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Espousable societies and infinite matroids

Jerzy Wojciechowski

Department of Mathematics, West Virginia University,
PO BOX 6310, Morgantown, WV 26506-6310, USA

Abstract. Given a society $\Gamma = (M, W, K)$, where M, W are disjoint sets and $K \subseteq M \times W$, and a matroid $\mathcal{T}$ on M, a $\mathcal{T}$-*espousal* of Γ is an injective partial function $E : M \to W$ such that $E \subseteq K$ and $\operatorname{dom} E$ is spanning in $\mathcal{T}$. The problem of characterizing $\mathcal{T}$-espousable societies is a generalization of characterizing systems of matroids with disjoint bases. Wojciechowski [15] formulated a criterion for a society $\Gamma = (M, W, K)$ to be $\mathcal{P}(M)$-espousable, where $\mathcal{P}(M)$ is the matroid on M consisting of all subsets of M. In this paper, we generalize this criterion to the case when $\mathcal{T}$ is an arbitrary matroid on M, and we prove that the obtained criterion is necessary for Γ to be $\mathcal{T}$-espousable.

1. Introduction

Let R and X be disjoint sets and, for each $r \in R$, let $\mathcal{M}_r$ be a possibly infinite matroid on the set X. (We assume that infinite matroids have finite character.) The system $\mathfrak{M} = (\mathcal{M}_r)_{r\in R}$ will be called a *system of matroids* on X. The system $\mathfrak{B} = (B_r)_{r\in R}$ of subsets of X is a *system of disjoint bases* for $\mathfrak{M}$ if B_r is a base of $\mathcal{M}_r$ for each $r \in R$, and $B_r \cap B_{r'} = \emptyset$ for every $r, r' \in R$ with $r \neq r'$.

The problem of finding a necessary and sufficient condition for a system of matroids to have a system of disjoint bases (known as the *packing problem*) has a long history. The first result, which motivated further development in this area, was a characterization of finite graphs having k edge-disjoint spanning trees proved independently by Tutte [12] and Nash-Williams [6]. Edmonds [4] generalized this result and characterized all finite matroids having k disjoint bases. The following characterization was proved by Brualdi [2] who generalized its finite version obtained by Edmonds and Fulkerson [5].

Theorem 1.1. *If $\mathfrak{M} = (\mathcal{M}_r)_{r\in R}$ is a finite system of rank-finite matroids or an arbitrary system of finite matroids on X, then $\mathfrak{M}$ has a system of disjoint bases if and only if*
($\star$) for every finite subset A of X we have

$$|A| \geq \sum_{r\in R} \rho(\mathcal{M}_r \cdot A),$$

where ρ is the rank function and $\mathcal{M}_r \cdot A$ is the contraction of $\mathcal{M}_r$ to A.

Unfortunately, condition ($\star$) is not sufficient for the existence of a system of disjoint bases for $\mathfrak{M}$ when $\mathfrak{M}$ is infinite and includes an infinite matroid, even when all the matroids in $\mathfrak{M}$ have rank at most one. For example, let $\mathfrak{M} = (\mathcal{M}_r)_{r\in R}$ be the system of matroid on

$$X = \{i \in \mathbb{Z} : i \geq 0\},$$

with

$$R = \{i \in \mathbb{Z} : i \geq 1\} \cup \{\infty\},$$

and

$$M_r = \{\{i\} : i < r\} \cup \{\emptyset\},$$

for every $r \in R$. Then $\mathfrak{M}$ is countable, contains matroids of rank one, and only one matroid in $\mathfrak{M}$ is infinite. Note that condition ($\star$) is satisfied, but there is no system of disjoint bases for $\mathfrak{M}$.

Oxley [9] formulated a sufficient (but not necessary) condition for a system of matroids to have a system of disjoint bases. Using an idea of Nash-Williams [7], Wojciechowski [13] formulated a condition which is a generalization of condition ($\star$) and is necessary for any system of matroids to have a system of disjoint bases. The condition is also sufficient in the case of countable systems of rank-finite matroids (see [14]).

