On Constructing Snakes in Powers of Complete Graphs

Jerzy Wojciechowski

Department of Mathematics, West Virginia University,
PO BOX 6310, Morgantown, WV 26506-6310, USA

E-mail: JERZY@MATH.WVU.EDU

Abstract. We prove the conjecture of Abbott and Katchalski that for every $m \geq 2$ there is a positive constant λ_m such that $S(K_{mn}^d) \geq \lambda_m n^{d-1} S(K_m^{d-1})$ where $S(K_m^d)$ is the length of the longest snake (cycle without chords) in the cartesian product K_m^d of d copies of the complete graph K_m. As a corollary, we conclude that for any finite set P of primes there is a constant $c = c(P) > 0$ such that $S(K_n^d) \geq cn^{d-1}$ for any n divisible by an element of P and any $d \geq 1$.

Supported by the WVU Senate Research Grant #R-93-033

1. Introduction

Let G be a graph. By a *path* in G we mean a sequence of distinct vertices of G with every pair of consecutive vertices being adjacent. A path will be called *closed* if its first vertex is adjacent to the last one.

Let P be a path in the graph G. By a *chord* of P we mean an edge of G joining two nonconsecutive vertices of P. If P is closed and e is a chord of P, then we say that e is a *proper chord* if it is not the edge joining the first vertex of P to its last vertex. Note that a proper chord of a closed path corresponds to the standard notion of a chord in a cycle. By a *snake* we mean a closed path without proper chords, and an *open snake* is a path without chords.

Let G and H be graphs. The *product* $G \times H$ of G and H is the graph with $V(G) \times V(H)$ as the vertex set and (g_1, h_1) adjacent to (g_2, h_2) if either $g_1 g_2 \in E(G)$ and $h_1 = h_2$, or else if $g_1 = g_2$ and $h_1 h_2 \in E(H)$.

Let K_n^d be the product of d copies of the complete graph K_n, $n \geq 2$, $d \geq 1$. It will be convenient to think of the vertices of K_n^d to be the d-tuples of n-ary digits, *i.e.* the elements of the set $\{0, 1, \ldots, n-1\}$, with edges between any two d-tuples differing at exactly one coordinate.

Let $S(K_n^d)$ be the length of the longest snake in K_n^d. The problem of estimating the value of $S(K_n^d)$ has a long history. It was first met by Kautz [9] in the case of $n = 2$ (known in the literature as the *snake-in-the-box problem*) in constructing a type of error-checking code for a certain analog-to-digital conversion systems. The evaluation of $S(K_2^d)$ has proven to be a notoriously difficult problem and, on the other hand, it has been demonstrated to be of importance in connection with several applied problems (see for example [10], [11]). As a consequence several authors became interested in estimating the value of $S(K_2^d)$ and a large literature has evolved (see [5] for a list of references). Subsequently, the general case of the problem with an arbitrary value of n has been introduced by Abbott and Dierker

[2] and developed further by Abbott and Katchalski [4], [6], and Wojciechowski [15]. The following theorem is a result of these investigations.

Theorem 1.1. *For any integer $n \geq 2$, there is a constant $c_n > 0$ such that*

$$S(K_n^d) \geq c_n n^d, \tag{1.1}$$

for any $d \geq 1$.

In the case when $n = 2$, Theorem 1.1 was first proved by Evdokimov [8]. Other shorter proofs, in that case, were given by Abbott and Katchalski [3] and Wojciechowski [13]. The largest value of the constant $c_2 = \frac{77}{256} = 0.300781\ldots$ was obtained by Abbott and Katchalski [5].

In the case when $n \equiv 0 \bmod 4$, Theorem 1.1 has been proved by Abbott and Katchalski [6]. Actually, they proved the following theorem that allows for this case of Theorem 1.1 to be deduced from the case when $n = 2$.

Theorem 1.2. *If $n \equiv 0 \bmod 4$, then*

$$S(K_n^d) \geq \left(\frac{n}{2}\right)^{d-1} S(K_2^{d-1}),$$

for every $d \geq 3$.

As remarked by Abbott and Katchalski [6], a modification of their technique can be used to prove that the following more general theorem holds.

Theorem 1.3. *There is a constant $\lambda > 0$ such that if $n \geq 2$ is an even integer, then*

$$S(K_n^d) \geq \lambda \left(\frac{n}{2}\right)^{d-1} S(K_2^{d-1}),$$

for any $d \geq 2$.

Theorem 1.3 implies that Theorem 1.1 holds for every even integer $n \geq 2$. In the case of n being odd, Theorem 1.1 was proved by Wojciechowski [15]. He proved the following result which implies the corresponding case of Theorem 1.1.

Theorem 1.4. *If $n \geq 3$ is an odd integer, then*

$$S(K_n^d) \geq 2(n-1)n^{d-4},$$

for any $d \geq 5$.

The constant c_n in Theorem 1.1 cannot be made independent of n since Abbott and Katchalski [4] proved that

$$S(K_n^d) \leq \left(1 + \frac{1}{d-1}\right) n^{d-1}.$$

However the following conjecture seems plausible.

Conjecture 1.5. *There is a constant $c > 0$ such that*

$$S(K_n^d) \geq cn^{d-1}, \tag{1.2}$$

for any $n \geq 2$, $d \geq 1$.

It follows from Theorems 1.1 and 1.3 that if we restrict the range of values of n to even integers, then Conjecture 1.5 holds, *i.e.* the following theorem is true.

Theorem 1.6. *There is a constant $c > 0$ such that if $n \geq 2$ is an even integer, then*

$$S(K_n^d) \geq cn^{d-1},$$

for any $d \geq 1$.

In the general case, however, Conjecture 1.5 remains still open since in the case of n being odd, the value of c in (1.2) given by Theorem 1.4 ($c = 2(n-1)/n^3$) depends on n and approaches 0 when n tends to infinity.

The main result of this paper is the following generalization of Theorem 1.3 conjectured by Abbott and Katchalski [1].

Theorem 1.7. *For any integer $m \geq 2$, there is a constant $\lambda_m > 0$ such that*

$$S(K_{mn}^d) \geq \lambda_m n^{d-1} S(K_m^{d-1}),$$

for any $n \geq 1$ and $d \geq 2$.

As a corollary of Theorem 1.7, we get the following generalization of Theorem 1.6 which provides further evidence for Conjecture 1.5 to be true.

Theorem 1.8. *Let P be a finite set of primes. Then there is a constant $c = c(P) > 0$ such that*

$$S(K_n^d) \geq c n^{d-1},$$

for any integer n that is divisible by an element of P and for any $d \geq 1$.

Actually, we prove the following result that implies Theorem 1.7.

Theorem 1.9. *Let $m, n \geq 2$ and $d \geq 4$ be integers. Then*

$$S(K_{mn}^d) \geq n^{d-1} \left(S(K_m^{d-1}) + 1 \right),$$

if n is even, and

$$S(K_{mn}^d) \geq (n-1) n^{d-2} \left(S(K_m^{d-1}) + 1 \right),$$

if n is odd.

The proof of Theorem 1.9 is given in section 3.

