A CRITERION FOR THE EXISTENCE OF TRANSVERSALS OF SET SYSTEMS

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Abstract. Nash-Williams [6] formulated a condition that is necessary and sufficient for a countable family $\mathcal{A} = (A_i)_{i\in I}$ of sets to have a transversal. In [7] he proved that his criterion applies also when we allow the set I to be arbitrary and require only that $\bigcap_{i\in J} A_i = \emptyset$ for any uncountable $J \subseteq I$. In this paper, we formulate another criterion of a similar nature, and prove that it is equivalent to the criterion of Nash-Williams for any family $\mathfrak{A}$. We also present a self-contained proof that if $\bigcap_{i\in J} A_i = \emptyset$ for any uncountable $J \subseteq I$, then our condition is necessary and sufficient for the family $\mathfrak{A}$ to have a transversal.

1. Introduction

Let $\mathcal{A} = (A_i)_{i\in I}$ be a family of sets. A subset $T \subseteq \bigcup_{i\in I} A_i$ is a *transversal* of $\mathcal{A}$ if there is a bijection $f : I \to T$ with $f(i) \in A_i$, $i \in I$. P. Hall [3] proved that if the set I is finite, then $\mathcal{A}$ has a transversal if and only if

$$\left|\bigcup_{i\in J} A_i\right| \geq |J|$$

for any $J \subseteq I$. M. Hall [4] showed that in the case when I is arbitrary but A_i is finite for every $i \in I$, essentially the same criterion applies, namely that $\mathcal{A}$ has a transversal if and only if the above inequality holds for any finite $J \subseteq I$. The well-known "playboy" example (where $I = \{0, 1, 2, \dots\}$, with $A_0 = \{1, 2, \dots\}$ and $A_i = \{i\}$ for $i = 1, 2, \dots$) shows that this criterion fails if we allow the family to be infinite and to contain at least one infinite member.

In the case of I being countable, three necessary and sufficient criterions have been given for $\mathfrak{A}$ to possess a transversal. One, conjectured by Nash-Williams, was proved by Damerell and Milner [2]. Later, a simpler criterion (*q-admissibility*) of a similar type was proved by Nash-Williams [6]. The third criterion (*c-admissibility*), of a different nature, was given by Podewski and Steffens [8]. Subsequently, Nash-Williams [7] generalized his theorem and proved that his criterion applies also in the case when I is arbitrary but $\bigcap_{i\in J} A_i = \emptyset$ for any uncountable $J \subseteq I$ (call such a family *countably repetitive*). Aharoni [1] proved later that the criteria of Nash-Williams and of Podewski

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and Steffens are equivalent, that is, for every family $\mathcal{A}$ of sets, the family $\mathcal{A}$ is q-admissible if and only if it is c-admissible.

The criterion of Podewski and Steffens is simple. The family $\mathcal{A}$ is *c-admissible* if there do not exist a subset $J \subseteq I$ and $j \in I \setminus J$ with $A_j \subseteq \bigcup_{i \in J} A_i$ such that the subfamily $\mathcal{B} = (A_i)_{i \in J}$ of $\mathcal{A}$ has exactly one transversal $T = \bigcup_{i \in J} A_i$. However, this criterion has the philosophical defect of being circular in the sense that in the criterion itself we require that a certain subfamily of $\mathcal{A}$ has a transversal (nevertheless, this defect does not prevent the criterion to be very useful). The criterion of Nash-Williams (see Section 6 for the definition) does not have this defect of being circular, however, it is more complicated than the criterion of Podewski and Steffens.

In this paper, we formulate another criterion (we call it *μ-admissibility*) for a family of sets to have a transversal. Our criterion is of similar nature as the criterion of Nash-Williams (in particular it is not circular). We prove that for any family $\mathcal{A}$, it is μ-admissible if and only if it is q-admissible (hence also c-admissible). Therefore, in particular, the criterion provides a necessary and sufficient condition for $\mathcal{A}$ to have a transversal in the case when $\mathcal{A}$ is countably repetitive. However, we will present a direct self-contained proof of this result. That will provide an alternative proof of the result of Nash-Williams [7] (see Section 7).

The paper is structured in the following way: In Section 2, we formulate our criterion using the terminology of espousals in societies and state our first main result of this paper (Theorem 2.1). In Section 3, we prove the preliminary lemmas required for the proof of Theorem 2.1 given in Section 4. In Sections 5 and 6, we prove the second main result of this paper (Theorem 6.2).

2. μ-admissible societies

We are going now to reformulate the concept of transversals for families of sets using the language of *espousals in societies* appearing in [1], [6] and [7].

We shall use the following set-theoretic conventions. A *relation* is a set of ordered pairs. Given a relation R, a set A, and an element a, we have $R\langle a\rangle = \{y : (a, y) \in R\}$; if $|R\langle a\rangle| = 1$, then $R(a)$ is the single element of $R\langle a\rangle$; the set $R[A]$ is equal to $\bigcup_{a \in A} R\langle a\rangle$; and $R^{-1} = \{(y, x) : (x, y) \in R\}$. The *domain* of the relation R is the set $\operatorname{dom} R = \{x : R\langle x\rangle \neq \emptyset\}$ and the *range* of R is the set $\operatorname{rge} R = \left\{y : R^{-1}\langle y\rangle \neq \emptyset\right\}$. A *function* is a relation f such that $|f\langle x\rangle| = 1$ for every $x \in \operatorname{dom} f$. An *ordinal number* α is equal to the set of all ordinal numbers smaller than α and a *cardinal number* is an ordinal number α such that any ordinal number smaller than α has a smaller cardinality. We denote by ω the first infinite ordinal number and we call a set X *countable* if $|X| \leq \omega$.

A *society* Γ is an ordered triple (M, W, K), where M, W are disjoint sets and $K \subseteq M \times W$. The elements of M are called *men*, the elements of W are called *women* and if $(a, x) \in K$, then we say that *a knows x*. An *espousal* of Γ is an injective function $E : M \to W$ with $E \subseteq K$. The society Γ

is *espousable* if there is an espousal of Γ. Throughout this paper we assume that we are discussing a fixed society $\Gamma = (M, W, K)$ and the symbols Γ, M, W, K should be interpreted accordingly.

Assume that $A \subseteq M \cup W$, $A_M = A \cap M$ and $A_W = A \cap W$. Let $\Gamma[A]$ be the society $(A_M, A_W, K \cap (A_M \times A_W))$ and let $\Gamma - A = \Gamma[(M \cup W) \setminus A]$. A *saturated subsociety* of Γ is a society of the form $\Gamma[A]$ for some $A \subseteq M \cup W$ with $K[A_M] \subseteq A_W$.

A *string* is an injective function with its domain being an ordinal. In particular, the empty set $\emptyset$ is a string with domain $0 = \emptyset$. A *string* in S is a string f with $\operatorname{rge} f \subseteq S$ and an α*-string* is a string g with $\operatorname{dom} g = \alpha$. A string in the society Γ is a string in $M \cup W$.