In this paper we consider the following more general problem. A *society* is an ordered triple (M, W, K), where M, W are disjoint sets and $K \subseteq M \times W$. Given a society $\Gamma = (M, W, K)$ and a matroid $\mathcal{T}$ on M, a $\mathcal{T}$-*espousal* of Γ is an injective partial function $E : M \to W$ such that $E \subseteq K$ and $\operatorname{dom} E$ is spanning in $\mathcal{T}$. The society Γ is $\mathcal{T}$-*espousable* if there is a $\mathcal{T}$-espousal of Γ. The problem of characterizing systems of matroids with disjoint bases can be reduced to characterizing $\mathcal{T}$-espousable societies.

Let $\mathfrak{M} = (\mathcal{M}_r)_{r\in R}$ be a system of matroids on X. The *society generated by* $\mathfrak{M}$ is a society $\Gamma_{\mathfrak{M}} = (M_{\mathfrak{M}}, W_{\mathfrak{M}}, K_{\mathfrak{M}})$ with $M_{\mathfrak{M}} = R \times X$, $W_{\mathfrak{M}} = X$ and

$$K_{\mathfrak{M}} = \{((r, x), x) : r \in R \text{ and } x \in X\}.$$

Let $\mathcal{T}_{\mathfrak{M}}$ be the matroid on $M_{\mathfrak{M}}$ defined by

$$\mathcal{T}_{\mathfrak{M}} = \left\{ \bigcup_{r\in R} (\{r\} \times X_r) : X_r \in \mathcal{M}_r \text{ for every } r \in R \right\}.$$

It is easy to see that the system of matroids $\mathfrak{M}$ has a system of disjoint bases if and only if the society $\Gamma_{\mathfrak{M}}$ is $\mathcal{T}_{\mathfrak{M}}$-espousable. Indeed, if $\mathfrak{B} = (B_r)_{r\in R}$ is a system of disjoint bases for $\mathfrak{M}$, then

$$E = \bigcup_{r\in R} \{r\} \times B_r$$

is a $\mathcal{T}_{\mathfrak{M}}$-espousal in $\Gamma_{\mathfrak{M}}$. On the other hand, if E is a $\mathcal{T}_{\mathfrak{M}}$-espousal in $\Gamma_{\mathfrak{M}}$ and E' is the restriction of E to a base of $\mathcal{T}_{\mathfrak{M}}$, then the system $\mathfrak{B} = (B_r)_{r\in R}$ with

$$B_r = \{x : (r, x) \in \operatorname{dom} E'\},$$

for every $r \in R$, is a system of disjoint bases for $\mathfrak{M}$.

Given a society $\Gamma = (M, W, K)$, we say that Γ is *espousable* if it is $\mathcal{P}(M)$-espousable, where $\mathcal{P}(M)$ is the matroid on M consisting of all subsets of M. In the case of M being countable, four necessary and sufficient criterions have been given for the society Γ to be espousable. One, conjectured by Nash-Williams, was proved by Damerell and Milner [3]. Simpler criterions of a similar type were proved by Nash-Williams [7] (q-*admissibility*) and by Wojciechowski [15] (μ-*admissibility*). The fourth criterion (*c-admissibility*), of a different nature, was given by Podewski and Steffens [11]. The three types of admissibility criterions are actually all equivalent to each other (see [1] and [15]) and apply in the more general case when we allow M to be of any cardinality and assume only that $K^{-1}\langle x\rangle$ is countable for every $x \in W$ (see [8] and [15]).

