THE BASIS NUMBER OF THE POWERS OF THE COMPLETE GRAPH

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Abstract

A basis of the cycle space $\mathcal{C}(G)$ of a graph G is h-fold if each edge of G occurs in at most h cycles of the basis. The basis number $b(G)$ of G is the least integer h such that $\mathcal{C}(G)$ has an h-fold basis. MacLane [3] showed that a graph G is planar if and only if $b(G) \leq 2$. Schmeichel [4] proved that $b(K_n) \leq 3$, and Banks and Schmeichel [2] proved that $b(K_2^d) \leq 4$ where K_2^d is the d-dimesional hypercube. We show that $b(K_n^d) \leq 9$ for any n and d, where K_n^d is the cartesian d-th power of the complete graph K_n.

[0]Keywords: cycle space of a graph, basis number, powers of complete graphs

1 Introduction

Let G be a graph, and let $e_1, e_2, \cdots, e_q$ be an enumeration of its edges. Then, any subset S of $E(G)$ corresponds to a $(0,1)$-vector $(a_1, a_2, \ldots, a_q) \in (Z_2)^q$ with $a_i = 1$ if $e_i \in S$ and $a_i = 0$ if $e_i \notin S$. Let $\mathcal{C}(G)$, called the *cycle space* of G, be the subspace of $(Z_2)^q$ generated by the vectors corresponding to the cycles in G. We will identify the elements of $\mathcal{C}(G)$ with the corresponding subsets of $E(G)$. It is well known that if G is connected, then the dimension of $\mathcal{C}(G)$ is given by the following formula:

$$\dim(\mathcal{C}(G)) = q - p + 1$$

where p and q denote, respectively, the number of vertices and edges of G. In fact, if T is a spanning tree in G and for every $e \in E(G) \setminus E(T)$ we denote by C_e the unique cycle with $E(C_e) \subseteq E(T) \cup \{e\}$, then the collection

$$B_T = \{E(C_e) : e \in E(G) \setminus E(T)\}$$

forms a basis of $\mathcal{C}(G)$, called the *fundamental basis corresponding to* T.

Let B be any basis of $\mathcal{C}(G)$, and h be a positive integer. We say that B is *h-fold* if each edge of G occurs in at most h cycles of B. The *basis number* of G (denoted by $b(G)$) is the smallest integer h such that $\mathcal{C}(G)$ has an h-fold basis.

The first important result concerning the basis number of a graph was the following theorem of MacLane [3].

Theorem 1 *A graph G is planar if and only if $b(G) \leq 2$.*

Schmeichel [4] proved that there are graphs with arbitrary large basis numbers, and generalized the "only if" part of Theorem 1 by showing that

$$b(G) \leq 2\gamma(G) + 2$$

for any graph G, where $\gamma(G)$ is the genus of G. Moreover, Schmeichel [4] proved the following result.

Theorem 2 *For every integer $n \geq 5$, the basis number of the complete graph K_n is equal to* 3.

Note that Theorems 1 and 2 imply that the basis number of any complete graph is at most 3.

Let K_2^d be the *d-dimensional hypercube*, that is, the d-th cartesian power of the complete graph K_2. Banks and Schmeichel [2] proved the following result about the basis number of the hypercube.

Theorem 3 *For every integer $d \geq 7$, the basis number of the hypercube K_2^d is equal to* 4.

In this paper, we are interested in establishing an upper bound on the basis number of the d-th cartesian power of any complete graph.

For completeness, let us recall the definition of the cartesian product of two graphs. The *product* $G \times H$ of G and H is the graph with $V(G) \times V(H)$ as the vertex set and (g_1, h_1) adjacent to (g_2, h_2) if either $g_1g_2 \in E(G)$ and $h_1 = h_2$, or else if $g_1 = g_2$ and $h_1h_2 \in E(H)$. Let K_n^d be the product of d copies of the complete graph K_n, $n \geq 2$, $d \geq 1$.

Ali and Marougi [1] proved the following result on the cartesian product of graphs.

Theorem 4 *For any two connected graphs G and H, we have*

$$b(G \times H) \leq \max \{b(G) + \Delta(T_H), b(H) + \Delta(T_G)\}$$

where T_G denotes a spanning tree of G with maximal degree as small as possible, and $\Delta(T_G)$ denotes the maximal degree of T_G.

Applying Theorems 2 and 4, we get easily the following two corollaries.

