j$ $(1 \leq i < j \leq n)$ the paths $w_i$ and $w_j$ have disjoint sets of vertices. Let $\mathcal{W}$ be the set of all (not necessarily disjoint) path systems in $(G, A, B)$.

**Theorem 3.** *([1]) Let $p_{ij}$ be the number of paths leading from $a_i$ to $b_j$ in $G$, let $p^+$ be the number of disjoint path systems $W$ in $(G, A, B)$ for which $\sigma(W)$ is an even permutation, and let $p^-$ be the number of such systems with for which $\sigma(W)$ is odd. Then* $\det(p_{ij}) = p^+ - p^-$.

*Proof.* If $w$ is a path in $G$, then let $E(w)$ be the set of edges used by $w$. If $W = (w_1, w_2, \ldots, w_n) \in \mathcal{W}$, then let

$$E(W) = \bigcup_{i=1}^{n} E(w_i).$$

Given a set of edges $D \subseteq E$ and a permutation $\tau \in S_n$, let

$$P(D, \tau) = \{W \in \mathcal{W} : E(W) = D \text{ and } \sigma(W) = \tau\},$$

and $p(D, \tau) = |P(D, \tau)|$.

Since

$$\bigcup_{D \subseteq E} P(D, \tau) = \{W \in \mathcal{W} : \sigma(W) = \tau\},$$

we have

$$\sum_{D \subseteq E} p(D, \tau) = \prod_{i=1}^{n} p_{i\tau(i)},$$

and so

$$\begin{aligned} p &= \sum_{\tau \in S_n} \sum_{D \subseteq E} \operatorname{sgn}(\tau) p(D, \tau) = \sum_{\tau \in S_n} \operatorname{sgn}(\tau) \sum_{D \subseteq E} p(D, \tau) \\ &= \sum_{\tau \in S_n} \operatorname{sgn}(\tau) \prod_{i=1}^{n} p_{i\tau(i)} = \det\left(p_{ij}\right). \end{aligned}$$

To complete the proof it remains to show that $p = p^+ - p^-$.

Let $\mathcal{D}_1$ be the set of all $D \subseteq E$ such that $D = E(W)$ for some disjoint path system $W \in \mathcal{W}$, let $\mathcal{D}_2$ be the set of all $D \subseteq E$ such that $D = E(W)$ for some $W \in \mathcal{W}$ that is not disjoint, and let $\mathcal{D}_3$ be the set of all $D \subseteq E$ such that $D \neq E(W)$ for any $W \in \mathcal{W}$. Then we have

$$\begin{aligned} p &= \sum_{\tau \in S_n} \sum_{D \subseteq E} \operatorname{sgn}(\tau) p(D, \tau) = \sum_{D \subseteq E} \sum_{\tau \in S_n} \operatorname{sgn}(\tau) p(D, \tau) \\ &= \sum_{D \in \mathcal{D}_1} \sum_{\tau \in S_n} \operatorname{sgn}(\tau) p(D, \tau) + \sum_{D \in \mathcal{D}_2} \sum_{\tau \in S_n} \operatorname{sgn}(\tau) p(D, \tau) \\ &\quad + \sum_{D \in \mathcal{D}_3} \sum_{\tau \in S_n} \operatorname{sgn}(\tau) p(D, \tau). \end{aligned}$$

If $D = E(W) \in \mathcal{D}_1$, then $W$ is the only path system in $\mathcal{W}$ with $D = E(W)$, implying that

$$\sum_{\tau \in S_n} \operatorname{sgn}(\tau) p(D, \tau) = \operatorname{sgn}(\sigma(W)).$$

Therefore

$$\sum_{D\in\mathcal{D}_1}\sum_{\tau\in S_n}\operatorname{sgn}(\tau)p(D,\tau)=p^+-p^-.$$

Since obviously

$$\sum_{D\in\mathcal{D}_3}\sum_{\tau\in S_n}\operatorname{sgn}(\tau)p(D,\tau)=0,$$

the proof of the theorem will be complete when we show that

$$\sum_{D\in\mathcal{D}_2}\sum_{\tau\in S_n}\operatorname{sgn}(\tau)p(D,\tau)=0. \tag{*}$$

**Claim:** If $D\in\mathcal{D}_2$ and

$$\mathcal{W}_D=\bigcup_{\tau\in S_n}P(D,\tau)=\{W\in\mathcal{W}:E(W)=D\},$$

then there is a bijection $f:\mathcal{W}_D\to\mathcal{W}_D$ such that

$$\operatorname{sgn}(\sigma(f(W)))=-\operatorname{sgn}(\sigma(W)) \tag{†}$$

for every $W\in\mathcal{W}_D$.

It is clear that the claim implies (*). It remains to prove the claim.