Let f be a string and β, γ be ordinals with $\beta \leq \gamma \leq \operatorname{dom} f$. The $[\beta, \gamma)$*-segment* $f_{[\beta,\gamma)}$ of f is the string defined by

$$f_{[\beta,\gamma)}(\theta) = f(\beta + \theta),$$

for all θ with $\beta + \theta < \gamma$, that is, $f_{[\beta,\gamma)}$ is obtained from f by restricting it to $[\beta, \gamma)$ and shifting the domain to start at 0. For $\alpha \leq \operatorname{dom} f$, let $f_\alpha = f_{[0,\alpha)}$. If f and g are strings in Γ with domains α and β respectively, then the *concatenation* $f * g$ of f and g is defined to be the $(\alpha + \beta)$-string h such that $h_\alpha = f$ and $h_{[\alpha,\alpha+\beta)} = g$. For $u \in M \cup W$, let $[u]$ be the 1-string f with $f(0) = u$. A *saturated string* in Γ is a string in Γ such that $\Gamma[\operatorname{rge} f_\alpha]$ is a saturated subsociety of Γ for every $\alpha \leq \operatorname{dom} f$, that is, if for every man a appearing in f all the women that he knows (all the elements of $K\langle a \rangle$) appear in f before him. Note that if $\operatorname{dom} f$ is a limit ordinal and f_α is saturated for every $\alpha < \operatorname{dom} f$, then f is saturated as well.

Let $\mathbb{Z}^\infty = \mathbb{Z} \cup \{-\infty, \infty\}$ be the set of *quasi-integers*. If $a_1, \ldots, a_n \in \mathbb{Z}^\infty$, then let the sum $a_1 + \cdots + a_n$ be the usual sum if $a_1, \ldots, a_n$ are all integers, let the sum be ∞ if at least one of them is ∞, and let it be $-\infty$ if neither of $a_1, \ldots, a_n$ is ∞ but at least one of them is $-\infty$. Note that it follows immediately from the above definition that the operation of addition in $\mathbb{Z}^\infty$ is commutative and associative. The difference $a - b$ of two quasi-integers a, b means $a + (-b)$; and likewise, for example, $a - b + c - d$ means $a + (-b) + c + (-d)$, etc. Let $\mathbb{Z}^\infty$ be ordered in the obvious way. Note that if $a, b, c, d \in \mathbb{Z}^\infty$ satisfy $a \leq c$ and $b \leq d$, then $a + b \leq c + d$. Given a set S, let $\|S\| \in \mathbb{Z}^\infty$ be the cardinality of S if S is finite, and $\|S\| = \infty$ if S is infinite.

Assume that f is a string in Γ. Let $\operatorname{rge}_M f = (\operatorname{rge} f) \cap M$ and $\operatorname{rge}_W f = (\operatorname{rge} f) \cap W$. The μ*-margin* $\mu(f)$ of f is an element of $\mathbb{Z}^\infty$ defined by transfinite induction on $\alpha = \operatorname{dom} f$ as follows. Let $\mu(f) = 0$ if $\alpha = 0$, let

$$\mu(f) = \begin{cases} \mu(f_\beta) + 1 & \text{if } f(\beta) \in W, \\ \mu(f_\beta) - 1 & \text{if } f(\beta) \in M, \end{cases}$$

when $\alpha = \beta + 1$ is a successor ordinal, and

$$\mu(f) = \liminf_{\beta \to \alpha} \mu(f_\beta)$$

if α is a limit ordinal. We say that Γ is *μ-admissible* if $\mu(f) \geq 0$ for every saturated string f in Γ.

The following theorem is implied by Theorem 6.2, proved in Section 6, and a result (Theorem 7.1) of Nash-Williams [7]. We will present here (see Section 4) a self-contained proof of Theorem 2.1 because it provides an alternative proof of Theorem 7.1.

Theorem 2.1. *If $K^{-1}\langle x\rangle$ is countable for every $x \in W$, then Γ is espousable if and only if it is μ-admissible.*

3. Preliminary results

The following lemma implies that, in general, μ-admissibility is a necessary condition for the society Γ to be espousable.

Lemma 3.1. *If E is an espousal of Γ and f is a saturated string in Γ, then*

$$\|\operatorname{rge}_W f \setminus E[\operatorname{rge}_M f]\| \leq \mu(f). \tag{1}$$

Proof. We show that (1) holds using transfinite induction on $\alpha = \operatorname{dom} f$. If $\alpha = 0$, then both sides of (1) are equal to 0.

Assume that $\alpha > 0$ and that (1) holds when $\operatorname{dom} f < \alpha$. Assume also first that $\alpha = \beta + 1$ is a successor ordinal. If $f(\beta) \in W$, then it follows from the inductive hypothesis that

$$\begin{aligned} \mu(f) &= \mu(f_\beta) + 1 \\ &\geq \|\operatorname{rge}_W f_\beta \setminus E\,[\operatorname{rge}_M f_\beta]\| + 1 \\ &= \|(\operatorname{rge}_W f_\beta \setminus E\,[\operatorname{rge}_M f_\beta]) \cup \{f(\beta)\}\| \\ &= \|\operatorname{rge}_W f \setminus E[\operatorname{rge}_M f]\|\,. \end{aligned}$$

The last equality holds since f_β is saturated which implies that $f(\beta) \notin E[\operatorname{rge}_M f_\beta] = E\,[\operatorname{rge}_M f]$.

If $f(\beta) \in M$, then $E(f(\beta)) \in \operatorname{rge}_W f = \operatorname{rge}_W f_\beta$ since f is saturated. Therefore

$$\|\operatorname{rge}_W f \setminus E[\operatorname{rge}_M f]\| = \|\operatorname{rge}_W f_\beta \setminus E[\operatorname{rge}_M f_\beta]\| - 1,$$

and using the inductive hypothesis, we conclude that (1) holds as in the case above.

Assume now that α is a limit ordinal. Let

$$S_\beta = \operatorname{rge}_W f_\beta \setminus E[\operatorname{rge}_M f]$$

for every $\beta \leq \alpha$. Then $(S_\beta : \beta < \alpha)$ is an ascending sequence of subsets of S_α and

$$S_\alpha = \bigcup_{\beta<\alpha} S_\beta.$$

Hence

$$\|S_\alpha\| = \liminf_{\beta\to\alpha} \|S_\beta\|.$$

Moreover, for any $\beta < \alpha$ we have

$$\|S_\beta\| \le \|\mathrm{rge}_W f_\beta \setminus E[\mathrm{rge}_M f_\beta]\| \le \mu(f_\beta),$$

where the first inequality is obvious and the second follows from the inductive hypothesis. Hence

$$\begin{aligned} \|\mathrm{rge}_W f \setminus E[\mathrm{rge}_M f]\| &= \|S_\alpha\| \\ &= \liminf_{\beta\to\alpha} \|S_\beta\| \\ &\le \liminf_{\beta\to\alpha} \mu(f_\beta) = \mu(f), \end{aligned}$$

and the proof is complete. ∎

The following corollary follows immediately from Lemma 3.1.

Corollary 3.2. *If Γ is espousable, then it is μ-admissible.*

The following lemma holds.