In this paper, we generalize the criterion of μ-admissibility of a society $\Gamma = (M, W, K)$ to the case when $\mathcal{T}$ is an arbitrary matroid on M (we say then that the society Γ is *μ-admissible under* $\mathcal{T}$), and we prove that this criterion is necessary for Γ to be $\mathcal{T}$-espousable (Theorem 2.1). We also conjecture the sufficiency of our criterion in the case when $K^{-1}\langle x\rangle$ is countable for any $x \in W$ and the matroid $\mathcal{T}$ is rank-countable (Conjecture 2.2).

2. Preliminaries

We shall use the following set-theoretic conventions. A *relation* is a set of ordered pairs. Given a relation R, a set A, and an element a, we have $R\langle a\rangle = \{y : (a, y) \in R\}$; if $|R\langle a\rangle| = 1$, then $R(a)$ is the single element of $R\langle a\rangle$; the set $R[A]$ is equal to $\bigcup_{a\in A} R\langle a\rangle$; and $R^{-1} = \{(y, x) : (x, y) \in R\}$. The *domain* of the relation R is the set $\operatorname{dom} R = \{x : R\langle x\rangle \neq \emptyset\}$ and the *range* of R is the set $\operatorname{rge} R = \{y : R^{-1}\langle y\rangle \neq \emptyset\}$. A *function* is a relation f such that $|f\langle x\rangle| = 1$ for every $x \in \operatorname{dom} f$. A function f is *injective* if $\left|f^{-1}\langle y\rangle\right| = 1$ for every $y \in \operatorname{rge} f$. We say that $f : A \to X$ is a *partial function* if f is a function with $\operatorname{dom} f \subseteq A$ and $\operatorname{rge} f \subseteq X$. An *ordinal number* α is equal to the set of all ordinal numbers smaller than α and a *cardinal number* is an ordinal number α such that any ordinal number smaller than α has a smaller cardinality. We denote by ω the first infinite ordinal number and we call a set X *countable* if $|X| \leq \omega$.

Let X be a set and let $\mathcal{M}$ be a family of subsets of X. We say that $\mathcal{M}$ has *finite character* if a set A belongs to $\mathcal{M}$ if and only if every finite subset I of A belongs to $\mathcal{M}$. We say that $\mathcal{M}$ is a *matroid* on X if $\mathcal{M}$ is non-empty and satisfies the following conditions:

(i) If $A \in \mathcal{M}$ and $B \subseteq A$, then $B \in \mathcal{M}$.

(ii) If $I, J \in \mathcal{M}$ are finite and $|I| = |J| + 1$, then there is an element $y \in I \setminus J$ such that $J \cup \{y\} \in \mathcal{M}$.

(iii) $\mathcal{M}$ has finite character.

A matroid is *finite* if it is a finite family of sets or, equivalently, if it is a matroid on a finite set.

Let $\mathcal{M}$ be a matroid on a set X. A subset A of X is *independent* in $\mathcal{M}$ if $A \in \mathcal{M}$, and A is *dependent* if $A \notin \mathcal{M}$. A maximal independent set of $\mathcal{M}$ is called a *base* of $\mathcal{M}$. A subset A of X is *spanning* in $\mathcal{M}$ if it contains a base of $\mathcal{M}$. The cardinality of any base of $\mathcal{M}$ is called the *rank* of $\mathcal{M}$ and is denoted by $\rho(\mathcal{M})$. The matroid $\mathcal{M}$ is said to be *rank-finite* if $\mathcal{M}$ has finite rank, and to be *rank-countable* if it has countable rank ($\rho(\mathcal{M}) \leq \omega$). If $\mathcal{M}$ is a matroid on the set X and $A \subseteq X$, then let $\mathcal{M}|A$ be the *restriction* of $\mathcal{M}$ to A, that is, let $I \in \mathcal{M}|A$ iff $I \subseteq A$ and $I \in \mathcal{M}$. The restriction of $\mathcal{M}$ to $X \setminus A$ will be denoted by $\mathcal{M} - A$. Further, let $\mathcal{M} \cdot A$ be the *contraction* of $\mathcal{M}$ to A, that is, let $I \in \mathcal{M} \cdot A$ iff $I \subseteq A$ and $I \cup J \in \mathcal{M}$ for every $J \in \mathcal{M} - A$.