Corollary 5 *For every $n \geq 2$ and $k \geq 0$, we have*

$$b(K_n^{2^k}) \leq 2k + 3.$$

Proof. It follows from Theorems 1 and 2 that $b(K_n^{2^0}) \leq 3$. Assume now that $k \geq 1$ and that $b(K_n^{2^{k-1}}) \leq 2(k-1)+3$. Since $K_n^{2^k} = K_n^{2^{k-1}} \times K_n^{2^{k-1}}$ and since any power of a complete graph is hamiltonian, it follows from Theorem 4 that

$$b(K_n^{2^k}) \leq b(K_n^{2^{k-1}}) + 2 \leq 2k + 3,$$

and the proof is complete. ∎

Corollary 6 *For every $n \geq 2$ and $d \geq 1$, we have*

$$b(K_n^d) \leq 2 \log_2 d + 5.$$

Proof. We show, by induction on k, that if d is an integer satisfying $2^{k-1} \leq d \leq 2^k$, then $b(K_n^d) \leq 2k + 3$. For $k = 0$ and $k = 1$, the claim follows from Corollary 5. Assume that $k \geq 2$, and that our claim holds for smaller values of k. Let d satisfy $2^{k-1} \leq d \leq 2^k$, and let $d_1 = \lfloor d/2 \rfloor$ and $d_2 = \lceil d/2 \rceil$. Then

$$2^{k-2} \leq d_1, d_2 \leq 2^{k-1},$$

so it follows from the inductive hypothesis that $b(K_n^{d_1}) \leq 2\,(k-1) + 3$ and $b(K_n^{d_2}) \leq 2\,(k-1) + 3$. Using Theorem 4 and the fact that any power of a complete graph is hamiltonian, we get

$$b(K_n^{2^k}) \leq \max\left\{ b(K_n^{d_1}) + 2, b(K_n^{d_2}) + 2 \right\} \leq 2k + 3,$$

completing the proof of the claim.

It follows from the claim that

$$b(K_n^d) \leq 2\lceil \log_2 d \rceil + 3 \leq 2\log_2 d + 5,$$

completing the proof. ∎

In view of Theorem 3 saying that the basis number of a hypercube has a constant upper bound, a natural question arises: does there exist a constant c that is independent of n and d such that the basis number of K_n^d is bounded from above by c for arbitrary values of n and d? Using a generalization of the technique developed by Banks and Schmeichel [2], we will show that the answer to the above question is positive. We shall prove the following result.

Theorem 7 *For every $n \geq 2$ and $d \geq 1$, the basis number of K_n^d is at most 9.*

2 Preliminary lemmas

Let $n \geq 2$ be a fixed integer. We are going first to define, recursively, a collection $\mathcal{B}_d$ of cycles in K_n^d, for every $d \geq 1$. We will show later that $\mathcal{B}_d$ is a 9-fold basis of $\mathcal{C}(K_n^d)$.

It will be convenient to think of the vertices of K_n^d as d-tuples of n-ary digits, *i.e.,* the elements from the set $\{0, 1, \ldots, n-1\}$ with two such d-tuples being adjacent if and only if they differ at exactly one coordinate.

Given $i \in \{0, 1, \ldots, n-1\}$ and a vertex $u = (x_1, x_2, \ldots, x_d)$ of K_n^d, let $u^{(i)}$ denote the vertex of K_n^{d+1} that is obtained from u by adjoining the digit i at the end, *i.e.*, let

$$u^{(i)} = (x_1, x_2, \ldots, x_d, i)\,.$$

If G is any subgraph of the graph K_n^d, then $G^{(i)}$ will denote the isomorphic copy of G in K_n^{d+1} obtained by adjoining the digit i at the end of each vertex of G, and if $\mathcal{R}$ is a collection of subgraphs of K_n^d, then let

$$\mathcal{R}^{(i)} = \left\{ G^{(i)} : G \in \mathcal{R} \right\},$$

$i = 0, 1, \ldots, n-1$.

For any $d \geq 1$, let

$$\tau_d = \Big(t(d,1), t(d,2), \cdots, t(d, n^d - 1) \Big)$$

be a sequence of integers defined recursively as follows. Take

$$\tau_1 = (\underbrace{1, 1, \ldots, 1}_{n-1})$$

and for $d > 1$, define τ_d by:

$$t(d,i) = \begin{cases} t(d-1,k) & \text{if } i = nk, \\ d & \text{otherwise.} \end{cases}$$

Then, for example, we have

$$\tau_2 = (\underbrace{\underbrace{2, 2, \ldots, 2}_{n-1}, 1, \underbrace{2, 2, \ldots, 2}_{n-1}, 1, \ldots, 1, \underbrace{2, 2, \ldots, 2}_{n-1}, 1}_{n-1}, \underbrace{2, 2, \ldots, 2}_{n-1}).$$

We will use the sequence τ_d to define a Hamiltonian path in K_n^d.

Given a pair u, v of vertices of K_n^d and $i \in \{0, 1, \ldots, n-1\}$, we say that the vertex v is an *i-successor* of the vertex u if the i-th coordinate of v is equal to the i-th coordinate of u plus 1 modulo n and all other coordinates of u and v are the same. Let

$$W_d = (w_1, w_2, \ldots, w_{n^d})$$

be a sequence of vertices in K_n^d defined as follows. Take w_1 to be arbitrary and, for every i with $1 \leq i < n^d$, let w_{i+1} be the $t(d,i)$-successor of w_i. The sequence W_d will be called the *τ_d-sequence starting at w_1*.