Lemma 3.3. *Let α be a limit ordinal and $(t_\beta)_{\beta<\alpha}$, $(d_\beta)_{\beta<\alpha}$ be transfinite sequences in $\mathbb{Z}^\infty$. Then*

$$\liminf_{\beta\to\alpha} (t_\beta - d_\beta) \le \liminf_{\beta\to\alpha} t_\beta - \liminf_{\beta\to\alpha} d_\beta. \tag{2}$$

Moreover

$$\liminf_{\beta\to\alpha} (t_\beta + d_\beta) \ge \liminf_{\beta\to\alpha} t_\beta + \liminf_{\beta\to\alpha} d_\beta \tag{3}$$

unless $\liminf_{\beta\to\alpha} t_\beta$ *and* $\liminf_{\beta\to\alpha} d_\beta$ *are infinite with opposite signs.*

Proof. Clearly, we have

$$\inf\{t_\theta - d_\theta : \phi_1 \le \theta < \alpha\} \le t_\delta - \inf\{d_\theta : \phi_2 \le \theta < \alpha\}$$

for any $\phi_1 \le \phi_2 \le \delta < \alpha$. Therefore

$$\inf\{t_\theta - d_\theta : \phi_1 \le \theta < \alpha\} \le \liminf_{\beta\to\alpha} t_\beta - \inf\{d_\theta : \phi_2 \le \theta < \alpha\}$$

for any $\phi_1 \le \phi_2 < \alpha$. Hence

$$\inf\{t_\theta - d_\theta : \phi_1 \le \theta < \alpha\} \le \liminf_{\beta\to\alpha} t_\beta - \liminf_{\beta\to\alpha} d_\beta$$

for every $\phi_1 < \alpha$. It follows that (2) holds.

Assume now that it is not true that $\liminf_{\beta\to\alpha} t_\beta$ and $\liminf_{\beta\to\alpha} d_\beta$ are infinite with opposite signs. We will show that (3) holds. If $\liminf_{\beta\to\alpha} d_\beta = \infty$, then $\liminf_{\beta\to\alpha}(t_\beta + d_\beta) = \infty$ since $\liminf_{\beta\to\alpha} t_\beta > -\infty$, and hence (3) holds. If $\liminf_{\beta\to\alpha} d_\beta = -\infty$, then the right hand side of (3) is equal to $-\infty$ since $\operatorname{liminf}_{\beta\to\alpha} t_\beta < \infty$, and again (3) holds. It remains to consider the case when $\operatorname{liminf}_{\beta\to\alpha} d_\beta$ is finite. Let $s_\beta = t_\beta + d_\beta$ for every $\beta < \alpha$. It follows from (2) that

$$\operatorname*{liminf}_{\beta\to\alpha} (s_\beta - d_\beta) \le \operatorname*{liminf}_{\beta\to\alpha} s_\beta - \operatorname*{liminf}_{\beta\to\alpha} d_\beta.$$

Since $\operatorname{liminf}_{\beta\to\alpha} d_\beta$ is finite and addition in $\mathbb{Z}^\infty$ is associative, we conclude that

$$\begin{aligned} \operatorname*{liminf}_{\beta\to\alpha} (s_\beta - d_\beta) + \operatorname*{liminf}_{\beta\to\alpha} d_\beta &\le \operatorname*{liminf}_{\beta\to\alpha} s_\beta - \operatorname*{liminf}_{\beta\to\alpha} d_\beta + \operatorname*{liminf}_{\beta\to\alpha} d_\beta \\ &= \operatorname*{liminf}_{\beta\to\alpha} s_\beta \end{aligned}$$

Since $s_\beta - d_\beta = t_\beta + d_\beta - d_\beta \ge t_\beta$ for every $\beta < \alpha$, it follows that (3) holds, and hence the proof is complete. ∎

The following lemma is obvious.

Lemma 3.4. *Let $\Gamma[A]$ be a saturated subsociety of Γ. Then any saturated subsociety of $\Gamma[A]$ is a saturated subsociety of Γ. In particular, any saturated string in $\Gamma[A]$ is saturated in Γ and if Γ is μ-admissible, then $\Gamma[A]$ is μ-admissible.*

Let f, g be strings with domains α, β respectively. We say that g is a *substring* of f if $\operatorname{rge} g \subseteq \operatorname{rge} f$ and for any $x, y \in \operatorname{rge} g$ with $g^{-1}(x) < g^{-1}(y)$ we have $f^{-1}(x) < f^{-1}(y)$, that is, g is a substring of f if all the elements appearing in g appear in f and they appear in f in the same order as in g.

Assume that g is a substring of f. For every $\theta \le \alpha = \operatorname{dom} f$, there is the unique ordinal $\phi \le \beta = \operatorname{dom} g$ such that $\operatorname{rge} g_\phi = (\operatorname{rge} g) \cap (\operatorname{rge} f_\theta)$. The ordinal ϕ will be called the *f-projection of θ onto g.*

Lemma 3.5. *Let f be an α-string in Γ and g be a substring of f. If α is a limit ordinal and θ_g is the f-projection of θ onto g for every $\theta < \alpha$, then*

$$\operatorname*{liminf}_{\theta\to\alpha} \mu(g_{\theta_g}) = \mu(g). \tag{4}$$

Proof. Let $\beta = \operatorname{dom} g$. If $\theta_g = \beta$ for some $\theta < \alpha$, then (4) is obvious. Otherwise, for every $\theta < \alpha$ we have $\theta_g < \beta$. Hence β is a limit ordinal and

$$\{\mu(g_\tau) : \theta_g \le \tau < \beta\} = \left\{\mu(g_{\xi_g}) : \theta \le \xi < \alpha\right\}$$

for every $\theta < \alpha$. Therefore

$$\operatorname*{liminf}_{\theta\to\alpha} \mu(g_{\theta_g}) = \operatorname*{liminf}_{\delta\to\beta} \mu(g_\delta) = \mu(g)$$

and the proof is complete. ∎

Let f be a string and g, h be substrings of f. We say that the string f is a *shuffle* of g and h if the ranges of g and h form a partition of $\operatorname{rge} f$, that is, if $(\operatorname{rge} g) \cap (\operatorname{rge} h) = \emptyset$ and $\operatorname{rge} f = (\operatorname{rge} g) \cup (\operatorname{rge} h)$.

Note that if an α-string f is a shuffle of strings g and h and θ_g, θ_h are the f-projections of some $\theta \leq \alpha$ onto g and h respectively, then f_θ is a shuffle of g_{θ_g} and h_{θ_h}.

Lemma 3.6. *Let f, g and h be strings in the society Γ with f being a shuffle of g and h. If $\mu(g), \mu(h) > -\infty$, then*

$$\mu(f) \geq \mu(g) + \mu(h). \tag{5}$$

Proof. Let f, g and h have domains α, β and γ respectively. To show that (5) holds we use transfinite induction on α. If $\alpha = 0$, then $\beta = \gamma = 0$ and the inequality (5) holds since both its sides are equal to 0.