A *society* Γ is an ordered triple (M, W, K), where M, W are disjoint sets and $K \subseteq M \times W$. The elements of M are called *men*, the elements of W are called *women* and if $(a, x) \in K$, then we say that a *knows* x. Let $\Gamma = (M, W, K)$ be a society and $\mathcal{T}$ be a matroid on M. A $\mathcal{T}$-*espousal* of Γ is an injective partial function $E : M \to W$ such that $E \subseteq K$ and $\operatorname{dom} E$ is spanning in $\mathcal{T}$. The society Γ is $\mathcal{T}$-*espousable* if there is a $\mathcal{T}$-espousal of Γ. If $\mathcal{P}(M)$ is the collection of all subsets of M, then $\mathcal{P}(M)$ is clearly a matroid on M. We will call $\mathcal{P}(M)$ the *discrete matroid* on M. An *espousal* of Γ is a $\mathcal{P}(M)$-espousal of Γ and Γ is *espousable* if it is $\mathcal{P}(M)$-espousable. Throughout this paper we assume that we are discussing a fixed society $\Gamma = (M, W, K)$ with a fixed matroid $\mathcal{T}$ on M and the symbols Γ, M, W, K, $\mathcal{T}$ should be interpreted accordingly.

A *string* is an injective function with its domain being an ordinal. In particular, the empty set $\emptyset$ is a string with domain $0 = \emptyset$. A *string* in X is a string f with $\operatorname{rge} f \subseteq X$ and an α-*string* is a string g with $\operatorname{dom} g = \alpha$. Given a string f in X and

$Y \subseteq X$, let

$$\operatorname{rge}_Y f = \operatorname{rge} f \cap Y$$

and

$$\operatorname{dom}_Y f = \operatorname{dom} f \cap f^{-1}[Y].$$

A string in the society Γ is a string in $M \cup W$.

Let f be a string and β, γ be ordinals with $\beta \leq \gamma \leq \operatorname{dom} f$. The $[\beta, \gamma)$-*segment* $f_{[\beta,\gamma)}$ of f is the string defined by

$$f_{[\beta,\gamma)}(\theta) = f(\beta + \theta),$$

for all θ with $\beta + \theta < \gamma$, that is, $f_{[\beta,\gamma)}$ is obtained from f by restricting it to $[\beta, \gamma)$ and shifting the domain to start at 0. For $\alpha \leq \operatorname{dom} f$, let $f_\alpha = f_{[0,\alpha)}$.

Given a set $X \subseteq W$ of women, let the *demand set* $D(X)$ of X be the set of men that know no women outside X, that is, let

$$D(X) = \{a \in M : K\langle a \rangle \subseteq X\},$$

and if f is a string in Γ, then let

$$\Delta f = D(\operatorname{rge}_W f).$$

Let f be a λ-string in Γ and $\alpha \leq \lambda$. The set Δf_α will be called the *demand set* of f at α, and the set

$$\Delta^\alpha f = \Delta f_\alpha \setminus \operatorname{rge}_M f_{[\alpha,\lambda)}$$

will be called the *strong demand set* of f at α. Note that the demand set of f at α depends only on f_α but the strong demand set of f at α depends on the whole of f. The string f is *saturated at* α if the set of men appearing in f_α is a subset of Δf_α, and f is *regular at* α if the set of men appearing in f_α is independent in $\mathcal{T} \cdot \Delta^\alpha f$. Note that if f is regular at α, then f is also saturated at α. We say that f is *saturated* (*regular*) if it is saturated (regular) at every $\alpha \leq \lambda$.