Lemma 8 *For any vertex u of K_n^d, the τ_d-sequence starting at u is a Hamiltonian path in K_n^d.*

Proof. We use induction with respect to d. For $d = 1$, the result is obvious. Assume that $d \geq 2$, and that for any vertex u of K_n^{d-1} the τ_{d-1}-sequence starting at u is a Hamiltonian path in K_n^{d-1}. Let v be any vertex of K_n^d, and W_d be the τ_d-sequence starting at v. We will show that W_d is a Hamiltonian path in K_n^d. Let i be the last digit of v, let w_1 be the vertex of K_n^{d-1} such that $v = w_1^{(i)}$, and let

$$W_{d-1} = (w_1, w_2, \ldots, w_{n^{d-1}})$$

be the τ_{d-1}-sequence starting at w_1. It follows from the definition of τ_d that

$$W_d = \Big(w_1^{(i)}, w_1^{(i+1)}, \ldots, w_1^{(i+n-1)}, w_2^{(i-1)}, w_2^{(i)}, \ldots, w_2^{(i+n-2)}, \\ \ldots, w_{n^{d-1}}^{(i+1)}, w_{n^{d-1}}^{(i+2)}, \ldots, w_{n^{d-1}}^{(i+n)}\Big),$$

where the top indexes are taken modulo n. Since W_{d-1} is a Hamiltonian path in K_n^{d-1}, it is clear from the above representation that W_d is a Hamiltonian path in K_n^d, and so the proof is complete. ∎

The τ_d-sequence starting at $(1, 1, \ldots, 1, 0)$ will be called the *τ_d-path* and, in the remainder of this chapter, we reserve the symbol W_d for the τ_d-path, $d \geq 1$.

Given an integer $d \geq 2$, assume that the τ_{d-1}-path W_{d-1} is equal to the following sequence of vertices

$$W_{d-1} = (w_1, w_2, \ldots, w_{n^{d-1}}).$$

Then the sequence

$$W_{d-1}^{(j)} = \left(w_1^{(j)}, w_2^{(j)}, \ldots, w_{n^{d-1}}^{(j)}\right)$$

is a path in K_n^d, for every $j \in \{0, 1, \ldots, n-1\}$. For every $i \in \left\{1, 2, \ldots, n^{d-1}\right\}$, let J_i be the subgraph of K_n^d induced by the set of vertices

$$\left\{w_i^{(j)} : j \in \{0, 1, \ldots, n-1\}\right\}.$$

Clearly, the graph J_i is isomorphic to K_n for each i. As we saw earlier, $b(K_n) \leq 3$. For every $i \in \left\{1, 2, \ldots, n^{d-1}\right\}$, let D_i be a 3-fold basis of J_i, and let

$$\mathcal{D}_d = \bigcup_{i=1}^{n^{d-1}} D_i$$

For every $i \in \left\{1, 2, \ldots, n^{d-1} - 1\right\}$ and $j \in \{0, 1, \ldots, n-2\}$, let $C_{i,j}$ be the following 4-cycle

$$C_{i,j} = \left(w_i^{(j)}, w_{i+1}^{(j)}, w_{i+1}^{(j+1)}, w_i^{(j+1)}, w_i^{(j)}\right),$$

and let

$$\mathcal{F}_d = \left\{C_{i,j} : i \in \left\{1, 2, \ldots, n^{d-1} - 1\right\} \text{ and } j \in \{0, 1, \ldots, n-2\}\right\}.$$

Now we are ready to define the collection $\mathcal{B}_d$ of cycles in K_n^d. For $d = 1$, let $\mathcal{B}_1$ be any 3-fold basis of K_n. Assume now that $d \geq 2$ and that the collection $\mathcal{B}_{d-1}$ has been defined. Define $\mathcal{B}_d$ by:

$$\mathcal{B}_d = \bigcup_{i=0}^{n-1} \mathcal{B}_{d-1}^{(i)} \cup \mathcal{D}_d \cup \mathcal{F}_d.$$

To show that $\mathcal{B}_d$ is a 9-fold basis of $\mathcal{C}(K_n^d)$, we will need some prelimary results.