Suppose that $\alpha > 0$ and that (5) holds when $\operatorname{dom} f < \alpha$. Assume first that $\alpha = \delta + 1$ is a successor ordinal. Let θ and ϕ be the f-projections of δ onto g and h respectively. If $f(\delta) \in \operatorname{rge} g$, then $\beta = \theta + 1$ and $\phi = \gamma$. If moreover $f(\delta) \in W$, then $g(\theta) = f(\delta) \in W$ and, using the inductive hypothesis, we get

$$\begin{aligned} \mu(f) &= \mu(f_\delta) + 1 \\ &\geq \mu(g_\theta) + \mu(h) + 1 \\ &= \mu(g) + \mu(h). \end{aligned}$$

To complete the proof in the case of α being a successor ordinal it remains to consider the following three cases: $f(\delta) \in \operatorname{rge}_M g$, $f(\delta) \in \operatorname{rge}_W h$, and $f(\delta) \in \operatorname{rge}_M h$. All these cases can be handled similarly as the case $f(\delta) \in \operatorname{rge}_W g$ considered above.

Assume now that α is a limit ordinal. For every $\delta < \alpha$, let δ_g and δ_h be the f-projections of δ onto g and h respectively. Then

$$\begin{aligned} \mu(f) &= \liminf_{\delta \to \alpha} \mu(f_\delta) \\ &\geq \liminf_{\delta \to \alpha} (\mu(g_{\delta_g}) + \mu(h_{\delta_h})) \\ &\geq \liminf_{\delta \to \alpha} \mu(g_{\delta_g}) + \liminf_{\delta \to \alpha} \mu(h_{\delta_h}) \\ &= \mu(g) + \mu(h), \end{aligned}$$

where the first inequality follows from the inductive hypothesis, the second equality follows from Lemma 3.3, and the last equality follows from Lemma 3.5. Thus the proof is complete. ∎

Let $\mathfrak{T}$ be the set of all strings in Γ and let $\preceq$ be the relation on $\mathfrak{T}$ such that $g \preceq f$ if $g = f_\beta$ for some $\beta \leq \operatorname{dom} f$. Clearly $\preceq$ is a partial order on $\mathfrak{T}$. Let $\mathfrak{R}$ be the subset of $\mathfrak{T}$ consisting of saturated strings f in Γ such that $\mu(f_\beta) \geq 0$ for every $\beta \leq \operatorname{dom} f$ and $\mu(f) = 0$.

Lemma 3.7. *The set $\mathfrak{R}$ contains a maximal element with respect to $\preceq$.*

Proof. We are going to use Zorn lemma. Since the empty string belongs to $\mathfrak{R}$, the set $\mathfrak{R}$ is nonempty. Let $\mathfrak{B}$ be a nonempty chain in $\mathfrak{R}$. We show that there is an upper bound for $\mathfrak{B}$ in $\mathfrak{R}$.

Let $\Theta = \{\operatorname{dom} g : g \in \mathfrak{B}\}$ and $\alpha = \sup \Theta$. We are going to define an α-string f in Γ that belongs to $\mathfrak{R}$ and is an upper bound for $\mathfrak{B}$. If $\beta < \alpha$, then there is $g \in \mathfrak{B}$ with $\beta < \operatorname{dom} g$. Define $f(\beta) = g(\beta)$. Since $\mathfrak{B}$ is a chain, the value of $f(\beta)$ does not depend on the choice of g. It is clear that $g \preceq f$ for every $g \in \mathfrak{B}$ so f is an upper bound for $\mathfrak{B}$.

We show that $f \in \mathfrak{R}$. Since $\mathfrak{B} \subseteq \mathfrak{R}$, we can assume that $f \notin \mathfrak{B}$. Then α is a limit ordinal. If $\beta < \alpha$, then $f_\beta = g_\beta$ for some $g \in \mathfrak{B}$ so $\mu(f_\beta) = \mu(g_\beta) \geq 0$. Moreover, the society $\Gamma\,[\operatorname{rge} f_\beta] = \Gamma\,[\operatorname{rge} g_\beta]$ is a saturated subsociety of Γ for every $\beta < \alpha$, hence f is a saturated string in Γ. Since α is a limit ordinal and since $\mu(f_\beta) \geq 0$ for every $\beta < \alpha$, we have

$$\mu(f) = \liminf_{\beta \to \alpha} \mu(f_\beta) \geq 0.$$

Since $\alpha = \sup \Theta$ and since $\mu(f_\beta) = 0$ for every $\beta \in \Theta$, it follows that $\mu(f) = 0$. Therefore $f \in \mathfrak{R}$ and so $\mathfrak{B}$ has an upper bound in $\mathfrak{R}$.

Since $\mathfrak{B}$ was an arbitrary nonempty chain in $\mathfrak{R}$, it follows from Zorn lemma that $\mathfrak{R}$ contains a maximal element with respect to $\preceq$, and hence the proof is complete. ∎

Lemma 3.8. *Assume that the society Γ is μ-admissible. If $a \in M$, then there is $x \in W$ such that $(a, x) \in K$ and the society $\Gamma - \{a, x\}$ is μ-admissible.*

Proof. Let $a \in M$ and $\mathfrak{R}'$ be the subset of $\mathfrak{T}$ consisting of saturated strings f in $\Gamma - \{a\}$ such that $\mu(f_\beta) \geq 0$ for every $\beta \leq \operatorname{dom} f$ and $\mu(f) = 0$. By Lemma 3.7, there is a maximal element f in $\mathfrak{R}'$ with respect to $\preceq$. Since Γ is μ-admissible and

$$\mu(f * [a]) = \mu(f) - 1 = -1 < 0,$$

it follows that $f * [a]$ is not saturated in Γ. Since f is saturated in $\Gamma - \{a\}$, there is $x \in W \setminus \operatorname{rge} f$ with $(a, x) \in K$. We will show that $\Gamma - \{a, x\}$ is μ-admissible.

First, we show that $\Gamma - (\{a, x\} \cup \operatorname{rge} f)$ is μ-admissible. Suppose, by the way of contradiction, that there is a saturated string g in $\Gamma - (\{a, x\} \cup \operatorname{rge} f)$ with $\mu(g) < 0$. Let $h = f * [x] * g$. We will show that $h \in \mathfrak{R}'$, contradicting our assumption that f is maximal in $\mathfrak{R}'$.

Since f is saturated in Γ and g is saturated in $\Gamma - (\{a, x\} \cup \operatorname{rge} f)$, it follows that h is saturated in Γ and in $\Gamma - \{a\}$. Since Γ is μ-admissible, we have $\mu(h) \geq 0$. Moreover, for every $\delta \leq \operatorname{dom} h$, the string h_δ is saturated in Γ implying that $\mu(h_\delta) \geq 0$. Since $\mu(f) = 0$, it follows that $\mu(h) = \mu([x] * g) = \mu(g) + 1$. Since $\mu(g) < 0$, we conclude that $\mu(h) = 0$, and so $h \in \mathfrak{R}'$. The obtained contradiction implies that $\Gamma - (\{a, x\} \cup \operatorname{rge} f)$ is μ-admissible.