2. Basic Definitions

A k-*path* in a graph is a path consisting of k vertices, *i.e.*a path of length $k-1$. If P is a k-path, then we will write $k = |P|$. A *chain* $\mathcal{C}$ of paths is a sequence $(P_1, P_2, \ldots, P_k)$ of paths such that each path of $\mathcal{C}$ has at least two vertices, and the last vertex of P_i is equal to the first vertex of P_{i+1}, $i = 1, 2, \ldots, k-1$. A k-*chain* of paths is a chain consisting of k paths. A chain $\mathcal{C} = (P_i)_{i=1}^k$ of paths will be called *closed* if the first vertex of P_1 is equal to the last vertex of P_k. If $\mathcal{C}$ is a kr-chain of paths, then the r-*splitting* of $\mathcal{C}$ is the sequence $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r)$ of k-chains of paths which joined together (juxtaposed) give $\mathcal{C}$.

Let G and H be graphs. Given a k-chain of paths $\mathcal{C} = (P_i)_{i=1}^k$ in G with $P_i = (u_1, u_2, \ldots, u_{r_i})$ and a k-path $Q = (v_i)_{i=1}^k$ in H, let $\mathcal{C} \otimes Q$ be the $\left(\sum_{i=1}^k |P_i|\right)$-path in the graph $G \times H$ obtained by juxtaposing the paths P'_1, P'_2, $\ldots$, P'_k, where $P'_i = \big((u_1, v_i), (u_2, v_i), \ldots, (u_{r_i}, v_i)\big)$, $i = 1, 2, \ldots, k$.

Let $d, m, n \geq 2$ be integers. We assume d, m and n to be fixed throughtout the paper. Given an integer p with $1 \leq p \leq d$, let G_p be the graph

$$G_p = K_{mn}^p \times K_m^{d-p}.$$

In particular $G_d = K_{mn}^d$. Let $p \geq 1$ and $q \geq 1$ be integers with $p + q \leq d$. If $u = (a_1, a_2, \ldots, a_d)$ is a vertex of the graph G_p $(i.e.a_1, a_2, \ldots, a_p \in \{0, 1, \ldots, mn-1\}$, $a_{p+1}, a_{p+2}, \ldots, a_d \in \{0, 1, \ldots, m-1\})$ and $v = (b_1, b_2, \ldots, b_q)$ is a vertex of the graph K_n^q, then let $u \boxplus v$ be the vertex of $G_{p+q} = K_{mn}^{p+q} \times K_m^{d-p-q}$ defined by

$$u \boxplus v = (a_1, a_2, \ldots, a_p, a'_{p+1}, a'_{p+2}, \ldots, a'_{p+q}, a_{p+q+1}, a_{p+q+2}, \ldots, a_d),$$

where

$$a'_{p+i} = a_{p+i} + mb_i,$$

for $i = 1, 2, \ldots, q$. If $P = (u_i)_{i=1}^k$ is a k-path in G_p and v is a vertex in K_n^q, then let $P \boxplus v = (u_1 \boxplus v, u_2 \boxplus v, \ldots, u_k \boxplus v)$. Clearly $P \boxplus v$ is a k-path in G_{p+q}.

Given a k-chain of paths $\mathcal{C} = (P_i)_{i=1}^k$ in the graph G_p and a k-path $Q = (v_i)_{i=1}^k$ in the graph K_n^q, let $\mathcal{C} \boxtimes Q$ be the $\left(\sum_{i=1}^k |P_i|\right)$-path in the graph G_{p+q} obtained by juxtaposing the paths P_1', P_2', $\ldots$, P_k', where $P_i' = P_i \boxplus v_i$, $i = 1, 2, \ldots, k$. Note that if the chain $\mathcal{C}$ is closed and the path Q is closed, then the path $\mathcal{C} \boxtimes Q$ is also closed.

Given a kr-chain $\mathcal{C}$ of paths in G_p with the r-splitting $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r)$ and an r-chain $\mathcal{D} = (P_i)_{i=1}^r$ of k-paths in the graph K_n^q, set

$$\mathcal{C} \boxtimes \mathcal{D} = (\mathcal{C}_1 \boxtimes P_1, \mathcal{C}_2 \boxtimes P_2, \ldots, \mathcal{C}_r \boxtimes P_r).$$

Note that for each $i \in \{1, 2, \ldots, r-1\}$ the last vertex of the path $\mathcal{C}_i \boxtimes P_i$ is equal to the first vertex of the path $\mathcal{C}_{i+1} \boxtimes P_{i+1}$, hence the sequence $\mathcal{C} \boxtimes \mathcal{D}$ is an r-chain of paths in G_{p+q}. Note that if the chains $\mathcal{C}$ and $\mathcal{D}$ are closed, then the chain $\mathcal{C} \boxtimes \mathcal{D}$ is also closed. It is straightforward to verify that the following property holds.

Property 2.1. *Let q_1, q_2 be positive integers with $q_1 + q_2 = q$. If $\mathcal{C}$ is a kr-chain of paths in G_p, $\mathcal{D}$ is an r-chain of k-paths in $K_n^{q_1}$ and P is a k-path in $K_n^{q_2}$, then*

$$\mathcal{C} \boxtimes (\mathcal{D} \otimes P) = (\mathcal{C} \boxtimes \mathcal{D}) \boxtimes P.$$

□

If v_1, v_2 are vertices of G_p, then we say that v_1 and v_2 are *apart* if they differ either at one of the first p coordinates or at least at two coordinates. Let P_1 and P_2 be paths in G_p. We say that P_1 and P_2 are *apart* if for every pair of vertices v_1, v_2 of P_1, P_2 respectively, the vertices v_1 and v_2 are apart. We say that P_1 and P_2 are *almost apart* if they have one vertex v in common and for every pair of vertices v_1, v_2 of P_1, P_2 respectively, such that at least one of v_1, v_2 is different than v, the vertices v_1 and v_2 are apart.

When we refer to a pair s_i, s_j of elements of a sequence $(s_1, s_2, \ldots, s_t)$, we say that s_i and s_j are *cyclically consecutive* if either $j = i \pm 1$ or $\{i, j\} = \{1, t\}$.

Let $\mathcal{C} = (P_i)_{i=1}^k$ be a chain of paths in the graph G_p. We say that $\mathcal{C}$ is *openly separated* if any two consecutive paths of $\mathcal{C}$ are almost apart and any two nonconsecutive paths are apart. We say that $\mathcal{C}$ is *closely separated* if $\mathcal{C}$ is closed, any two cyclically consecutive paths of $\mathcal{C}$ are almost apart and any two cyclically nonconsecutive paths are apart. The following lemma holds.

Lemma 2.2. *Let $\mathcal{C}$ be a chain of open snakes in the graph G_p and Q be a path in K_n^q.*

(i) If $\mathcal{C}$ is openly separated, then the path $\mathcal{C} \boxtimes Q$ is an open snake in the graph G_{p+q}.

(ii) If $\mathcal{C}$ is closely separated and Q is closed, then the path $\mathcal{C} \boxtimes Q$ is a snake in G_{p+q}.

Proof. Let $\mathcal{C} = (P_i)_{i=1}^k$ and $Q = (v_i)_{i=1}^k$. Then the path $R = \mathcal{C} \boxtimes Q$ is obtained by juxtaposing the paths P_1', P_2', …, P_k', where $P_i' = P_i \boxplus v_i$, $i = 1, 2, \ldots, k$. Let w_1, w_2 be vertices of R that are adjacent in G_{p+q} and assume that $w_1 = u_1 \boxplus v_i$, $w_2 = u_2 \boxplus v_j$ where u_1 is a vertex of P_i and u_2 is a vertex of P_j, $i, j \in \{1, 2, \ldots, k\}$. If $i = j$, then the vertices u_1, u_2 are adjacent in G_p, hence they must be consecutive in P_i since P_i is an open snake. Therefore w_1, w_2 are consecutive in R and so w_1w_2 is not a chord of R.