Let $\mathbb{Z}^\infty = \mathbb{Z} \cup \{-\infty, \infty\}$ be the set of *quasi-integers*. If $a_1, \ldots, a_n \in \mathbb{Z}^\infty$, then let the sum $a_1 + \ldots + a_n$ be the usual sum if $a_1, \ldots, a_n$ are all integers, let the sum be ∞ if at least one of them is ∞, and let it be $-\infty$ if neither of $a_1, \ldots, a_n$ is ∞ but at least one of them is $-\infty$. Note that it follows immediately from the above definition that the operation of addition in $\mathbb{Z}^\infty$ is commutative and associative. The difference $a - b$ of two quasi-integers a, b means $a + (-b)$; and likewise, for example, $a - b + c - d$ means $a + (-b) + c + (-d)$, etc. Let $\mathbb{Z}^\infty$ be ordered in the obvious way. Note that if $a, b, c, d \in \mathbb{Z}^\infty$ satisfy $a \le c$ and $b \le d$, then $a + b \le c + d$. Given a set X, let $\|X\| \in \mathbb{Z}^\infty$ be the cardinality of X if X is finite, and $\|X\| = \infty$ if X is infinite.

Assume that f is a string in Γ. The *μ-margin* $\mu(f)$ of f is an element of $\mathbb{Z}^\infty$ defined by transfinite induction on $\alpha = \operatorname{dom} f$ as follows. Let $\mu(f) = 0$ if $\alpha = 0$, let

$$\mu(f) = \begin{cases} \mu(f_\beta) + 1 & \text{if } f(\beta) \in W, \\ \mu(f_\beta) - 1 & \text{if } f(\beta) \in M, \end{cases}$$

when $\alpha = \beta + 1$ is a successor ordinal, and

$$\mu(f) = \liminf_{\beta \to \alpha} \mu(f_\beta)$$

if α is a limit ordinal. We say that Γ is *μ-admissible under* $\mathcal{T}$ if $\mu(f) \ge 0$ for every regular string f in Γ.

The following result will be proved later.

Theorem 2.1. *If Γ is $\mathcal{T}$-espousable, then it is μ-admissible under $\mathcal{T}$.*

Moreover, we conjecture that the following result is true.

Conjecture 2.2. *If the set $K^{-1}\langle x \rangle$ of men knowing the woman x is countable for every $x \in W$ and the matroid $\mathcal{T}$ is rank-countable, then the society Γ is $\mathcal{T}$-espousable if and only if it is μ-admissible under $\mathcal{T}$.*

To present some motivation for our definition of μ-admissibility, we show in the following example that a certain natural strengthening of the notion of μ-admissibility leads to a condition that is not necessary for Γ to be $\mathcal{T}$-espousable. We say that a string f in Γ is *weakly regular* if the set $\operatorname{rge}_M f_\alpha$ is independent in $\mathcal{T} \cdot \Delta f_\alpha$ for every $\alpha \le \operatorname{dom} f$. We will show that the condition that $\mu(f) \ge 0$ for every weakly regular string f in Γ is not necessary for Γ to be $\mathcal{T}$-espousable.

Example 2.3. *Let G be the graph from Figure 2.1, and let*

$$E = \left\{ e_i^j : i = 1, 2, \dots , \quad j = 1, 2, 3, 4 \right\}$$

be the set of edges of G. Let $\mathcal{M}$ be the cycle matroid *of G, that is, a matroid on E such that $A \in \mathcal{M}$ if and only if A contains no cycle of G.*

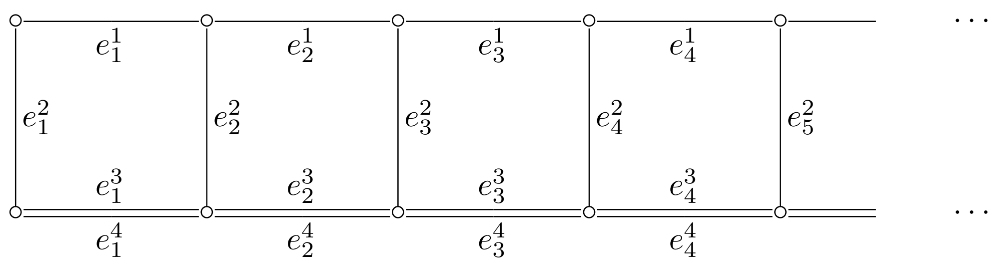

Fig. 2.1. The graph G.