Now we show that $\Gamma - \{a, x\}$ is μ-admissible. Let φ be any saturated string in $\Gamma - \{a, x\}$. We will show that $\mu(\varphi) \geq 0$. Let g be the string in $\Gamma - (\{a, x\} \cup \operatorname{rge} f)$ and h be the string in $\Gamma\,[\operatorname{rge} f]$ such that φ is the shuffle of g and h. Since φ is saturated in $\Gamma - \{a, x\}$, it follows that g is saturated in $\Gamma - (\{a, x\} \cup \operatorname{rge} f)$ and h is saturated in $\Gamma\,[\operatorname{rge} f]$. Since f is saturated in Γ, if follows from Lemma 3.4 that h is saturated in Γ. Since both societies $\Gamma - (\{a, x\} \cup \operatorname{rge} f)$ and Γ are μ-admissible, it follows that $\mu(g) \geq 0$ and $\mu(h) \geq 0$. By Lemma 3.6, $\mu(\varphi) \geq 0$ completing the proof. ∎

4. Proof of Theorem 2.1

Assume that $K^{-1}\langle x\rangle$ is countable for every $x \in W$. By Corollary 3.2, if Γ is espousable, then it is μ-admissible. Assume that Γ is μ-admissible, we will show that it is espousable.

We define a partial order $\leq$ on the product $\omega \times \omega$ by setting $(i, j) \leq (k, \ell)$ if either $i + j < k + \ell$ or $i + j = k + \ell$ and $i \leq k$. If $x \in W$, then let $\rho_0(x), \rho_1(x), \ldots$ be a fixed (possibly finite) enumeration of the countable set $K^{-1}\langle x\rangle$. Let f be a fixed string in M with $\operatorname{rge} f = M$.

We are going to prove that for some ordinal number α there are α-strings g and h in M and W respectively with $\operatorname{rge} g = M$, such that $(g(\beta), h(\beta)) \in K$ for every $\beta < \alpha$. It is clear that the existence of the strings g and h satisfying the above conditions implies that Γ is espousable.

We define the strings g and h using transfinite induction on ξ to specify the values of the initial segments g_ξ and h_ξ. Given an ordinal number ξ, we say that g and h are *ξ-defined* if the values of g_ξ and h_ξ are known (with $(g(\delta), h(\delta)) \in K$ for every $\delta < \xi$) and the society $\Gamma - (\operatorname{rge} g_\xi \cup \operatorname{rge} h_\xi)$ is μ-admissible. Since Γ is μ-admissible, at the beginning of the construction (when no values of g and h have been specified) the strings g and h are 0-defined.

Assume that ξ is an ordinal and that the strings g and h are β-defined for every $\beta < \xi$, that is, the values $g(\delta)$, $h(\delta)$ are known for any $\delta < \beta$ with $\beta < \xi$ and the society $\Gamma - (\operatorname{rge} g_\beta \cup \operatorname{rge} h_\beta)$ is μ-admissible for every $\beta < \xi$. If $\xi = \beta + 1$ is a successor ordinal and $\operatorname{rge} g_\beta \neq M$, then we show that the values $g(\beta)$, $h(\beta)$ can be defined in such a way that g and h become ξ-defined. If ξ is a limit ordinal, then we show that g and h are already ξ-defined.

Assume first that $\xi = \beta + 1$ is a successor ordinal. Then g and h are β-defined. We can assume that $\operatorname{rge} g_\beta \neq M$ since otherwise $\alpha = \beta$ and the definition of g and h is complete. Clearly, we can express $\beta = \gamma + i$ where $i < \omega$ and γ is either a limit ordinal or is equal to 0. Consider the set

$$A_\beta = \left\{(j, k) : 0 \leq j < i,\ \left|K^{-1}\langle h(\gamma + j)\rangle\right| \geq k, \text{ and } \rho_k(h(\gamma + j)) \notin \operatorname{rge} g_\beta\right\}$$

If $A_\beta = \emptyset$ (in particular, this is the case if β is a limit ordinal), let $g(\beta)$ be equal to $f(\theta)$ where θ is the minimum ordinal number with $f(\theta) \notin \operatorname{rge} g_\beta$. If $A_\beta \neq \emptyset$, then let $g(\beta) = \rho_k(h(\gamma + j))$ where (j, k) is the minimum element in A_β. By Lemma 3.8, there is $x \in W$ such that $(g(\beta), x) \in K$ and the society $(\Gamma - (\operatorname{rge} g_\beta \cup \operatorname{rge} h_\beta)) - \{g(\beta), x\}$ is μ-admissible. Let $h(\beta) = x$. It follows immediately from the definition of $g(\beta)$ and $h(\beta)$ that g and h are ξ-defined.

Assume now that ξ is a limit ordinal. Since for every $\beta < \xi$ the strings g and h are β-defined, it follows that the values of g_ξ and h_ξ are known (with $(g(\delta), h(\delta)) \in K$ for every $\delta < \xi$). To show that g and h are ξ-defined it remains to show that the society $\Gamma_\xi = \Gamma - (\operatorname{rge} g_\xi \cup \operatorname{rge} h_\xi)$ is μ-admissible. Because of Lemma 3.4, it suffices to show that Γ_ξ is a saturated subsociety of Γ.

Let $\Gamma_\xi = (M_\xi, W_\xi, K_\xi)$ and suppose, by the way of contradiction, that there is $(a, x) \in K$ with $a \in M_\xi$ and $x \notin W_\xi$. Then $x \in \operatorname{rge} h_\xi$ so $x = h(\delta)$ for some $\delta < \xi$. Let $\delta = \gamma + j$ where $j < \omega$ and γ is either equal to zero or is a limit ordinal. Since $(a, x) \in K$, it follows that $a = \rho_k(x)$ for some $k < \omega$. Since ξ is a limit ordinal, we have $\gamma + i < \xi$ for every $i < \omega$. Since $a = \rho_k(h(\gamma + j)) \in M_\xi = M \setminus \operatorname{rge} g_\xi$, we conclude that $(j, k) \in A_{\gamma+i}$ for every i with $j < i < \omega$. Let

$$B_i = \{(\ell, m) \in A_{\gamma+i} : (\ell, m) \leq (j, k)\}$$

for every i with $j + k < i < \omega$. Then the set B_i is finite and $(j, k) \in B_i$ for every i with $j + k < i < \omega$. Defining $g(\gamma + i)$ we delete the least element of $A_{\gamma+i}$ and hence of B_i. Defining $h(\gamma + i)$ we may add some elements (i, m) to $A_{\gamma+i}$ but these new elements (i, m) are not in B_{i+1} since the inequality $i > j + k$ implies that $(i, m) > (j, k)$. Therefore $B_{i+1} = B_i \setminus \{\min B_i\}$ for every i with $j + k < i < \omega$. Since the sets B_i are finite, we have a contradiction. Thus Γ_ξ is saturated and the proof is complete.

5. A deferment of a string in Γ

Let f be a string in Γ. Then there are unique strings h' and h in M and W respectively such that f is a shuffle of h' and h. The string h' will be called the *men substring* of f and h the *woman substring* of f. Let $\lambda = \operatorname{dom} h$. If $\alpha \leq \lambda$, then the *f-lift* of α is the unique ordinal $\alpha' \in f^{-1}[W] \cup \{\operatorname{dom} f\}$ such that α is the f-projection of α' onto h, that is, such that $\operatorname{rge}_W f_{\alpha'} = \operatorname{rge} h_\alpha$. Note that

$$\alpha' = \begin{cases} f^{-1}(h(\alpha)) & \text{if } \alpha < \operatorname{dom} h, \\ \operatorname{dom} f & \text{if } \alpha = \operatorname{dom} h. \end{cases}$$

For every limit ordinal $\alpha \leq \lambda$, let $\overline{\alpha} = \sup \{\beta' : \beta < \alpha\}$ (where β' denotes the f-lift of β). Clearly, $\overline{\alpha}$ is a limit ordinal with $\overline{\alpha} \leq \alpha'$.