Assume now that $i \neq j$. Since $w_1w_2 \in E(G_{p+q})$, the vertices w_1, w_2 differ at exactly one coordinate $t \in \{1, 2, \ldots, d\}$. Hence u_1, u_2 must agree at each coordinate in $\{1, 2, \ldots, d\} \setminus \{t\}$. Since $v_i \neq v_j$, it follows that $t \in \{p+1, p+2, \ldots, p+q\}$, so u_1, u_2 are not apart. Therefore the paths P_i, P_j are not apart.

If $\mathcal{C}$ is openly separated, then any two nonconsecutive paths of $\mathcal{C}$ are apart, hence the paths P_i, P_j are consecutive in $\mathcal{C}$ (say P_j follows P_i) and they are almost apart in G_p. Thus $u_1 = u_2$ is the last vertex of P_i and the first vertex of P_j, thus w_1, w_2 are consecutive in R. Hence w_1w_2 is not a chord of R, and the proof of (i) is complete.

Similarly, if $\mathcal{C}$ is closely separated, then the paths P_i, P_j are cyclically consecutive in $\mathcal{C}$ (say P_j follows P_i) and $u_1 = u_2$ must be the last vertex of P_i and the first vertex of P_j, hence w_1, w_2 are cyclically consecutive in R. Thus w_1w_2 is not a proper cord of R and the proof of (ii), hence of the lemma, is complete. □

If P is a path, then let $-P$ be the path obtained from P by reversing the order of vertices, and if $\mathcal{C} = (P_i)_{i=1}^r$ is a chain of paths, then let $-\mathcal{C} = (-P_r, -P_{r-1}, \ldots, -P_1)$ be the chain of paths obtained from $\mathcal{C}$ by reversing the order of paths and reversing every path. The expression $(-1)^i X$, where X is a path or a chain of paths, will mean X for i even and $-X$ for i odd. Obviously, the following property holds.

Property 2.3. *If $\mathcal{C}$ is an r-chain of paths in the graph G_p and P is an r-path in the graph K_n^q, then $\mathcal{C} \boxtimes (-Q) = -(-\mathcal{C} \boxtimes Q)$.* □

Let $\mathcal{C}$ be a kr-chain of paths, and let $\mathcal{S} = \big(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r\big)$ be the r-splitting of $\mathcal{C}$. By the *alternate matrix* of the splitting $\mathcal{S}$ we mean the following $(r \times k)$-matrix $\mathcal{A}$ of paths:

$$\mathcal{A} = \begin{pmatrix} \mathcal{C}_1 \\ -\mathcal{C}_2 \\ \vdots \\ (-1)^{r-1}\mathcal{C}_r \end{pmatrix} = \begin{pmatrix} Q_1^1 & Q_1^2 & \ldots & Q_1^k \\ -Q_2^1 & -Q_2^2 & \ldots & -Q_2^k \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{r-1}Q_r^1 & (-1)^{r-1}Q_r^2 & \ldots & (-1)^{r-1}Q_r^k \end{pmatrix}$$

where $\mathcal{C}_i = (Q_i^1, Q_i^2, \ldots, Q_i^k)$ for i odd and $\mathcal{C}_i = (Q_i^k, Q_i^{k-1}, \ldots, Q_i^1)$ for i even, $i = 1, 2, \ldots, r$. The splitting $\mathcal{S}$ will be called *openly alternating* if for any $\ell \in \{1, 2, \ldots, k\}$ and for any two distinct paths Q_i^ℓ, Q_j^ℓ appearing in the ℓ-th column of $\mathcal{A}$, the paths Q_i^ℓ, Q_j^ℓ are almost apart when they are consecutive in $\mathcal{C}$ and they are apart otherwise.

Assume now that the chain $\mathcal{C}$ is closed and r is even. Then, we say that the splitting $\mathcal{S}$ is *closely alternating* if for any $\ell \in \{1, 2, \ldots, k\}$ and for any two distinct paths Q_i^ℓ, Q_j^ℓ appearing in the ℓ-th column of $\mathcal{A}$, the paths Q_i^ℓ, Q_j^ℓ are almost apart when they are cyclically consecutive in $\mathcal{C}$ and they are apart otherwise. The following lemma holds.

Lemma 2.4. *Let $\mathcal{C}$ be a kr-chain of paths in the graph G_p and P be a k-path in K_n^q.*

(i) If the r-splitting of $\mathcal{C}$ is openly alternating and $\mathcal{D}$ is the r-chain $(P, -P, \ldots, (-1)^{r-1}P)$, then the r-chain $\mathcal{C} \boxtimes \mathcal{D}$ of paths in the graph G_{p+q} is openly separated.

(ii) If r is even, the r-splitting of $\mathcal{C}$ is closely alternating and $\mathcal{D}$ is the closed r-chain

$(P, -P, P, -P, \dots, -P)$, *then the closed r-chain $\mathcal{C} \boxtimes \mathcal{D}$ of paths in the graph G_{p+q} is closely separated.*

Proof. Let $\mathcal{S} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_r)$ be the r-splitting of $\mathcal{C}$ and let

$$\mathcal{A} = \begin{pmatrix} \mathcal{C}_1 \\ -\mathcal{C}_2 \\ \vdots \\ (-1)^{r-1}\mathcal{C}_r \end{pmatrix} = \begin{pmatrix} Q_1^1 & Q_1^2 & \dots & Q_1^k \\ -Q_2^1 & -Q_2^2 & \dots & -Q_2^k \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{r-1}Q_r^1 & (-1)^{r-1}Q_r^2 & \dots & (-1)^{r-1}Q_r^k \end{pmatrix}$$

be the alternate matrix of $\mathcal{S}$. Then we have

$$\mathcal{E} = \mathcal{C} \boxtimes \mathcal{D} = (\mathcal{C}_1 \boxtimes P, \mathcal{C}_2 \boxtimes (-P), \dots, \mathcal{C}_r \boxtimes (-1)^{r-1}P) = (R_1, R_2, \dots, R_r).$$

Let R_i, R_j be distinct paths of $\mathcal{E}$ and let u_1 be a vertex of the path R_i and u_2 be a vertex of R_j. Assume that u_1, u_2 are not apart in G_{p+q}. To complete the proof of (i), we need to show that the paths R_i, R_j are consecutive in $\mathcal{E}$ and that $u_1 = u_2$ is their common vertex.

Assume that $P = (v_\ell)_{\ell=1}^k$. Then $u_1 = w_1 \boxplus v_s$ where w_1 is a vertex of the path Q_i^s and $u_2 = w_2 \boxplus v_t$ where w_2 is a vertex of Q_j^t, for some $s, t \in \{1, 2, \dots, k\}$. Since u_1, u_2 are not apart in G_{p+q}, they agree at each coordinate $1, 2, \dots, p+q$, hence $v_s = v_t$ and $s = t$. Thus the paths Q_i^s and Q_j^t appear in the same column of $\mathcal{A}$. Since u_1, u_2 are not apart in G_{p+q}, it follows that the vertices w_1, w_2 are not apart in G_p and hence the paths Q_i^s and Q_j^t are not apart in G_p. Since $\mathcal{S}$ is openly alternating, Q_i^s and Q_j^t are consecutive in $\mathcal{C}$ and they are almost apart in G_p. It follows that R_i, R_j are consecutive in $\mathcal{E}$ and that $w_1 = w_2$. Hence $u_1 = u_2$ and the proof of (i) is complete.