Assume that $M = E \times \{1, 2\}$, $W = E$,

$$K = \{((a, b), a) : a \in E, \, b \in \{1, 2\}\},$$

and

$$\mathcal{T} = \{A \times \{1\} \cup B \times \{2\} : A, B \in \mathcal{M}\}.$$

We claim that if Γ and $\mathcal{T}$ are as in Example 2.3, then Γ is $\mathcal{T}$-espousable but there is a weekly regular string f in Γ with $\mu(f) < 0$. Indeed, let

$$E_1 = \left\{ e_i^3 : i = 1, 2, \dots \right\} \cup \left\{ e_{2i}^2 : i = 1, 2, \dots \right\} \cup \left\{ e_{2i-1}^1 : i = 1, 2, \dots \right\},$$

and

$$E_2 = \left\{ e_i^4 : i = 1, 2, \dots \right\} \cup \left\{ e_{2i-1}^2 : i = 1, 2, \dots \right\} \cup \left\{ e_{2i}^1 : i = 1, 2, \dots \right\}.$$

Then

$$F = \{((a, b), a) \in M \times W : (a, b) \in E_1 \times \{1\} \cup E_2 \times \{2\}\}$$

is a $\mathcal{T}$-espousal of Γ, so Γ is $\mathcal{T}$-espousable.

Consider the following $(\omega+1)$-string f in Γ. Let the sequence $f(0)$, $f(1), \ldots$ be equal to

$$e_1^1, e_1^2, \left(e_1^1, 1\right), \left(e_1^1, 2\right), e_1^3, e_1^4, \left(e_1^3, 1\right), \left(e_1^3, 2\right), e_2^1, e_2^2, \left(e_2^1, 1\right), \left(e_2^1, 2\right), \\ e_2^3, e_2^4, \left(e_2^3, 1\right), \left(e_2^3, 2\right), e_3^1, e_3^2, \left(e_3^1, 1\right), \left(e_3^1, 2\right), \ldots$$

and let $f(\omega) = \left(e_1^2, 1\right)$. Note that the string f is weakly regular (but not regular) and that $\mu(f) = -1$.

3. Necessity of μ-admissibility

In this section we will prove Theorem 2.1. It is well known that the following lemmas hold (see Oxley [10]).

Lemma 3.1. *If $\mathcal{M}$ is a matroid, the set I is independent in $\mathcal{M}$, the set P is spanning in $\mathcal{M}$, and $a \in I \setminus P$, then there is $b \in P \setminus I$ such that $(I \setminus \{a\}) \cup \{b\} \in \mathcal{M}$.*

Lemma 3.2. *If $\mathcal{M}$ is a matroid on X, the set A is a subset of X, and the set B is spanning in $\mathcal{M}$, then $A \cap B$ is spanning in $\mathcal{M} \cdot A$.*

Let f, g be strings in Γ. We say that g is a *men replacement* of f if $\mathrm{dom}_M f = \mathrm{dom}_M g$, $\mathrm{dom}_W f = \mathrm{dom}_W g$, and $f(\alpha) = g(\alpha)$ for every $\alpha \in \mathrm{dom}_W f$. Note that if g is a men replacement of f, then $\mu(f) = \mu(g)$. We will need the following lemma.