Lemma 5.1. *If $\alpha \leq \lambda$ is a limit ordinal, then*

$$\liminf_{\theta \to \overline{\alpha}} \mu(f_\theta) = \liminf_{\beta \to \alpha} \mu(f_{\beta'}).$$

Proof. Clearly, if $\theta < \overline{\alpha}$ and ζ is the smallest ordinal with $\zeta \geq \theta$ and $f(\zeta) \in W$, then $\mu(f_\theta) \geq \mu(f_\zeta)$. Therefore

$$\inf \{\mu(f_\theta) : \gamma' \leq \theta < \overline{\alpha}\} = \inf \{\mu(f_{\beta'}) : \gamma \leq \beta < \alpha\} \tag{6}$$

for every $\gamma < \alpha$. Since the values of $\inf \{\mu(f_\theta) : \phi \leq \theta < \overline{\alpha}\}$ do not decrease as ϕ increases, we have

$$\sup \{\inf \{\mu(f_\theta) : \phi \leq \theta < \overline{\alpha}\} : \phi < \overline{\alpha}\} = \sup \{\inf \{\mu(f_\theta) : \gamma' \leq \theta < \overline{\alpha}\} : \gamma < \alpha\}. \tag{7}$$

It follows from (6) and (7) that

$$\begin{aligned}\liminf_{\theta\to\overline{\alpha}} \mu(f_\theta) &= \sup\{\inf\{\mu(f_\theta) : \phi \le \theta < \overline{\alpha}\} : \phi < \overline{\alpha}\}\\ &= \sup\{\inf\{\mu(f_\theta) : \gamma' \le \theta < \overline{\alpha}\} : \gamma < \alpha\}\\ &= \sup\left\{\inf\left\{\mu(f_{\beta'}) : \gamma \le \beta < \alpha\right\} : \gamma < \alpha\right\}\\ &= \liminf_{\beta\to\alpha} \mu(f_{\beta'}).\end{aligned}$$

completing the proof. ∎

Let g be another string in Γ with the same women substring as f. For every ordinal $\alpha \le \lambda = \operatorname{dom} h$ let α'' be the g-lift of α. We say that g is a *deferment* of f if for every $\alpha \le \lambda$ we have

$$\operatorname{rge}_M g_{\alpha''} \subseteq \operatorname{rge}_M f_{\alpha'}.$$

Informally, g is a deferment of f if each man appearing in f either does not appear at all in g or appears in the same or a later segment determined by the appearance of women.

Assume that g is a deferment of f. If $\alpha \le \lambda$ is a limit ordinal, then let

$$\overline{\overline{\alpha}} = \sup\left\{\beta'' : \beta < \alpha\right\}.$$

Lemma 5.2. *If $\alpha \le \lambda$ is a limit ordinal, then*

$$\liminf_{\beta\to\alpha} \left\|\operatorname{rge}_M f_{\beta'} \setminus \operatorname{rge}_M g_{\beta''}\right\| \ge \left\|\operatorname{rge}_M f_{\overline{\alpha}} \setminus \operatorname{rge}_M g_{\overline{\overline{\alpha}}}\right\|.$$

Proof. Let $\alpha \le \lambda$ and for each $\beta < \alpha$ let $A_\beta = \operatorname{rge}_M f_{\beta'} \setminus \operatorname{rge}_M g_{\overline{\overline{\alpha}}}$. Clearly $(A_\beta)_{\beta<\alpha}$ is a nondecreasing transfinite sequence of subsets of $\operatorname{rge}_M f_{\overline{\alpha}} \setminus \operatorname{rge}_M g_{\overline{\overline{\alpha}}}$ with

$$\operatorname{rge}_M f_{\overline{\alpha}} \setminus \operatorname{rge}_M g_{\overline{\overline{\alpha}}} = \bigcup_{\beta<\alpha} A_\beta.$$

Hence

$$\left\|\operatorname{rge}_M f_{\overline{\alpha}} \setminus \operatorname{rge}_M g_{\overline{\overline{\alpha}}}\right\| = \liminf_{\beta\to\alpha} \|A_\beta\|.$$

Since $A_\beta \subseteq \operatorname{rge} f_{\beta'} \setminus \operatorname{rge} g_{\beta''}$ for every $\beta < \alpha$, it follows that

$$\liminf_{\beta\to\alpha} \left\|\operatorname{rge}_M f_{\beta'} \setminus \operatorname{rge}_M g_{\beta''}\right\| \ge \liminf_{\beta\to\alpha} \|A_\beta\|.$$

Therefore

$$\liminf_{\beta\to\alpha} \left\|\operatorname{rge}_M f_{\beta'} \setminus \operatorname{rge}_M g_{\beta''}\right\| \ge \left\|\operatorname{rge}_M f_{\overline{\alpha}} \setminus \operatorname{rge}_M g_{\overline{\overline{\alpha}}}\right\|$$

completing the proof. ∎

A corollary of the following lemma is the main result of this section.

Lemma 5.3. *For every ordinal $\alpha \le \lambda$ we have*

$$\mu(f_{\alpha'}) \le \mu(g_{\alpha''}) - \left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M g_{\alpha''}\right\| \tag{8}$$

Proof. We use transfinite induction on α. Let $\alpha = 0$. Since $\mu(f_{0'})$ is the negative of the size of the set of men appearing before the first woman in f, we have

$$\mu(f_{0'}) = -\left\|\operatorname{rge}_M f_{0'}\right\|.$$

Similarly

$$\mu(g_{0''}) = -\left\|\operatorname{rge}_M g_{0''}\right\|.$$

If the set $\operatorname{rge}_M f_{0'}$ is infinite, then $\mu(f_{0'}) = -\infty$ and so (8) is obvious. Assume that $\operatorname{rge}_M f_{0'}$ is finite. Since g is a deferment of f, it follows that $\operatorname{rge}_M g_{0''} \subseteq \operatorname{rge}_M f_{0'}$. Therefore the set $\operatorname{rge}_M g_{0''}$ is also finite and (8) holds.