The proof of (ii) is similar. □

Let $\langle n \rangle = 2\lfloor \frac{n}{2} \rfloor$, *i.e.*let $\langle n \rangle = n$ if n is even and $\langle n \rangle = n - 1$ if n is odd. Let $t \geq 1$ and $\mathcal{C}$ be be an n^t-chain of paths in G_p. We say that $\mathcal{C}$ is *openly well assembled* if either $t = 1$ and $\mathcal{C}$ is an openly separated chain of open snakes, or $t \geq 2$, every chain $\mathcal{C}_i$ in the n-splitting $\mathcal{S} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n)$ of $\mathcal{C}$ is openly well assembled and $\mathcal{S}$ is openly alternating.

Let $\mathcal{D}$ be an $\langle n\rangle n^t$-chain of paths in G_p. We say that $\mathcal{D}$ is *closely well assembled* if every chain $\mathcal{D}_i$ in the $\langle n\rangle$-splitting $\mathcal{S}' = (\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_{\langle n\rangle})$ of $\mathcal{D}$ is openly well assembled and $\mathcal{S}'$ is closely alternating. The following property can be proved by a straightforward induction with respect to t.

Property 2.5. *If $t \geq 1$, $\mathcal{C}$ is an openly well assembled n^t-chain of paths in the graph G_p, then the chain $-\mathcal{C}$ is also openly well assembled.* □

For every $t \geq 1$ we are going now to define the n^t-path π_n^t in K_n^t, and the closed $\langle n\rangle n^{t-1}$-path γ_n^t in K_n^t. These paths will be used in the construction of long snakes. Let π_n^1 be the n-path $(0, 1, \dots, n-1)$ and γ_n^1 be the closed $\langle n\rangle$-path $(0, 1, \dots, \langle n\rangle - 1)$ in K_n^t. Assuming that the path π_n^t in K_n^t is defined, let

$$\pi_n^{t+1} = (\pi_n^t, -\pi_n^t, \pi_n^t, -\pi_n^t, \dots, (-1)^{n-1}\pi_n^t) \otimes \pi_n^1$$

and

$$\gamma_n^{t+1} = (\pi_n^t, -\pi_n^t, \pi_n^t, -\pi_n^t, \dots, -\pi_n^t) \otimes \gamma_n^1.$$

The following lemma holds.

Lemma 2.6. *If $\mathcal{C}$ is an openly well assembled n^q-chain of paths in the graph G_p, then the path $\mathcal{C} \boxtimes \pi_n^q$ is an open snake in the graph G_{p+q}.*

Proof. We are going to use induction with respect to q. For $q = 1$, the lemma is true by Lemma 2.2 (i). Assume that $p+q+1 \leq d$ and $\mathcal{C}$ is an openly assembled n^{q+1}-chain of paths in the graph G_p. We have $\mathcal{C} \boxtimes \pi_n^{d+1} = \mathcal{C} \boxtimes (\mathcal{D} \otimes \pi_n^1)$, where $\mathcal{D} = (\pi_n^q, -\pi_n^q, \dots, (-1)^{n-1}\pi_n^q)$. By Property 2.1, the chain $\mathcal{C} \boxtimes \pi_n^{q+1}$ is equal to $(\mathcal{C} \boxtimes \mathcal{D}) \boxtimes \pi_n^1$. Let $\mathcal{S} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n)$ be the n-splitting of $\mathcal{C}$. Then

$$\mathcal{C} \boxtimes \mathcal{D} = \big(\mathcal{C}_1 \boxtimes \pi_n^q, \mathcal{C}_2 \boxtimes (-\pi_n^q), \dots, \mathcal{C}_n \boxtimes (-1)^{n-1}\pi_n^q\big).$$

By Property 2.3,

$$\mathcal{C} \boxtimes \mathcal{D} = \big(\mathcal{C}_1 \boxtimes \pi_n^q, -(-\mathcal{C}_2 \boxtimes \pi_n^q), \dots, (-1)^{n-1}((-1)^{n-1}\mathcal{C}_n \boxtimes \pi_n^q)\big).$$

Since the chain $\mathcal{C}$ is openly well assembled, the chains $\mathcal{C}_1$, $\mathcal{C}_2$, ..., $\mathcal{C}_n$ are also openly well assembled. By Property 2.5, the chains $\mathcal{C}_1$, $-\mathcal{C}_2$, ..., $(-1)^{n-1}\mathcal{C}_n$ are openly well assembled, so by the inductive hypothesis, the paths $\mathcal{C}_1 \boxtimes \pi_n^q$, $-(-\mathcal{C}_2 \boxtimes \pi_n^q)$, ..., $(-1)^{n-1}((-1)^{n-1}\mathcal{C}_n \boxtimes \pi_n^q)$ are open snakes in G_{p+q}. The splitting $\mathcal{S}$ is openly alternating, so by Lemma 2.4 (i), the chain $\mathcal{C} \boxtimes \mathcal{D}$ is openly separated. Hence by Lemma 2.2 (i), $(\mathcal{C} \boxtimes \mathcal{D}) \boxtimes \pi_n^1 = \mathcal{C} \boxtimes \pi_n^q$ is an open snake in G_{p+q+1}, and the proof is complete. □

The following lemma will be used in the proof of the main result.

Lemma 2.7. *If $q \geq 2$ and $\mathcal{C}$ is a closely well assembled $\langle n \rangle n^{q-1}$-chain of paths in the graph G_p, then the path $\mathcal{C} \boxtimes \gamma_n^q$ is a snake in the graph G_{p+q}.*

Proof. We have $\mathcal{C} \boxtimes \gamma_n^q = \mathcal{C} \boxtimes (\mathcal{D} \otimes \gamma_n^1)$, where $\mathcal{D}$ is the $\langle n \rangle$-chain $(\pi_n^{q-1}, -\pi_n^{q-1}, \dots, -\pi_n^{q-1})$. By Property 2.1, the chain $\mathcal{C} \boxtimes \gamma_n^q$ is equal to $(\mathcal{C} \boxtimes \mathcal{D}) \boxtimes \gamma_n^1$. Let $\mathcal{S} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{\langle n \rangle})$ be the $\langle n \rangle$-splitting of $\mathcal{C}$. Then

$$\mathcal{C} \boxtimes \mathcal{D} = \big(\mathcal{C}_1 \boxtimes \pi_n^{q-1}, \mathcal{C}_2 \boxtimes (-\pi_n^{q-1}), \dots, \mathcal{C}_{\langle n \rangle} \boxtimes (-\pi_n^{q-1})\big).$$

By Property 2.3,

$$\mathcal{C} \boxtimes \mathcal{D} = \big(\mathcal{C}_1 \boxtimes \pi_n^{q-1}, -(-\mathcal{C}_2 \boxtimes \pi_n^{q-1}), \dots, -(-\mathcal{C}_{\langle n \rangle} \boxtimes \pi_n^{q-1})\big).$$

By Property 2.5 and Lemma 2.6, arguing as in the proof of Lemma 2.6, we conclude that $\mathcal{C} \boxtimes \mathcal{D}$ is a chain of open snakes in G_{p+q-1}. The splitting $\mathcal{S}$ is closely alternating so by Lemma 2.4 (ii), the chain $\mathcal{C} \boxtimes \mathcal{D}$ is closely separated. Hence by Lemma 2.2 (ii), $(\mathcal{C} \boxtimes \mathcal{D}) \boxtimes \gamma_n = \mathcal{C} \boxtimes \gamma_n^q$ is a snake in G_{p+q}, and the proof is complete. □

3. Construction of long snakes

Assume that $d \geq 4$. Let C be a snake of length $S(K_m^{d-1})$ in K_m^{d-1} and let C' be the open snake obtained from C by deleting the last vertex. Given any pair u_1, u_2 of vertices of K_m^{d-1} that differ at exactly two coordinates, we can get an open snake in K_m^{d-1} with endpoints u_1, u_2 by permuting the coordinates and permuting the entries at some coordinates of the open snake C'. Let a_1, a_2, a_3 and a_4 be four vertices in K_m^{d-1} such that any two of them differ at exactly two coordinates. For example, let $a_1 = (10000\ldots,0)$, $a_2 = (01000\ldots0)$, $a_3 = (00100\ldots0)$ and $a_4 = (11100\ldots0)$. Let C_{ij} be an open snake in K_m^{d-1} with $S(K_m^{d-1}) - 1$ vertices such that a_i is the first and a_j is the last vertex of C_{ij}, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.