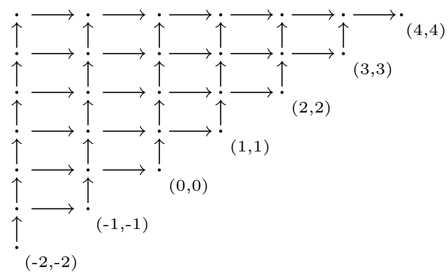
Let $D\in\mathcal{D}_2$. If $W=\{w_1,w_2,\dots,w_n\}\in\mathcal{W}_D$, then let $i\in\{1,2,\dots,n\}$ be the smallest integer with $w_i$ having a common vertex with another path of $W$, let $v$ be the first vertex along $w_i$ that is also a vertex of another path of $W$, and let $j\in\{1,2,\dots,n\}\setminus\{i\}$ be the smallest integer such that $v$ is a vertex of $w_j$. Let $f(W)=\{w_1',w_2',\dots,w_n'\}$ be a path system in $\mathcal{W}_D$ such that $w_k'=w_k$ for $k\notin\{i,j\}$ and $w_i'$, $w_j'$ are obtained from $w_i$ and $w_j$ respectively, by exchanging the segments leading from $v$ to $b_{\tau(i)}$ and from $v$ to $b_{\tau(j)}$, where $\tau=\sigma(W)$. Clearly (†) is satisfied and since $f\circ f$ is the identity map on $\mathcal{W}_D$, it follows that $f$ is a bijection. Hence the proof of the claim, and thus of the theorem, is complete. ∎

## 3. Proof of the main result

In this section we are going to prove Theorem 1. Let $k,t$ be positive integers with $k\geq 1$, $t\geq 0$, and $G_k^t$ be the subgraph of the directed acyclic graph $G$ (see section 1) that is induced by the following set of vertices:

$$V_k^t=\{(i,j)\in\mathbb{Z}\times\mathbb{Z}:-k+1\leq i\leq t+k-1 \text{ and } i\leq j\leq t+k-1\}.$$

For example, the graph $G_3^2$ looks as follows




Note that for every $i, j \in \mathbb{Z}$ with

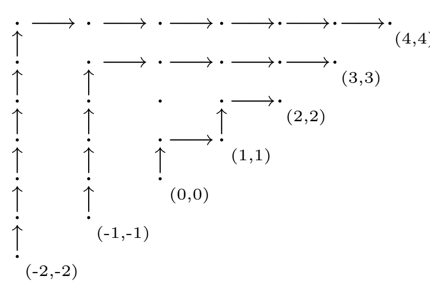
$$-k+1 \leq i \leq j \leq t+k-1$$

the number of directed paths in $G_k^t$ from $d_i$ to $d_j$ is the same as in $G$, and so it is equal to the Catalan number $c_{j-i}$.

Let $A_k = \{a_1, a_2, \ldots, a_k\}$ and $B_k = \{b_1, b_2, \ldots, b_k\}$ be $k$-element sets that are disjoint from $V_k^t$ and from each other. Let $H_k^t$ be the directed acyclic graph obtained from $G_k^t$ by adding $A_k \cup B_k$ to the set of vertices and adding new directed arcs from $a_i$ to $d_{-i+1}$ and from $d_{t+i-1}$ to $b_i$, $i = 1, 2, \ldots, k$. Then, for every $i, j = 1, 2, \ldots, k$, the number of directed paths in $H_k^t$ from $a_i$ to $b_j$ is equal to the Catalan number $c_r$ with

$$r = (t+j-1) - (-i+1) = t+i+j-2,$$

which is $m_{ij}$. Consider disjoint path systems in $\left(H_k^t, A_k, B_k\right)$. It is easy to see that for each such system $W$ the permutation $\sigma(W)$ is the identity permutation. See the following picture for an example of such a system in $H_3^2$ (the vertices $a_i$ and $b_j$ are not pictured).




Therefore, we have $p^+ = \left|Q_k^t\right|$ and $p^- = 0$. It follows from Theorem 3 that

$$\det M_k^t = p^+ - p^- = \left|Q_k^t\right|.$$

Thus the proof of Theorem 1 is complete.

## 4. Applications

**Corollary 4.** *Consider the infinite array $A = \{a_{i\,j}\}$ in which the rows are interpreted as sequences satisfying the following properties:*

(1) $a_{i\,j} = a_{i-1\,j}$ *for* $j \leq 2i-2$
(2) *If* $j > 2i-2$, $a_{i\,j}$ *satisfies*

$$\det \begin{bmatrix} a_{i\,j-2i+2} & \cdots & a_{i\,j-i} & a_{i\,j-i+1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{i\,j-i+2} & \cdots & a_{i\,j-2} & a_{i\,j-1} \\ a_{i\,j-i+1} & \cdots & a_{i\,j-1} & a_{i\,j} \end{bmatrix} = 1,$$

*then* $a_{1\,1}, a_{1\,2}, a_{2\,3}, a_{2\,4}, a_{3\,5}, a_{3\,6}, \ldots$ *is the sequence of Catalan numbers.*

The array defined by these properties is

$$N = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ 1 & 1 & 2 & 5 & 13 & 34 & 89 & \ldots \\ 1 & 1 & 2 & 5 & 14 & 42 & 99 & \ldots \\ 1 & 1 & 2 & 5 & 14 & 42 & 132 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This array is not persymmetric itself, of course, but $i \times i$ persymmetric arrays built from any $2i-1$ successive elements of the $i$th row have determinant 1. Successive rows of $N$ may be interpreted as sequences that converge to the sequence of Catalan numbers, in the sense that row $i$ matches the Catalan sequence for the first $2i$ terms. The $2 \times 2$ determinants that determine the second row make it the sequence of every other Fibonacci number.