Now assume that $\alpha = \beta + 1$ is a successor ordinal. Then we have

$$\begin{aligned}
\mu(f_{\alpha'}) &= \mu(f_{\beta'}) + 1 - \left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M f_{\beta'}\right\|, \\
\mu(g_{\alpha''}) &= \mu(g_{\beta''}) + 1 - \left\|\operatorname{rge}_M g_{\alpha''} \setminus \operatorname{rge}_M g_{\beta''}\right\|.
\end{aligned}$$

Using the inductive hypothesis and the following obvious equalities

$$\begin{aligned}
-\left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M g_{\beta''}\right\| &= -\left\|\operatorname{rge}_M f_{\beta'} \setminus \operatorname{rge}_M g_{\beta''}\right\| - \left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M f_{\beta'}\right\| \\
&= -\left\|\operatorname{rge}_M g_{\alpha''} \setminus \operatorname{rge}_M g_{\beta''}\right\| - \left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M g_{\alpha''}\right\|
\end{aligned}$$

and remembering that addition in $\mathbb{Z}^\infty$ is commutative and associative, we conclude that

$$\begin{aligned}
\mu(f_{\alpha'}) &= \mu(f_{\beta'}) + 1 - \left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M f_{\beta'}\right\| \\
&\le \mu(g_{\beta''}) - \left\|\operatorname{rge}_M f_{\beta'} \setminus \operatorname{rge}_M g_{\beta''}\right\| + 1 - \left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M f_{\beta'}\right\| \\
&= \mu(g_{\beta''}) + 1 - \left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M g_{\beta''}\right\| \\
&= \mu(g_{\beta''}) + 1 - \left\|\operatorname{rge}_M g_{\alpha''} \setminus \operatorname{rge}_M g_{\beta''}\right\| - \left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M g_{\alpha''}\right\| \\
&= \mu(g_{\alpha''}) - \left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M g_{\alpha''}\right\|,
\end{aligned}$$

implying that (8) holds.

Finally, assume that α is a limit ordinal. First, we show that

$$\mu(f_{\overline{\alpha}}) \le \mu(g_{\overline{\overline{\alpha}}}) - \left\|\operatorname{rge}_M f_{\overline{\alpha}} \setminus \operatorname{rge}_M g_{\overline{\overline{\alpha}}}\right\| \tag{9}$$

By Lemma 5.1, we have

$$\mu(f_{\overline{\alpha}}) = \liminf_{\theta \to \overline{\alpha}} \mu(f_\theta) = \liminf_{\beta \to \alpha} \mu(f_{\beta'}).$$

By the inductive hypothesis

$$\mu(f_{\overline{\alpha}}) \le \liminf_{\beta \to \alpha} \left(\mu(g_{\beta''}) - \left\|\operatorname{rge}_M f_{\beta'} \setminus \operatorname{rge}_M g_{\beta''}\right\|\right).$$

Using Lemma 3.3, then Lemma 5.2 and finally Lemma 5.1 again, we conclude that

$$\begin{aligned}
\mu(f_{\overline{\alpha}}) &\leq \liminf_{\beta\to\alpha} \mu(g_{\beta''}) - \liminf_{\beta\to\alpha} \left\|\mathrm{rge}_M f_{\beta'} \setminus \mathrm{rge}_M g_{\beta''}\right\| \\
&\leq \liminf_{\beta\to\alpha} \mu(g_{\beta''}) - \|\mathrm{rge}_M f_{\overline{\alpha}} \setminus \mathrm{rge}_M g_{\overline{\overline{\alpha}}}\| \\
&= \liminf_{\theta\to\overline{\overline{\alpha}}} \mu(g_\theta) - \|\mathrm{rge}_M f_{\overline{\alpha}} \setminus \mathrm{rge}_M g_{\overline{\overline{\alpha}}}\| \\
&= \mu(g_{\overline{\overline{\alpha}}}) - \|\mathrm{rge}_M f_{\overline{\alpha}} \setminus \mathrm{rge}_M g_{\overline{\overline{\alpha}}}\|
\end{aligned}$$

completing the proof of (9).

Since $f(\beta) \in M$ for every β with $\overline{\alpha} \leq \beta < \alpha'$, we have

$$\mu(f_{\alpha'}) = \mu(f_{\overline{\alpha}}) - \|\mathrm{rge}_M f_{\alpha'} \setminus \mathrm{rge}_M f_{\overline{\alpha}}\|\,.$$

Similarly

$$\mu(g_{\alpha''}) = \mu(g_{\overline{\overline{\alpha}}}) - \|\mathrm{rge}_M g_{\alpha''} \setminus \mathrm{rge}_M g_{\overline{\overline{\alpha}}}\|\,.$$

Since addition in $\mathbb{Z}^\infty$ is associative, using the above two equalities and the equality (9) we conclude that

$$\begin{aligned}
\mu(f_{\alpha'}) &= \mu(f_{\overline{\alpha}}) - \|\mathrm{rge}_M f_{\alpha'} \setminus \mathrm{rge}_M f_{\overline{\alpha}}\| \\
&\leq \mu(g_{\overline{\overline{\alpha}}}) - \|\mathrm{rge}_M f_{\overline{\alpha}} \setminus \mathrm{rge}_M g_{\overline{\overline{\alpha}}}\| - \|\mathrm{rge}_M f_{\alpha'} \setminus \mathrm{rge}_M f_{\overline{\alpha}}\| \\
&= \mu(g_{\overline{\overline{\alpha}}}) - \|\mathrm{rge}_M f_{\alpha'} \setminus \mathrm{rge}_M g_{\overline{\overline{\alpha}}}\| \\
&= \mu(g_{\overline{\overline{\alpha}}}) - \|\mathrm{rge}_M g_{\alpha''} \setminus \mathrm{rge}_M g_{\overline{\overline{\alpha}}}\| - \|\mathrm{rge}_M f_{\alpha'} \setminus \mathrm{rge}_M g_{\alpha''}\| \\
&= \mu(g_{\alpha''}) - \|\mathrm{rge}_M f_{\alpha'} \setminus \mathrm{rge}_M g_{\alpha''}\|
\end{aligned}$$

completing the proof of the lemma. ∎

Lemma 5.3 implies immediately the following corollary.

Corollary 5.4. *Let f and g be strings in Γ that have the same women substring. If g is a deferment of f, then*

$$\mu(f) \leq \mu(g).$$

Proof. Let λ be the domain of the women substring of f (and of g). Taking $\alpha = \lambda$ in (8) gives

$$\mu(f) \leq \mu(g) - \|\mathrm{rge}_M f \setminus \mathrm{rge}_M g\|\,.$$

Since $\|\mathrm{rge}_M f \setminus \mathrm{rge}_M g\| \geq 0$, the desired inequality follows. ∎

6. q-ADMISSIBLE SOCIETIES

In this section we are going to present the definition of a q-admissible society and prove that the society Γ is q-admissible if and only if it is μ-admissible. The concept of q-admissible societies was introduced by Nash-Williams [6].

Given a set $X \subseteq W$ of women, let $D(X) = \{a \in M : K\langle a\rangle \subseteq X\}$ be the set of men that know no women outside X. If f is a string in W, then let $\Delta(f) = D(\operatorname{rge} f)$. The *$q$-margin* $q(f)$ of f is the element of $\mathbb{Z}^\infty$ defined by transfinite induction on $\alpha = \operatorname{dom} f$ as follows. Let $q(f) = -\|D(\emptyset)\|$ if $\alpha = 0$, let

$$q(f) = q(f_\beta) + 1 - \|\Delta(f) \setminus \Delta(f_\beta)\|$$

when $\alpha = \beta + 1$ is a successor ordinal, and

$$q(f) = \liminf_{\beta\to\alpha} q(f_\beta) - \left\|\Delta(f) \setminus \bigcup_{\beta<\alpha} \Delta(f_\beta)\right\|$$

if α is a limit ordinal. The society Γ is *q-admissible* if $q(f) \geq 0$ for every string f in W.