For each $i \in \{1, 2, 3, 4\}$ and $k \in \{0, 1, \ldots, n-1\}$, let a_i^k be the vertex of the graph $G_1 = K_{mn} \times K_m^{d-1}$ obtained from a_i by adjoining the digit k as the first coordinate, *i.e.*let

$$a_1^k = (k\,1\,0\,0\,0\,0\ldots0),$$

$$a_2^k = (k\,0\,1\,0\,0\,0\ldots0),$$

$$a_3^k = (k\,0\,0\,1\,0\,0\ldots0),$$

$$a_4^k = (k\,1\,1\,1\,0\,0\ldots0),$$

and let $A = \{a_i^k : i \in \{1, 2, 3, 4\}, k \in \{0, 1, \ldots, n-1\}\}$.

For each $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$ and each $r \in \{0, 1, \ldots, n-1\}$, let C_{ij}^r be the open snake in G_1 obtained from the open snake C_{ij} in K_m^{d-1} by adjoining the digit $r + n$ to every vertex of C_{ij} as the first coordinate. For example $C_{1\,2}^0 = ((n\,1\,0\,0\,0\,0\ldots0), \ldots, (n\,0\,1\,0\,0\,0\ldots0))$.

For each $a_i^k, a_j^\ell \in A$ with $i \neq j$ and each $r \in \{0, 1, \ldots, n-1\}$, let $C(a_i^k, r, a_j^\ell)$ be the open snake in G_1 with $S(K_m^{d-1}) + 1$ vertices obtained from C_{ij}^r by adjoining the vertex a_i^k in front and the vertex a_j^ℓ at the end. For example, if $n \geq 5$, then

$$C(a_1^3, 0, a_2^4) = ((3\,1\,0\,0\,0\,0\ldots0), (n\,1\,0\,0\,0\,0\ldots0), \ldots, (n\,0\,1\,0\,0\,0\ldots0), (4\,0\,1\,0\,0\,0\ldots0)).$$

Let $\mathcal{M} = \{C(a_i^k, r, a_j^\ell) : a_i^k, a_j^\ell \in A, i \neq j, r \in \{0, 1, \ldots, n-1\}\}$ and let $\mathcal{M}_t = \{C(a_i^k, r, a_j^\ell) \in \mathcal{M} : t \in \{i, j\}\}$, for any $t \in \{1, 2, 3, 4\}$.

If $\mathcal{C}$ is a chain of paths in a graph H and u_1, u_2 are vertices of H, then we say that $\mathcal{C}$ *joins* u_1 *to* u_2 if u_1 is the first vertex of the first path of $\mathcal{C}$ and u_2 is the last vertex of the last path of $\mathcal{C}$. Given $\mathcal{M}' \subseteq \mathcal{M}$, we say that a chain $\mathcal{C}$ of paths in G_1 is $\mathcal{M}'$-*built* if every path of $\mathcal{C}$ belongs to $\mathcal{M}'$.

Let $\mathcal{C}$ and $\mathcal{C}'$ be $\mathcal{M}$-built n^q-chains of paths with

$$\mathcal{C} = (C(u_0, r_0, u_1), C(u_1, r_1, u_2), \ldots, C(u_{n^q-1}, r_{n^q-1}, u_{n^q})),$$

and

$$\mathcal{C}' = (C(u'_0, r'_0, u'_1), C(u'_1, r'_1, u'_2), \ldots, C(u'_{n^q-1}, r'_{n^q-1}, u'_{n^q})),$$

where $u_i, u'_i \in A$ and $r_j, r'_j \in \{0, 1, \ldots, n-1\}$, $i = 0, 1, \ldots n^q$, $j = 0, 1, \ldots, n^q - 1$. Then we say that $\mathcal{C}$, $\mathcal{C}'$ are *internally compatible* if $r_i = r'_i$ for every $i = 0, 1, \ldots, n^q - 1$ and $u_i = u'_i$ for every $i = 1, 2, \ldots, n^q - 1$.

For any $t \in \{0, 1, \ldots, n-1\}$ and for any permutation $\tau \in S_4$, let $\sigma^t_\tau : \mathcal{M} \to \mathcal{M}$ be defined by

$$\sigma^t_\tau(C(a^k_i, r, a^\ell_j)) = C(a^{k\oplus t}_{\tau(i)}, r \oplus t, a^{\ell\oplus t}_{\tau(j)}),$$

where $\oplus$ denotes addition mod n. If $\mathcal{C}$ is an $\mathcal{M}$-built chain, then let $\sigma^t_\tau(\mathcal{C})$ be obtained by applying σ^t_τ to each path of $\mathcal{C}$. The following property can be proved by a straightforward induction on s.

Property 3.1. *If $\mathcal{C}$ is an $\mathcal{M}$-built openly well assembled n^s-chain, $\tau \in S_4$ and $t \in \{0, 1, \ldots$ $\ldots, n-1\}$, then the chains $\pm\sigma^t_\tau(\mathcal{C})$ are also openly well assembled.*

Let $\mathcal{M}' = \mathcal{M}_1$ if n is is odd and $\mathcal{M}' = \mathcal{M}_3$ if n is even. If $1 \le q \le d-2$, then a *q-network in* G_1 is a family $\mathcal{N}_q$ of $\mathcal{M}'$-built openly well assembled n^q-chains $\mathcal{C}^{k\ell}_q$ such that $\mathcal{C}^{k\ell}_q$ joins a^k_1 to a^ℓ_2, $k \in \{0, 1\}$, $\ell \in \{0, 1, \ldots, n-1\}$, and any two chains in $\mathcal{N}_q$ are internally compatible.

For each q, $1 \le q \le d-2$, we shall construct now a q-network $\mathcal{N}_q$ in G_1. Let $\mathcal{N}_1 = \{\mathcal{C}_1^{k\ell} : k \in \{0,1\}, \ell \in \{0,1,\dots,n-1\}\}$ with

$$\begin{aligned}\mathcal{C}_1^{k\ell} =&(C(a_1^k, 0, a_{i_1}^1), C(a_{i_1}^1, 1, a_{i_2}^2), C(a_{i_2}^2, 2, a_{i_3}^3), \dots \\ &\dots, C(a_{i_{n-2}}^{n-2}, n-2, a_{i_{n-1}}^{n-1}), C(a_{i_{n-1}}^{n-1}, n-1, a_2^\ell)),\end{aligned}$$

where $i_s = 1$ for s even and $i_s = 3$ for s odd, $s = 1, 2, \dots, n-1$.