Let $d \geq 2$ be an integer. Assume that

$$W_{d-1} = (w_1, w_2, \ldots, w_{n^{d-1}})$$

is the τ_{d-1}-path. For every $i \in \left\{1, 2, \ldots, n^{d-1} - 1\right\}$ and every $j, k \in \{0, 1, \ldots, n-1\}$ with $j \neq k$, let $C_{i,j,k}$ be the following 4-cycle

$$C_{i,j,k} = \left(w_i^{(j)}, w_{i+1}^{(j)}, w_{i+1}^{(k)}, w_i^{(k)}, w_i^{(j)}\right).$$

Note that $C_{i,j,(j+1)}$ is equal to the cycle $C_{i,j}$ defined before. Let

$$\mathcal{E}_d = \left\{C_{i,j,k} : i \in \left\{1, 2, \cdots, n^{d-1} - 1\right\},\ \ j, k \in \{0, 1, \cdots, n-2\},\ j < k\right\}.$$

Recall that $J_{n^{d-1}}$ is the subgraph of K_n^d induced by the set of vertices

$$\{w_{n^{d-1}}^{(j)} : j = 0, 1, \ldots, n-1\}.$$

Let $\mathcal{D}'_d = D_{n^{d-1}}$ be the 3-fold basis of $J_{n^{d-1}}$ that is contained in $\mathcal{D}_d$.

If $\mathcal{R}_{d-1} \subseteq \mathcal{C}(K_n^{d-1})$ and $\mathcal{R}_{d-1}^{(i)}$ is obtained from $\mathcal{R}_{d-1}$ by adding the digit i at the end of each vertex of each cycle, then the collection

$$\mathcal{R}_{d-1}^{+} = \bigcup_{i=0}^{n-1} \mathcal{R}_{d-1}^{(i)}$$

of cycles in K_n^d will be called the *lift* of $\mathcal{R}_{d-1}$.

Lemma 9 *If $\mathcal{R}_{d-1}$ is a basis of $\mathcal{C}(K_n^{d-1})$, then $\mathcal{R}_{d-1}^{+} \cup \mathcal{D}'_d \cup \mathcal{E}_d$ is a basis of $\mathcal{C}(K_n^d)$.*

Proof. Let $\mathcal{R}_{d-1}$ be any basis of K_n^{d-1} and let

$$\mathcal{R}_d = \mathcal{R}_{d-1}^{+} \cup \mathcal{D}'_d \cup \mathcal{E}_d$$

Since the graph K_n^d has n^d vertices and is $(n-1)d$-regular, it has $n^d(n-1)d/2$ edges, and so

$$\dim \mathcal{C}(K_n^d) = \frac{n^d(n-1)d}{2} - n^d + 1 = n^d \left(\frac{(n-1)d}{2} - 1\right) + 1. \qquad (1)$$

Thus

$$|\mathcal{R}_{d-1}| = \dim \mathcal{C}(K_n^{d-1}) = n^{d-1}\left(\frac{(n-1)(d-1)}{2} - 1\right) + 1,$$

and

$$|\mathcal{R}_{d-1}^+| = n|\mathcal{R}_{d-1}| = n^d\left(\frac{(n-1)(d-1)}{2} - 1\right) + n.$$

Moreover, we have

$$|\mathcal{E}_d| = \left(n^{d-1} - 1\right)\frac{n(n-1)}{2} = \frac{n^d(n-1)}{2} - \frac{n(n-1)}{2},$$

and

$$|\mathcal{D}'_d| = \dim \mathcal{C}(K_n) = \frac{n(n-1)}{2} - n + 1.$$

Therefore

$$\begin{aligned} |\mathcal{R}_d| &= |\mathcal{R}_{d-1}^+| + |\mathcal{D}'_d| + |\mathcal{E}_d| \\ &= n^d\left(\frac{(n-1)(d-1)}{2} - 1\right) + \frac{n^d(n-1)}{2} + 1 \\ &= n^d\left(\frac{(n-1)d}{2} - 1\right) + 1 \\ &= \dim \mathcal{C}(K_n^d). \end{aligned}$$

Thus to prove that $\mathcal{R}_d$ is a basis of $\mathcal{C}(K_n^d)$, it suffices to show that the cycles of $\mathcal{R}_d$ are linearly independent.

Suppose, by way of contradiction, that there is a nonempty subset $S \subseteq \mathcal{R}_d$ such that

$$\sum_{C \in S} C = 0 \bmod 2.$$

Since $\mathcal{R}_{d-1}$ is a basis of $\mathcal{C}(K_n^{d-1})$, the set $\mathcal{R}_{d-1}^{(i)}$ is linearly independent in $\mathcal{C}(K_n^d)$, for every $i \in \{0, 1, \dots, n-1\}$. Since any cycle in $\mathcal{R}_{d-1}^{(i)}$ is edge-disjoint from any cycle in $\mathcal{R}_{d-1}^{(j)}$ for $j \neq i$, the set $\mathcal{R}_{d-1}^+$ is linearly independent in $\mathcal{C}(K_n^d)$. Since the set $\mathcal{D}'_d$ is a basis of $J_{n^{d-1}}$, and since no cycle in $\mathcal{D}'_d$ share an edge with a cycle in $\mathcal{R}_{d-1}^{+}$, it follows that $\mathcal{R}_{d-1}^{+} \cup \mathcal{D}_d'$ is linearly independent in $\mathcal{C}(K_n^d)$. Therefore S must include at least one cycle $\mathcal{E}_d$, *i.e.* there are $i \in \left\{1, 2, \ldots, n^{d-1} - 1\right\}$ and $j, k \in \{0, 1, \cdots, n-2\}$ with $j \neq k$ such that $C_{i,j,k} \in S$.