Let f be a saturated string in Γ, h be the women substring of f and $\lambda = \operatorname{dom} h$. For each $\alpha \leq \lambda$ let α' be the f-lift of α. We say that f is *prompt* if for every $\alpha \leq \lambda$ we have

$$\operatorname{rge}_M f_{\alpha'} = \Delta(h_\alpha),$$

that is, if for every $\beta \in f^{-1}[W] \cup \{\operatorname{dom} f\}$ the set of men appearing in f_β is as large as possible without violating the condition that f is saturated. Note that if f is prompt and g is a saturated string in Γ with the same women substring as f, then g is a deferment of f.

The following lemma will be used to prove that q-admissibility is equivalent to μ-admissibility.

Lemma 6.1. *Let f be a prompt string in Γ and h be the women substring of f. Then*

$$\mu(f) = q(h).$$

Proof. Let $\lambda = \operatorname{dom} h$. For each $\alpha \leq \lambda$ let α' be the f-lift of α. We will use transfinite induction on α to show that

$$\mu(f_{\alpha'}) = q(h_\alpha) \tag{10}$$

holds for every $\alpha \leq \lambda$. Since f is prompt, we have

$$\mu(f_{0'}) = -\|\operatorname{rge}_M f_{0'}\| = -\|\Delta(h_0)\| = -\|D(\emptyset)\| = q(h_0)$$

showing that (10) holds for $\alpha = 0$.

Assume that $\alpha = \beta + 1$ is a successor ordinal. Then

$$\mu(f_{\alpha'}) = \mu(f_{\beta'}) + 1 - \left\|\operatorname{rge}_M f_{\alpha'} \setminus \operatorname{rge}_M f_{\beta'}\right\|.$$

Since f is prompt, we have

$$\mathrm{rge}_M f_{\alpha'} \setminus \mathrm{rge}_M f_{\beta'} = \Delta(h_\alpha) \setminus \Delta(h_\beta).$$

Using the inductive hypothesis, we conclude that

$$\mu(f_{\alpha'}) = q(f_\beta) + 1 - \|\Delta(h_\alpha) \setminus \Delta(h_\beta)\| = q(g_\alpha)$$

Now assume that α is a limit ordinal. Let

$$\overline{\alpha} = \sup \left\{\beta' : \beta < \alpha\right\}.$$

By Lemma 5.1, we have

$$\mu(f_{\overline{\alpha}}) = \liminf_{\theta \to \overline{\alpha}} \mu(f_\theta) = \liminf_{\beta \to \alpha} \mu(f_{\beta'}).$$

Using the inductive hypothesis, we get

$$\mu(f_{\overline{\alpha}}) = \liminf_{\beta \to \alpha} q(h_\beta)$$

Since $f(\beta) \in M$ for every β with $\overline{\alpha} \leq \beta < \alpha'$, we conclude that

$$\begin{aligned} \mu(f_{\alpha'}) &= \mu(f_{\overline{\alpha}}) - \|\mathrm{rge}_M f_{\alpha'} \setminus \mathrm{rge}_M f_{\overline{\alpha}}\| \\ &= \liminf_{\beta \to \alpha} q(h_\beta) - \|\mathrm{rge}_M f_{\alpha'} \setminus \mathrm{rge}_M f_{\overline{\alpha}}\|. \end{aligned}$$

Since the string f is prompt, $\mathrm{rge}_M f_{\beta'} = \Delta(h_\beta)$ for every $\beta \leq \alpha$. Therefore

$$\mathrm{rge}_M f_{\overline{\alpha}} = \bigcup_{\beta < \alpha} \mathrm{rge}_M f_{\beta'} = \bigcup_{\beta < \alpha} \Delta(h_\beta)$$

and

$$\mathrm{rge}_M f_{\alpha'} \setminus \mathrm{rge}_M f_{\overline{\alpha}} = \Delta(h_\alpha) \setminus \bigcup_{\beta < \alpha} \Delta(h_\beta).$$

It follows that

$$\mu(f_{\alpha'}) = \liminf_{\beta \to \alpha} q(h_\beta) - \left\| \Delta(h_\alpha) \setminus \bigcup_{\beta < \alpha} \Delta(h_\beta) \right\| = q(h_\alpha)$$

completing the proof. ∎

The following theorem is the main result of this section.

Theorem 6.2. *The society* Γ *is* q*-admissible if and only if it is* μ*-admissible.*

Proof. Assume that Γ is q-admissible. Let g be any saturated string in Γ, h be the woman substring of g and f be a prompt string in Γ with woman substring h. Then g is a deferment of f and it follows from Corollary 5.4 that

$$\mu(g) \geq \mu(f).$$

By Lemma 6.1

$$\mu(f) = q(h).$$

Since Γ is q-admissible, we have $q(h) \geq 0$. Hence $\mu(g) \geq 0$ and so Γ is μ-admissible.

Now assume that Γ is μ-admissible. Let h be any string in W and f be any prompt string in Γ with h being its women substring. Since Γ is μ-admissible and f is saturated, we have $\mu(f) \geq 0$. It follows from Lemma 6.1 that

$$q(h) = \mu(f) \geq 0.$$

Therefore Γ is q-admissible and the proof is complete. ∎

7. Concluding Remarks

The following theorem was proved by Nash-Williams [7].

Theorem 7.1. *If $K^{-1}\langle x\rangle$ is countable for every $x \in W$, then Γ is espousable if and only if it is q-admissible.*

The proof given by Nash-Williams is based on a result of Milner and Shelah [5] and a weaker version of Theorem 7.1 (proved by Nash-Williams [6]) saying that espousability is equivalent to q-admissibility for societies Γ with M being countable. Our proofs of Theorems 2.1 and 6.2 provide an alternative self-contained proof of Theorem 7.1.

Aharoni [1] (Lemmas 11 and 12) proved the following properties of q-admissibility.

Theorem 7.2. *Let $Q \subseteq K$ and let Γ' be the society (M, W, Q). Then, if Γ' is q-admissible so is also Γ.*

Theorem 7.3. *Suppose that $M = A \cup B$, $A \cap B = \emptyset$ and $W = X \cup Y$, $X \cap Y = \emptyset$. If $\Gamma_1 = \Gamma[A \cup X]$ and $\Gamma_2 = \Gamma[B \cup Y]$ are q-admissible, then Γ is also q-admissible.*

Note that if we replace q-admissibility by μ-admissibility in Theorem 7.2, then the obtained statement is trivial since any saturated string in Γ is also a saturated string in Γ'. Therefore Theorem 6.2 implies Theorem 7.2.

To prove Theorem 7.3, Aharoni used the equivalence between q-admissibility and c-admissibility. Note that Theorem 7.3 can be alternatively proved using the equivalence between q-admissibility and μ-admissibility. Indeed, if f is a string in Γ, then there are uniquely determined strings h_1 and h_2 in Γ_1 and Γ_2 respectively (using the notation from Theorem 7.3) such that f is a shuffle of h_1 and h_2. Moreover, if the string f is saturated in Γ, then the strings h_1 and h_2 are saturated in Γ_1 and Γ_2 respectively. Therefore, Lemma 3.6 implies that if Γ_1 and Γ_2 are μ-admissible, then so is Γ. This establishes the analog of Theorem 7.3 with q-admissibility replaced by μ-admissibility which, by Theorem 6.2, implies Theorem 7.3 itself.

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