Lemma 3.2. *The set $\mathcal{N}_1$ is a 1-network in G_1.*

Proof. It is clear that $\mathcal{N}_1$ is a family of $\mathcal{M}'$-built n-chains such that $\mathcal{C}_1^{k\ell}$ joins a_1^k to a_2^ℓ, $k \in \{0,1\}$, $\ell \in \{0,1,\dots,n-1\}$, and any two chains in $\mathcal{N}_1$ are internally compatible. It remains to show that the chains in $\mathcal{N}_1$ are openly well assembled, and since the paths in $\mathcal{M}$ are open snakes, it suffices to show that every chain in $\mathcal{N}_1$ is openly separated.

Let $k \in \{0,1\}$, $\ell \in \{0,1,\dots,n-1\}$, let P, P' be distinct paths of the chain $\mathcal{C}_1^{k\ell}$ and let u, u' be vertices of P, P' respectively. Assume that u, u' are not apart. To complete the proof we need to show that P, P' are consecutive in $\mathcal{C}_1^{k\ell}$ and that $u = u'$ is their common vertex.

Since u, u' are not apart in G_1 the first coordinates of u and u' are the same. Since $P \neq P'$, it follows immediately from the definition of $\mathcal{N}_1$ that

$$u, u' \in \{a_1^k, a_{i_1}^1, a_{i_2}^2, \dots, a_{i_{n-1}}^{n-1}, a_2^\ell\}.$$

Since $i_1, i_2, \dots, i_{n-1} \in \{1,3\}$, $k \in \{0,1\}$ and $i_1 = 3$, it follows that all the vertices in the sequence $(a_1^k, a_{i_1}^1, a_{i_2}^2, \dots, a_{i_{n-1}}^{n-1}, a_2^\ell)$ are distinct. Since, clearly, any two distinct vertices of A are apart in G_1, it follows that $u = u'$ and that the paths P, P' are consecutive in $\mathcal{C}_1^{k\ell}$ completing the proof. □

Assume now that $q > 1$ and that $\mathcal{N}_{q-1}$ is a $(q-1)$-network in G_1. Given $k \in \{0,1\}$ and $\ell \in \{0,1,\dots,n-1\}$, let $\mathcal{C}_q^{k\ell}$ be the n^q-chain with the n-splitting $\mathcal{S}$ defined as follows.

If n is odd, then let

$$\mathcal{S} = (\sigma^0_{\tau_0}(\mathcal{C}^{k\,1}_{q-1}), -\sigma^1_{\tau_1}(\mathcal{C}^{1\,0}_{q-1}), \sigma^2_{\tau_2}(\mathcal{C}^{0\,1}_{q-1}), -\sigma^3_{\tau_3}(\mathcal{C}^{1\,0}_{q-1}), \dots, -\sigma^{n-2}_{\tau_{n-2}}(\mathcal{C}^{1\,0}_{q-1}), \sigma^{n-1}_{\tau_{n-1}}(\mathcal{C}^{0\,\ell\oplus 1}_{q-1})),$$

where τ_i is the transposition $(2\ \ 3)$ for $i = 0, 1, \dots, n-2$ and τ_{n-1} is the identity permutation. If n is even, then let

$$\mathcal{S} = (-\sigma^0_{\tau_0}(\mathcal{C}^{1\,k}_{q-1}), \sigma^1_{\tau_1}(\mathcal{C}^{0\,1}_{q-1}), -\sigma^2_{\tau_2}(\mathcal{C}^{1\,0}_{q-1}), \sigma^3_{\tau_3}(\mathcal{C}^{0\,1}_{q-1}), \dots, -\sigma^{n-2}_{\tau_{n-2}}(\mathcal{C}^{1\,0}_{q-1}), \sigma^{n-1}_{\tau_{n-1}}(\mathcal{C}^{0\,\ell\oplus 1}_{q-1})),$$

where τ_i is the 3-cycle $(2\ \ 1\ \ 4)$ for $i = 0, 1, \dots, n-2$, and τ_{n-1} is the transposition $(1\ \ 4)$. Let $\mathcal{N}_q = \{\mathcal{C}^{k\ell}_q : k \in \{0,1\}, \ell \in \{0, 1, \dots, n-1\}\}$.

The following lemma holds.

Lemma 3.3. *For every $q \in \{1, 2, \dots, d-2\}$ the set $\mathcal{N}_q$ is a q-network in G_1.*

Proof. If $q = 1$, then $\mathcal{N}_1$ is a 1-network in G_1 by Lemma 3.2. Assume now that $q > 1$ and that $\mathcal{N}_{q-1}$ is an $(q-1)$-network in G_1. It is clear that $\mathcal{N}_q$ is a family of $\mathcal{M}$-built n^q-chains such that $\mathcal{C}^{k\ell}_q$ joins a^k_1 to a^ℓ_2, $k \in \{0,1\}$, $\ell \in \{0, 1, \dots, n-1\}$. Since the chains in $\mathcal{N}_{q-1}$ are internally compatible, it immediately follows from the definition of $\mathcal{N}_q$ that any two chains in $\mathcal{N}_q$ are internally compatible. Since, in the case of n being odd, the chains in $\mathcal{N}_{q-1}$ are $\mathcal{M}_1$-built and since 1 is a fixed point of the permutation τ_i for each $i = 0, 1, \dots, n-1$, it follows that the chains in $\mathcal{N}_q$ are $\mathcal{M}_1$-built. Similarly, in the case of n being even, the chains in $\mathcal{N}_{q-1}$ are $\mathcal{M}_3$-built and 3 is a fixed point of τ_i, $i = 0, 1, \dots, n-1$, implying that the chains in $\mathcal{N}_q$ are $\mathcal{M}_3$-built. Thus, in general, the chains in $\mathcal{N}_q$ are $\mathcal{M}'$-built. It remains to show that the chains in $\mathcal{N}_q$ are openly well assembled.

Let $\mathcal{C}^{k\ell}_q$ be a chain in $\mathcal{N}_q$. Since each chain in $\mathcal{N}_{q-1}$ is openly well assembled and since Property 3.1 holds, it suffices to show that the n-splitting of $\mathcal{C}^{k\ell}_q$ is openly alternating.

Assume that n is odd. Let

$$\mathcal{A} = \begin{pmatrix} \sigma^0_{\tau_0}(\mathcal{C}^{k\,1}_{q-1}) \\ \sigma^1_{\tau_1}(\mathcal{C}^{1\,0}_{q-1}) \\ \sigma^2_{\tau_2}(\mathcal{C}^{0\,1}_{q-1}) \\ \vdots \\ \sigma^{n-2}_{\tau_{n-2}}(\mathcal{C}^{1\,0}_{q-1}) \\ \sigma^{n-1}_{\tau_{n-1}}(\mathcal{C}^{0\,\ell\oplus 1}_{q-1}) \end{pmatrix} = \begin{pmatrix} \sigma^0_{\tau_0}(C(a^k_1,\ldots)) & \ldots & \sigma^0_{\tau_0}(C(\ldots,a^1_2)) \\ \sigma^1_{\tau_1}(C(a^1_1,\ldots)) & \ldots & \sigma^1_{\tau_1}(C(\ldots,a^0_2)) \\ \sigma^2_{\tau_2}(C(a^0_1,\ldots)) & \ldots & \sigma^2_{\tau_2}(C(\ldots,a^1_2)) \\ \vdots & \ddots & \vdots \\ \sigma^{n-2}_{\tau_{n-2}}(C(a^1_1,\ldots)) & \ldots & \sigma^{n-2}_{\tau_{n-2}}(C(\ldots,a^0_2)) \\ \sigma^{n-1}_{\tau_{n-1}}(C(a^0_1,\ldots)) & \ldots & \sigma^{n-1}_{\tau_{n-1}}(C(\ldots,a^{\ell\oplus 1}_2)) \end{pmatrix}$$

be the alternate matrix of the n-splitting of $\mathcal{C}^{k\ell}_q$. Then

$$\mathcal{A} = \begin{pmatrix} C(a^k_1,\ldots) & \ldots & C(\ldots,a^1_3) \\ C(a^2_1,\ldots) & \ldots & C(\ldots,a^1_3) \\ C(a^2_1,\ldots) & \ldots & C(\ldots,a^3_3) \\ \vdots & \ddots & \vdots \\ C(a^{n-1}_1,\ldots) & \ldots & C(\ldots,a^{n-2}_3) \\ C(a^{n-1}_1,\ldots) & \ldots & C(\ldots,a^{\ell}_2) \end{pmatrix}.$$