Lemma 3.3. *Let α, λ be ordinals with $\alpha < \lambda$, and let f and g be λ-strings in Γ such that g is a men replacement of f and*

$$f(\delta) = g(\delta) \text{ whenever } \alpha \leq \delta < \lambda.$$

If f is regular at every δ with $\alpha \leq \delta \leq \lambda$ and g is regular at α, then g is regular at every δ with $\alpha \leq \delta \leq \lambda$.

Proof. Let δ be such that $\alpha < \delta \leq \lambda$. We will show that g is regular at δ, that is, that $\mathrm{rge}_M\, g_\delta \in \mathcal{T} \cdot \Delta^\delta g$. Let $X = \Delta^\delta f = \Delta^\delta g$, $\mathcal{M} = \mathcal{T} \cdot X$, $Y = \Delta^\alpha f = \Delta^\alpha g$, $B = \mathrm{rge}_M\, f_\alpha$, $C = \mathrm{rge}_M\, g_\alpha$, and $D = \mathrm{rge}_M\, f_{[\alpha,\delta)} = \mathrm{rge}_M\, g_{[\alpha,\delta)}$. We have

$$B \cup D = \mathrm{rge}_M\, f_\delta \in \mathcal{M}$$

since f is regular at δ, implying that $D \in \mathcal{M}$. Since $D \cap Y = \emptyset$, it follows that $D \in \mathcal{M} - Y$. Since g is regular at α, we have $C \in \mathcal{M} \cdot Y$, implying that

$$C \cup D = \operatorname{rge}_M g_\delta \in \mathcal{M} = \mathcal{T} \cdot \Delta^\delta g.$$

Therefore g is regular at δ and the proof is complete.

□

The following lemma will provide the main step in the proof of Theorem 2.1.

Lemma 3.4. *If E is a $\mathcal{T}$-espousal of Γ and f is a regular string in Γ, then there is a saturated string g in Γ with $\operatorname{dom}_M g \subseteq \operatorname{dom} E$ that is a men replacement of f.*

Proof. Let $\lambda = \operatorname{dom} f$. We will show by transfinite induction on α that, for each $\alpha \leq \lambda$, there is a saturated string g^α in Γ such that

(i) g^α is a men replacement of f,

(ii) g^α is regular at every δ with $\alpha \leq \delta \leq \lambda$,

(iii) $\operatorname{dom}_M g^\alpha_\alpha \subseteq \operatorname{dom} E$ (where $g^\alpha_\alpha = (g^\alpha)_\alpha$),

(iv) $g^\alpha(\delta) = f(\delta)$ for every δ with $\alpha \leq \delta < \lambda$, and

(v) $g^\alpha(\delta) = g^\gamma(\delta)$ whenever $\delta < \gamma < \alpha$.

Taking $g = g^\lambda$ will then complete the proof.

Note that to prove that g^α satisfies (i)–(v) it is enough to prove that g^α satisfies (i), (iii)–(v) and

(ii') g^α is regular at α,

since (ii) is implied by the regularity of f, conditions (i), (ii') and (iv), and Lemma 3.3.

If $\alpha = 0$, then $g^0 = f$ obviously satisfies conditions (i)–(v). Assume that $\alpha = \beta + 1 \leq \lambda$ is a successor ordinal. If $g^\beta(\beta) \in W$ or $g^\beta(\beta) \in \operatorname{dom} E$, then it follows from the inductive hypothesis that if we take $g^\alpha = g^\beta$, then conditions (i)–(v) will be satisfied, so we can assume that $g^\beta(\beta) \in M \setminus \operatorname{dom} E$. Let $A = \Delta^\alpha g^\beta$. By condition (ii) of the inductive hypothesis, the string g^β is regular at α implying that the set $I = \operatorname{rge}_M g^\beta_\alpha$ is independent in $\mathcal{T} \cdot A$. Since $\operatorname{dom} E$ is spanning in $\mathcal{T}$, it follows from Lemma 3.2 that $P = \operatorname{dom} E \cap A$ is spanning in $\mathcal{T} \cdot A$. Therefore, by Lemma 3.1, there is