We claim that $C_{1,j,k} \in S$. Indeed, if $i = 1$, then we are done. If $i > 1$, then since $C_{i,j,k}$ contains the edge $w_i^{(j)} w_i^{(k)}$ and the only other cycle in $\mathcal{R}_d$ containing the edge $w_i^{(j)} w_i^{(k)}$ is $C_{i-1,j,k}$, we conclude that $C_{i-1,j,k} \in S$. Continuing by induction we get $C_{1,j,k} \in S$, and so the proof of the claim is complete.

Since the cycle $C_{1,j,k}$ contains the edge $w_1^{(j)} w_1^{(k)}$ which occurs in no other cycle of $\mathcal{R}_d$, and in particular in no other cycle of S we have a contradiction. Thus $\mathcal{R}_d$ is a basis of $\mathcal{C}(K_n^d)$, and the proof of the lemma is complete. ∎

Let G_d be the spanning subgraph of K_n^d with $E(G_d)$ consisting of all the edges of the paths

$$W_{d-1}^{(j)} = \left(w_1^{(j)}, w_2^{(j)}, \ldots, w_{n^{d-1}}^{(j)}\right),$$

for $j \in \{0, 1, \ldots, n-1\}$, and all the edges of the graphs J_i for $i \in \left\{1, 2, \ldots, n^{d-1}\right\}$, *i.e.*, let

$$\begin{aligned} E(G_d) &= \left\{w_i^{(j)} w_i^{(k)} : i \in \left\{1, 2, \ldots, n^{d-1}\right\}; j, k \in \{0, 1, \ldots, n-1\}; j \neq k\right\} \\ &\quad \cup \left\{w_i^{(j)} w_{i+1}^{(j)} : i \in \left\{1, 2, \ldots, n^{d-1} - 1\right\}, j \in \{0, 1, \ldots, n-1\}\right\}. \end{aligned}$$

Note that the graph G_d is isomorphic to the cartesian product of the complete graph K_n with a path of length n^{d-1}.

Lemma 10 *The union $\mathcal{D}_d \cup \mathcal{F}_d$ is a basis of $\mathcal{C}(G_d)$.*

Proof. Since for each $i \in \left\{1, 2, \ldots, n^{d-1}\right\}$ the graph J_i has $n(n-1)/2$ edges, it follows that

$$|E(G_d)| = n^d (n-1)/2 + n\left(n^{d-1} - 1\right).$$

Therefore

$$\begin{aligned}\dim\mathcal{C}(G_d) &= \left(\frac{n^d(n-1)}{2}+n(n^{d-1}-1)\right)-n^d+1\\ &= \left(\frac{n-1}{2}\right)n^d-n+1.\end{aligned}$$

Since $\mathcal{D}_d$ consists of n^{d-1} disjoint copies of a basis of K_n, it follows that

$$|\mathcal{D}_d| = n^{d-1}\left(\frac{n(n-1)}{2}-n+1\right).$$

Since

$$|\mathcal{F}_d| = (n^{d-1}-1)(n-1)$$

we conclude that

$$\begin{aligned}|\mathcal{D}_d\cup\mathcal{F}_d| &= n^{d-1}\left(\frac{n(n-1)}{2}-n+1\right)+(n^{d-1}-1)(n-1)\\ &= \left(\frac{n-1}{2}\right)n^d-n+1\\ &= \dim\mathcal{C}(G_d).\end{aligned}\tag{2}$$

Therefore, to show that $\mathcal{D}_d\cup\mathcal{F}_d$ is a basis of $\mathcal{C}(G_d)$ it suffices to show that the cycles of $\mathcal{D}_d\cup\mathcal{F}_d$ are linearly independent. Suppose, by way of contradiction, that there is $S\subseteq\mathcal{D}_d\cup\mathcal{F}_d$ such that

$$\sum_{C\in S} C = 0 \bmod 2.$$

Since the graphs J_1, J_2, …, $J_{n^{d-1}}$ are mutually vertex disjoint and D_i is a basis of J_i for each i, it follows that the set $\mathcal{D}_d$ is linearly independent in $\mathcal{C}(G_d)$. Therefore S must contain at least one cycle from $\mathcal{F}_d$, *i.e.* the are $i\in\left\{1,2,\ldots,n^{d-1}-1\right\}$ and $j\in\{0,1,\ldots,n-2\}$ such that

$$C_{i,j}=\left(w_i^{(j)},w_{i+1}^{(j)},w_{i+1}^{(j+1)},w_i^{(j+1)},w_i^{(j)}\right)\in S.$$

We claim that $C_{i,1} \in S$. Indeed, if $j > 0$, then the cycle

$$C_{i,j-1} = \left(w_i^{(j-1)}, w_{i+1}^{(j-1)}, w_{i+1}^{(j)}, w_i^{(j)}, w_i^{(j-1)}\right)$$

is the only other cycle of $\mathcal{D}_d \cup \mathcal{F}_d$ containing the edge $w_i^{(j)} w_{i+1}^{(j)}$. Since $C_{i,j} \in S$, it follows that $C_{i,j-1} \in S$. Continuing by induction, we see that S must contain the cycle $C_{i,1}$, and so the proof of the claim is complete.