Let $j \in \{1,2,\ldots,n^{q-1}\}$ and let P, P' be distinct paths in the j-th column of the matrix $\mathcal{A}$. We need to show that the paths P, P' are almost apart in G_1 if they are consecutive in $\mathcal{C}^{k\ell}_q$ and that they are apart otherwise. Assume first that $2 \le j \le n^{q-1}-1$. Since the chains in $\mathcal{N}_{q-1}$ are internally compatible, there is $Q = C(a^u_i, r, a^{u'}_{i'}) \in \mathcal{M}$ such that

$$P = \sigma^s_{\tau_s}(Q) = C(a^{u\oplus s}_{\tau_s(i)}, r\oplus s, a^{u'\oplus s}_{\tau_s(i')}),$$
$$P' = \sigma^t_{\tau_t}(Q) = C(a^{u\oplus t}_{\tau_t(i)}, r\oplus t, a^{u'\oplus t}_{\tau_t(i')}),$$

for some $s,t \in \{0,1,\ldots,n-1\}$, $s \neq t$. Let w, w' be vertices of P, P' respectively. We will show that w and w' are apart. Consider the following three cases:

(i) $w = a^{u\oplus s}_{\tau_s(i)}$ and $w' = a^{u'\oplus t}_{\tau_t(i')}$,

(ii) $w = a^{u'\oplus s}_{\tau_s(i')}$ and $w' = a^{u\oplus t}_{\tau_t(i)}$,

(iii) neither (i) nor (ii) holds.

If (iii) holds, then the first coordinates of w and w' are different, hence w, w' are apart in G_1. If (i) holds, then since the chains in $\mathcal{N}_{q-1}$ are $\mathcal{M}_1$-built, it follows that exactly one of i, i' is equal to 1. Since 1 is a fixed point of both τ_s and τ_t, it follows that exactly one of $\tau_s(i)$, $\tau_t(i')$ is equal to 1. Hence w, w' differ at least at two coordinates, and so they are apart. Similarly, w, w' are apart if (ii) holds.

Assume now that $j = 1$. Since the chains in $\mathcal{N}_{q-1}$ are internally compatible, we have

$$P = \sigma^s_{\tau_s}(C(a_1^u, r, a_i^v)) = C(a_1^{u\oplus s}, r \oplus s, a_{\tau_s(i)}^{v\oplus s}),$$
$$P' = \sigma^t_{\tau_t}(C(a_1^{u'}, r, a_i^v)) = C(a_1^{u'\oplus t}, r \oplus t, a_{\tau_t(i)}^{v\oplus t}),$$

for some $i \in \{2,3,4\}$, $u, u' \in \{0,1\}$ and $r, s, t, v \in \{0, 1, \dots, n-1\}$ with $s \neq t$. We can assume that $s < t$. Let w, w' be vertices of P, P' respectively and assume that w, w' are not apart in G_1. We will show that P, P' are consecutive in $\mathcal{C}_q^{k\ell}$ and that $w = w'$ is their common vertex. Since the vertices w, w' are not apart in G_1, the first coordinates of w and w' must be equal and hence one of the following cases holds:

(i) $w = a_1^{u\oplus s}$ and $w' = a_{\tau_t(i)}^{v\oplus t}$,

(ii) $w = a_{\tau_s(i)}^{v\oplus s}$ and $w' = a_1^{u'\oplus t}$,

(iii) $w = a_1^{u\oplus s}$ and $w' = a_1^{u'\oplus t}$.

Since $\tau_s(i) \neq 1$ and $\tau_t(i) \neq 1$, we conclude that if (i) or (ii) holds, then w, w' differ at least at two coordinates so they are apart. Thus (iii) holds. Since w, w' are not apart, we have $u \oplus s = u' \oplus t$, and so $w = w'$. Since $u, u' \in \{0,1\}$, it follows that $t = s+1$ implying that $u = 1$ and $u' = 0$. Therefore s is odd, and so P, P' are consecutive in $\mathcal{C}_q^{k\ell}$.

To complete the proof in the case of n being odd, it remains to consider the case when $j = n^q$. Then

$$P = \sigma^s_{\tau_s}(C(a_1^u, r, a_2^v)) = C(a_1^{u\oplus s}, r \oplus s, a_{\tau_s(2)}^{v\oplus s}),$$
$$P' = \sigma^t_{\tau_t}(C(a_1^u, r, a_2^{v'})) = C(a_1^{u\oplus t}, r \oplus t, a_{\tau_t(2)}^{v'\oplus t}),$$

for some $r, s, t, u, v, v' \in \{0, 1, \dots, n-1\}$, $s \neq t$. We can assume that $s < t$. Let w, w' be vertices of P, P' respectively and assume that w, w' are not apart. Arguing as in the case when $j = 1$, we conclude that $w = a_{\tau_s(2)}^{v\oplus s}$ and $w' = a_{\tau_t(2)}^{v'\oplus t}$. Since w, w' are not apart and since any two distinct elements of A are apart, it follows that $w = w'$. Hence $v \oplus s = v' \oplus t$ and $\tau_s(2) = \tau_t(2)$, implying that $t \neq n-1$. Therefore $v, v' \in \{0,1\}$ and so $t = s+1$. Thus

$v = 1$, $v' = 0$ implying that s is even. Hence P, P' are consecutive in $\mathcal{C}_q^{k\ell}$ completing the proof in the case of n being odd.

If n is even and $\mathcal{A}$ is the alternate matrix of the n-splitting of $\mathcal{C}_q^{k\ell}$, then

$$\mathcal{A} = \begin{pmatrix} -\sigma_{\tau_0}^0(\mathcal{C}_{q-1}^{1\,k}) \\ -\sigma_{\tau_1}^1(\mathcal{C}_{q-1}^{0\,1}) \\ -\sigma_{\tau_2}^2(\mathcal{C}_{q-1}^{1\,0}) \\ \vdots \\ -\sigma_{\tau_{n-2}}^{n-2}(\mathcal{C}_{q-1}^{1\,0}) \\ -\sigma_{\tau_{n-1}}^{n-1}(\mathcal{C}_{q-1}^{0\,\ell\oplus 1}) \end{pmatrix} = \begin{pmatrix} \sigma_{\tau_0}^0(C(a_2^k,\ldots)) & \ldots & \sigma_{\tau_0}^0(C(\ldots,a_1^1)) \\ \sigma_{\tau_1}^1(C(a_2^1,\ldots)) & \ldots & \sigma_{\tau_1}^1(C(\ldots,a_1^0)) \\ \sigma_{\tau_2}^2(C(a_2^0,\ldots)) & \ldots & \sigma_{\tau_2}^2(C(\ldots,a_1^1)) \\ \vdots & \ddots & \vdots \\ \sigma_{\tau_{n-2}}^{n-2}(C(a_2^0,\ldots)) & \ldots & \sigma_{\tau_{n-2}}^{n-2}(C(\ldots,a_1^1)) \\ \sigma_{\tau_{n-1}}^{n-1}(C(a_2^{\ell\oplus 1},\ldots)) & \ldots & \sigma_{\tau_{n-1}}^{n-1}(C(\ldots,a_1^0)) \end{pmatrix}.$$