**Corollary 5.** *In the infinite persymmetric array whose first row is the Catalan sequence*

$$M = \begin{bmatrix} 1 & 1 & 2 & 5 & 14 & \ldots \\ 1 & 2 & 5 & 14 & 42 & \ldots \\ 2 & 5 & 14 & 42 & 132 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

*any finite square submatrix has positive determinant. Furthermore, the Catalan sequence is the smallest sequence with this property, in the sense that among all sequences of integers with this property it is lexicographically least.*

*Proof.* Any square $k \times k$ submatrix of $M$ is equal to the matrix $M_k^t$ for some $t \geq 0$. It follows from Theorem 1 that $\det M_k^t > 0$.

To conclude the proof it remains to show that among all sequences of integers with the stated property the Catalan sequence is lexicographically least. Suppose that $c_0', c_1', \ldots$ has the stated property and is not the Catalan sequence. Let $r$ be the smallest integer such that $c_r \neq c_r'$. We need to show that $c_r < c_r'$. Obviously $c_0', c_1' \geq 1$ so we can assume that $r \geq 2$. There are integers $k, t$ with $k \geq 2$ and $t \in \{0, 1\}$ such that $r = t + 2k - 2$. Expanding the determinant with respect to the last row we get

$$\begin{aligned} 0 &< \det \begin{bmatrix} c_t' & c_{t+1}' & \cdots & c_{t+k-1}' \\ c_{t+1}' & c_{t+2}' & \cdots & c_{t+k}' \\ \vdots & \vdots & \ddots & \vdots \\ c_{t+k-1}' & c_{t+k}' & \cdots & c_{t+2k-2}' \end{bmatrix} \\ &= \det M_k^t + \left(c_{2k-1}' - c_{2k-1}\right) \det M_{k-1}^t \\ &= 1 + \left(c_{2k-1}' - c_{2k-1}\right), \end{aligned}$$

since $\det M_k^t = \det M_{k-1}^t = 1$ by Corollary 2. Thus $c_{2k-1} < c_{2k-1}'$, completing the proof. ∎

The corollaries provide novel characterizations of the Catalan sequence, somewhat removed from the enumerative settings in which the sequence usually arises. In particular, Corollary 4 generates the Catalan sequence two terms at a time, as a limiting sequence of a family of sequences, each given via linear recurrences of slow growing order. Corollary 5 specifies a determinant property that many sequences may possess, but for which the Catalan sequence is least in a natural order.

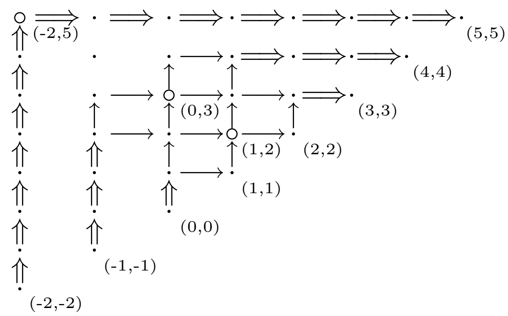
Finally, let us prove the following easy observation mentioned in section 1.

**Proposition 6.** *For every positive integer $k$ we have*

$$\det M_k^3 = \left|Q_k^3\right| = \sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

*Proof.* To see that $\left|Q_k^3\right| = \sum_{i=1}^{k+1} i^2$, partition $Q_k^3$ into $k+1$ sets $R_0, R_1, \ldots, R_k$ such that $R_i$ consists of those sets of paths $\left\{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}\right\} \in Q_k^3$, where $\gamma_i$ joins $d_{-i}$ with $d_{t+i}$, that satisfy the extra condition that $\gamma_j$ goes through $(-j, t+j)$ for $j = i, i+1, \ldots, k-1$ and through $(-j+1, t+j-1)$ for $j = 0, 1, \ldots, i-1$. For example, if $k = 3$, then $R_2$ consists of the sets $\{\gamma_0, \gamma_1, \gamma_2\} \in Q_3^3$ such that $\gamma_0$ goes through $(1, 2)$, $\gamma_1$ goes through $(0, 3)$, and $\gamma_2$

goes through $(-2,5)$; see the following picture where double arrows denote arcs that have to be taken and single arrows denote possible arcs.




It is easy to see that $|R_i| = (i+1)^2$. For example, in the picture above there are three possibilities for the paths $\gamma_0$ and $\gamma_1$ to reach vertices $(1,2)$ and $(0,3)$ respectively, and three possibilities for the remaining parts of the paths, implying that $|R_2| = 3^2$. Thus the proof is complete. ∎

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Department of Mathematics, West Virginia University, PO Box 6310, Morgantown, WV 26506-6310