Since $C_{i,1}$ is the only cycle of $\mathcal{D}_d \cup \mathcal{F}_d$ containing the edge $w_i^{(0)} w_{i+1}^{(0)}$, we have a contradiction. Thus $\mathcal{D}_d \cup \mathcal{F}_d$ is a basis of $\mathcal{C}(G_d)$, and the proof of the lemma is complete. ∎

3 Proof of the main result

The aim of this section is to prove Theorem 7. It suffices to establish that $\mathcal{B}_d$ is a 9-fold basis of K_n^d.

Theorem 11 *For each $d \geq 1$, the set $\mathcal{B}_d$ is a basis for K_n^d.*

Proof. We are going to use induction with respect to d. For $d = 1$, $\mathcal{B}_1$ is a basis of K_n by the definition. Assume that $d \geq 2$ and that $\mathcal{B}_{d-1}$ is a basis of K_n^{d-1}. By the definition, we have

$$\mathcal{B}_d = \mathcal{B}_{d-1}^{+} \cup \mathcal{D}_d \cup \mathcal{F}_d.$$

It follows from (2) and (1) that

$$\begin{aligned}
|\mathcal{B}_d| &= n \dim \mathcal{C}(K_n^{d-1}) + \left(\frac{n-1}{2}\right) n^d - n + 1 \\
&= n \left(n^{d-1} \left(\frac{(n-1)(d-1)}{2} - 1 \right) + 1 \right) + \left(\frac{n-1}{2}\right) n^d - n + 1 \\
&= n^d \left(\frac{(n-1)(d-1)}{2} - 1 \right) + \left(\frac{n-1}{2}\right) n^d + 1 \\
&= n^d \left(\frac{(n-1)d}{2} - 1 \right) + 1 \\
&= \dim \mathcal{C}(K_n^d).
\end{aligned}$$

Therefore, to prove that $\mathcal{B}_d$ is a basis of $\mathcal{C}(K_n^d)$ it suffices to show that $\mathcal{B}_d$ spans $\mathcal{C}(K_n^d)$. Since it follows from Lemma 9 that $\mathcal{B}_{d-1}^+ \cup \mathcal{D}'_d \cup \mathcal{E}_d$ is a basis of $\mathcal{C}(K_n^d)$, it is enough to show that $\mathcal{B}_d$ spans $\mathcal{B}_{d-1}^+ \cup \mathcal{D}'_d \cup \mathcal{E}_d$. Since $\mathcal{B}_{d-1}^+ \cup \mathcal{D}'_d \subseteq \mathcal{B}_d$, we only need to show that $\mathcal{B}_d$ spans $\mathcal{E}_d$. Since each cycle of $\mathcal{E}_d$ is a cycle in the graph G_d, it follows from Lemma 10 that $\mathcal{D}_d \cup \mathcal{F}_d$ spans $\mathcal{E}_d$. Since $\mathcal{D}_d \cup \mathcal{F}_d \subseteq \mathcal{B}_d$, the proof is complete. ∎

Given $k \in \{0, 1, \ldots, n-1\}$, an edge e of K_n^d is said to be of *type* k if and only if the vertices of e differ in exactly the k-th coordinate. Two edges e_1 and e_2 of K_n^d are said to be *k-correspondent* if and only if the two vertices of e_2 can be obtained by changing the k-th coordinate of the vertices of e_1. The edge e_2 is an *k-successor* of e_1 if and only if the k-coordinates of the endpoints of e_2 are obtained from the k-coordinates of the endpoints of e_1 by adding 1 modulo n. Note that if two edges of K_n^d are ℓ-correspondent, then they must be of the same type k, with $k \neq \ell$.

Assume that $d \geq 3$. Recall that, for each $i = 0, 1, \ldots, n-1$, the path $W_{d-1}^{(i)}$ is the path in K_n^d obtained by adding the digit i at the end of each vertex of the τ_{d-1}-path W_{d-1}.