Hence

$$\mathcal{A} = \begin{pmatrix} C(a_1^k,\ldots) & \ldots & C(\ldots,a_4^1) \\ C(a_1^2,\ldots) & \ldots & C(\ldots,a_4^1) \\ C(a_1^2,\ldots) & \ldots & C(\ldots,a_4^3) \\ \vdots & \ddots & \vdots \\ C(a_1^{n-2},\ldots) & \ldots & C(\ldots,a_4^{n-1}) \\ C(a_2^\ell,\ldots) & \ldots & C(\ldots,a_4^{n-1}) \end{pmatrix}.$$

Similarly as in the proof in the case of n being odd, we show that if $j \in \{1, 2, \ldots, n^{q-1}\}$ and P, P' are distinct paths in the j-th column of the matrix $\mathcal{A}$, then P, P' are almost apart in G_1 if they are consecutive in $\mathcal{C}_q^{k\ell}$ and they are apart otherwise. Therefore the n-splitting of $\mathcal{C}_q^{k\ell}$ is openly alternating and the proof is complete. $\square$

Assume that $q = d - 2$. By Lemma 3.3, the set $\mathcal{N}_q$ is a q-network in G_1. Define $\mathcal{D}$ to be the $\mathcal{M}$-built $\langle n\rangle n^q$-chain with the $\langle n\rangle$-splitting

$$(-\sigma^0(\mathcal{C}_q^{1\,0}), \sigma^1(\mathcal{C}_q^{0\,1}), -\sigma^2(\mathcal{C}_q^{1\,0}), \ldots, -\sigma^{\langle n\rangle-2}(\mathcal{C}_q^{1\,0}), \sigma^{\langle n\rangle-1}(\mathcal{C}_q^{0\,n-\langle n\rangle+1})),$$

where $\sigma^i = \sigma_\tau^i$ with τ being the identity permutation, $i = 0, 1, \ldots, \langle n\rangle - 1$. The following lemma together with Lemma 2.7 will allow us to construct long snakes in K_{mn}^d.

Lemma 3.4. *$\mathcal{D}$ is a closely well assembled $\langle n\rangle n^{d-2}$-chain of paths in G_1.*

Proof. Since each chain in $\mathcal{N}_q$ is openly well assembled and since Property 3.1 holds, it suffices to show that the $\langle n\rangle$-splitting of $\mathcal{D}$ is closely alternating. Let

$$\mathcal{A} = \begin{pmatrix} -\sigma^0(\mathcal{C}_q^{1\,0}) \\ -\sigma^1(\mathcal{C}_q^{0\,1}) \\ -\sigma^2(\mathcal{C}_q^{1\,0}) \\ \vdots \\ -\sigma^{\langle n\rangle-2}(\mathcal{C}_q^{1\,0}) \\ -\sigma^{\langle n\rangle-1}(\mathcal{C}_q^{0\,n-\langle n\rangle+1}) \end{pmatrix} = \begin{pmatrix} \sigma^0(C(a_2^0,\ldots)) & \ldots & \sigma^0(C(\ldots,a_1^1)) \\ \sigma^1(C(a_2^1,\ldots)) & \ldots & \sigma^1(C(\ldots,a_1^0)) \\ \sigma^2(C(a_2^0,\ldots)) & \ldots & \sigma^2(C(\ldots,a_1^1)) \\ \vdots & \ddots & \vdots \\ \sigma^{\langle n\rangle-2}(C(a_2^0,\ldots)) & \ldots & \sigma^{\langle n\rangle-2}(C(\ldots,a_1^1)) \\ \sigma^{\langle n\rangle-1}(C(a_2^{n-\langle n\rangle+1},\ldots)) & \ldots & \sigma^{\langle n\rangle-1}(C(\ldots,a_1^0)) \end{pmatrix}$$

be the alternate matrix of the $\langle n\rangle$-splitting of $\mathcal{D}$. Then

$$\mathcal{A} = \begin{pmatrix} C(a_2^0,\ldots) & \ldots & C(\ldots,a_1^1) \\ C(a_2^2,\ldots) & \ldots & C(\ldots,a_1^1) \\ C(a_2^2,\ldots) & \ldots & C(\ldots,a_1^3) \\ \vdots & \ddots & \vdots \\ C(a_2^{\langle n\rangle-2},\ldots) & \ldots & C(\ldots,a_1^{\langle n\rangle-1}) \\ C(a_2^0,\ldots) & \ldots & C(\ldots,a_1^{\langle n\rangle-1}) \end{pmatrix}.$$

Similarly as in the proof of Lemma 3.3, we show that if $j \in \{1,2,\ldots,n^q\}$ and P, P' are distinct paths in the j-th column of the matrix $\mathcal{A}$, then P, P' are almost apart in G_1 if they are cyclically consecutive in $\mathcal{D}$ and they are apart otherwise. Therefore the $\langle n\rangle$-splitting of $\mathcal{D}$ is closely alternating and the proof is complete. $\square$

Now we are ready to prove Theorem 1.9.

Proof of **Theorem 1.9.** It follows from Lemma 2.7 and Lemma 3.4 that $C = \mathcal{D} \boxtimes \gamma_n^{d-1}$ is a snake in $G_d = K_{mn}^d$. It is clear that the length of C is equal to $\langle n\rangle n^{d-2}(S(K_m^{d-1})+1)$ so the proof is complete. $\square$

4. Concluding remarks

The assumption that $d \geq 4$ in Theorem 1.9 can be slighty relaxed. If we assume either that $m \geq 3$ and n is odd, or that $m \geq 4$, then it suffices to require that $d \geq 3$. Indeed, if $m \geq 4$, then we can use the vertices $a_1 = (0000 \dots 0)$, $a_2 = (1100 \dots 0)$, $a_3 = (2200 \dots 0)$, $a_4 = (3300 \dots 0)$ in our construction. If n is odd, then the vertex a_4 is not needed so the construction works for $m \geq 3$.

Although Theorem 1.8 is a significant strengthening of Theorem 1.6, Conjecture 1.5 remains still open. We would like to formulate some more conjectures that are generalizations of the result of Wojciechowski [14] saying that for any $d \geq 2$, the hypercube K_2^d can be vertex-covered by at most 16 vertex-disjoint snakes.

Conjecture 4.1. *For any integer $n \geq 2$ there is an integer r_n such that the graph K_n^d can be vertex-covered by at most r_n vertex-disjoint snakes for any $d \geq 2$.*

In the case of n being odd, a weaker version of Conjecture 4.1 (without requiring that the snakes are vertex-disjoint) has been recently proved by Alsardary [7]. The following conjecture implies both Conjecture 4.1 and Conjecture 1.5.

Conjecture 4.2. *There is a constant c such that for any $n \geq 2$ and any $d \geq 1$, the graph K_n^d can be vertex-covered by at most cn vertex-disjoint snakes.*

The best upper bound on $S(K_2^d)$ has been given by Snevily [12].

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