$$a \in P \setminus I = \left(\operatorname{dom} E \setminus \operatorname{rge}_M g^\beta\right) \cap A$$

such that $\left(I \setminus \{g^\beta(\beta)\}\right) \cup \{a\}$ is independent in $\mathcal{T} \cdot A$. Let g^α be defined by

$$g^\alpha(\gamma) = \begin{cases} a & \text{if } \gamma = \beta, \\ g^\beta(\gamma) & \text{otherwise,} \end{cases}$$

for every $\gamma < \lambda$. It is clear that g^α is a saturated string in Γ satisfying conditions (i) and (iii)–(v). Moreover, by the choice of a, the set $\operatorname{rge}_M g^\alpha_\alpha = \left(I \setminus \{g^\beta(\beta)\}\right) \cup \{a\}$ is independent in $\mathcal{T} \cdot A$ where $A = \Delta^\alpha g^\beta = \Delta^\alpha g^\alpha$ implying that the string g^α is regular at α. Therefore, g^α satisfies condition (ii').

Now assume that α is a limit ordinal. Then for every $\gamma < \alpha$ we have $\gamma + 1 < \alpha$. Let g^α be defined by

$$g^\alpha(\gamma) = \begin{cases} g^{\gamma+1}(\gamma) & \text{if } \gamma < \alpha, \\ f(\gamma) & \text{otherwise.} \end{cases}$$

It follows from (v) that $g^\alpha(\gamma) = g^\beta(\gamma)$ for every β with $\gamma < \beta < \alpha$ and that g^α is a string in Γ. It is clear that the string g^α is saturated and that conditions (i) and (iii)—(v) are satisfied. It remains to establish (ii'), that is, that $\operatorname{rge}_M g^\alpha_\alpha$ is independent in $\mathcal{T} \cdot \Delta^\alpha g^\alpha$.

To show that $\operatorname{rge}_M g^\alpha_\alpha$ is independent in $\mathcal{T} \cdot \Delta^\alpha g^\alpha$ it suffices to show that any finite subset $A \subseteq \operatorname{rge}_M g^\alpha_\alpha$ is independent in $\mathcal{T} \cdot \Delta^\alpha g^\alpha$. Let $A \subseteq \operatorname{rge}_M g^\alpha_\alpha$ be finite. Then there is $\beta < \alpha$ such that

$$A \subseteq \operatorname{rge}_M g^\alpha_\beta = \operatorname{rge}_M g^\beta_\beta \subseteq \operatorname{rge}_M g^\beta_\alpha.$$

Since g^β is regular at α, the set $\operatorname{rge}_M g^\beta_\alpha$ is independent in $\mathcal{T} \cdot \Delta^\alpha g^\beta = \mathcal{T} \cdot \Delta^\alpha g^\alpha$. Thus A is independent in $\mathcal{T} \cdot \Delta^\alpha g^\alpha$ and so the proof is complete. $\square$

The following lemma was proved in [15].

Lemma 3.5. *If the society Γ is espousable, then $\mu(g) \geq 0$ for every saturated string g in Γ.*

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let E be a $\mathcal{T}$-espousal of Γ and let Γ' be the society $\Gamma' = (\operatorname{dom} E, W, K')$ with $K' = K \cap (\operatorname{dom} E \times W)$. Let f be a regular string in Γ. We need to show that $\mu(f) \geq 0$. By Lemma 3.4, there is a saturated string g in Γ with $\operatorname{dom}_M g \subseteq \operatorname{dom} E$ that is a men replacement of f. Then $\mu(g) = \mu(f)$ and g is a Γ'-saturated string in Γ'. Since E is an espousal of Γ', it follows from Lemma 3.5 that $\mu(g) \geq 0$. Therefore $\mu(f) \geq 0$ and so the proof is complete. □

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