Lemma 12 *If A is the set of all n mutually $(d-1)$-correspondent edges of K_n^d, then at most one edge of A can be an edge of the path $W_{d-1}^{(0)}$.*

Proof. Let $A = \{e_0, e_1, \ldots, e_{n-1}\}$ be the set of all mutually $(d-1)$-correspondent edges, with $e_i = u_i v_i$ and such that the $(d-1)$-coordinate of u_i and v_i is equal to i for $i \in \{0, 1, \ldots, n-1\}$. Assume that the edges in A are of type k. Since the path $W_{d-1}^{(0)}$ does not have any edges of type d, we can assume that $k \leq d-2$.

Suppose that e_0 occurs in $W_{d-1}^{(0)}$. Since W_{d-1} is the τ_{d-1}-path, and since each occurrence of k in τ_{d-1} is followed by $n-1$ entries equal to $d-1$, the vertices following v_0 in $W_{d-1}^{(0)}$ must be v_1, v_2, …, v_{n-1}, *i.e.*,

$$W_{d-1}^{(0)} = (\ldots, u_0, v_0, v_1, v_2, \ldots, v_{n-1}, \ldots)\,.$$

Since $n \geq 3$, the vertex u_{n-1} cannot follow v_{n-1} in $W_{d-1}^{(0)}$. Therefore neither of the edges e_1, e_2, …, e_{n-1} can occur in $W_{d-1}^{(0)}$, and the proof of the lemma is complete. ∎

Lemma 13 *If $d \geq 3$ and $k \leq d-2$, then the number of terms with value d preceeding any occurence of k in τ_d is divisible by n.*

Proof. Let α_d be the sequence defined by

$$\alpha_d = (\underbrace{\underbrace{d, d, \ldots, d}_{n-1}, d-1, \underbrace{d, d, \ldots, d}_{n-1}, d-1, \ldots, \underbrace{d, d, \ldots, d}_{n-1}, d-1,}_{n-1} \underbrace{d, d, \ldots, d}_{n-1}).$$

It is clear from the construction that the sequence τ_d consists of copies of the sequence α_d with a single term having value at most $d-2$ between any two consecutive copies of α_d. Since the number of terms with value d in α_d

is equal to $n\,(n-1)$, each term with value $k \leq d-2$ in τ_d is preceeded by a multiple of n of terms with value d. This completes the proof. ∎

For each $k \in \{1, 2, \ldots, d\}$, let $e_{j,k}$ denote the j-th edge of type k of the path W_d and let $e'_{j,k}$ be the j-th edge of type k in $W_{d-1}^{(0)}$. Note that if $k \leq d-1$, then the number edges of type k in W_d is the same as the number of edges of type k in $W_{d-1}^{(0)}$, so for each j, the edge $e_{j,k}$ exists if and only if $e'_{j,k}$ exists.

Lemma 14 *If $k \leq d-2$, then the edge $e_{j,k}$ is the $(d-1)$-successor of $e'_{j,k}$, for every j such that both $e_{j,k}$ and $e'_{j,k}$ exist.*

Proof. Assume that $k \leq d-2$ and that j is such that both $e_{j,k}$ and $e'_{j,k}$ exist. Let u, v be the endpoints of $e_{j,k}$ (with u appearing before v in W_d) and u', v' be the endpoints of $e'_{j,k}$ (with u' appearing before v' in $W_{d-1}^{(0)}$). It is enough to show that u is the $(d-1)$-successor of u'. Note that the first vertex of W_d is equal to $(1, 1, \ldots, 1, 0)$, and the first vertex of $W_{d-1}^{(0)}$ is equal to $(1, 1, \ldots, 1, 0, 0)$. Thus the the first vertex of $W_{d-1}^{(0)}$ is the $(d-1)$-successor of the first vertex of $W_{d-2}^{(0,0)}$.

Let ξ_ℓ^m be the number of terms with value m preceeding the j-th occurrence of k in the sequence τ_ℓ, for every $m = 1, 2, \cdots, d$ and $\ell = d-1, d$. To prove that u is the $(d-1)$-successor of u', we need to show that

$$\xi_d^m \equiv \xi_{d-1}^m \bmod n$$

for every $m = 1, 2, \ldots, d$.

Indeed, by the definition of τ_d, we have $\xi_d^m = \xi_{d-1}^m$ for every $m \leq d-1$. Clearly $\xi_{d-1}^d = 0$, and since $k \leq d-2$, it follows from Lemma 13 that ξ_d^d is divisible by n. Therefore, u is the $(d-1)$-successor of u' and the proof is complete. ∎

Theorem 7 follows immediately from the following result.

Theorem 15 *The set $\mathcal{B}_d$ is a 9-fold basis of $\mathcal{C}(K_n^d)$ for every $d \geq 1$.*

Proof. For every $i \in \left\{1, 2, \cdots, n^{d-1} - 1\right\}$ let $C_{i,n-1}$ be the 4-cycle

$$C_{i,n-1} = \left(w_i^{(n-1)}, w_{i+1}^{(n-1)}, w_{i+1}^{(0)}, w_i^{(0)}, w_i^{(n-1)}\right),$$

and let

$$\begin{aligned}\overline{\mathcal{F}}_d &= \mathcal{F}_d \cup \left\{C_{i,n-1} : i \in \left\{1, 2, \ldots, n^{d-1} - 1\right\}\right\} \\ &= \left\{C_{i,j} : i \in \left\{1, 2, \ldots, n^{d-1} - 1\right\} \text{ and } j \in \{0, 1, \ldots, n-1\}\right\}.\end{aligned}$$

Define recursively the collection $\overline{\mathcal{B}}_d$ of cycles in K_n^d by $\overline{\mathcal{B}}_1 = \mathcal{B}_1$ and

$$\overline{\mathcal{B}}_d = \overline{\mathcal{B}}_{d-1}^{+} \cup \mathcal{D}_d \cup \overline{\mathcal{F}}_d,$$

where

$$\overline{\mathcal{B}}_{d-1}^{+} = \bigcup_{i=0}^{n-1} \overline{\mathcal{B}}_{d-1}^{(i)}$$

is the lift of $\overline{\mathcal{B}}_{d-1}$. Clearly, $\mathcal{B}_d \subseteq \overline{\mathcal{B}}_d$ for every $d \geq 1$. We will show, by induction with respect to d, that each edge e of K_n^d is used at most 9 times in the cycles of $\overline{\mathcal{B}}_d$, and moreover, that it is used at most 7 times in the cycles of $\overline{\mathcal{B}}_d$ when e is an edge of the τ_d-path W_d, and that e is used at most 5 times in the cycles of $\overline{\mathcal{B}}_d$ if e is of type d. The basis $\overline{\mathcal{B}}_1$ is 3-fold by its definition, and it is clear from the construction that each edge of K_n^2 appears in at most 5 cycles of $\overline{\mathcal{B}}_2$, so our claim holds for $d = 1$ and $d = 2$.

Assume that $d \geq 3$, and that the set $\overline{\mathcal{B}}_{d-1}$ satisfies the specified condition. Since any edge e of K_n^d of type d appears in at most 3 cycles of the set $\mathcal{D}_d$, in at most 2 cycles of the set $\overline{\mathcal{F}}_d$ and in no cycles of $\overline{\mathcal{B}}_{d-1}^{+}$, the edge e is used at most 5 times in the cycles of $\overline{\mathcal{B}}_d$. Assume now that e is an edge of K_n^d of type $k \leq d-1$. Clearly, the edge e appears in no cycles of $\mathcal{D}_d$. If e is an edge of the path $W_{d-1}^{(i)}$ for some $i \in \{0, 1, \ldots, n-1\}$, then it appears in exactly 2 cycles of $\overline{\mathcal{F}}_d$, and by the inductive hypothesis, it appears in at most 7 cycles of $\overline{\mathcal{B}}^+_{d-1}$. Otherwise, the edge e appears in no cycles of $\overline{\mathcal{F}}_d$ and in at most 9 cycles of $\overline{\mathcal{B}}^+_{d-1}$. To complete the proof, it remains to show that if e is an edge of W_d of type $k \leq d-1$, then it appears in at most 7 cycles of $\overline{\mathcal{B}}_d$.

Let e be an edge of type $k \leq d-1$ in the path W_d. Suppose first that $k = d-1$. Then it follows from the inductive hypothesis that e appears in at most 5 cycles of $\overline{\mathcal{B}}^+_{d-1}$. Since e appears in at most 2 cycles of $\overline{\mathcal{F}}_d$, it appears in at most 7 cycles of $\overline{\mathcal{B}}_d$.

Now assume that $k \leq d-2$, and that $e = e_{j,k}$, *i.e.,* that e is the j-th edge of type k in the path W_d. Consider the j-th edge $e'_{j,k}$ of type k in $W^{(0)}_{d-1}$. By the inductive hypothesis, the edge $e'_{j,k}$ occurs in at most 7 cycles of $\overline{\mathcal{B}}^{(0)}_{d-1}$. By Lemma 14, $e_{j,k}$ is the $(d-1)$-successor of $e'_{j,k}$. Since any two $(d-1)$-correspondent edges of K_n^{d-1} appear in the same number of cycles of $\overline{\mathcal{B}}_{d-1}$, it follows that the edges $e_{j,k}$ and $e'_{j,k}$ appear in the same number of cycles of $\overline{\mathcal{B}}^{(0)}_{d-1}$. Therefore, $e_{j,k}$ appears in at most 7 cycles of $\overline{\mathcal{B}}^{(0)}_{d-1}$. Since $e'_{j,k}$ belongs to $W^{(0)}_{d-1}$, it follows from Lemma 12 that $e_{j,k}$ does not belong to $W^{(0)}_{d-1}$. Hence $e_{j,k}$ occurs in at most 7 cycles of $\overline{\mathcal{B}}_d$ and the proof is complete. ∎

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