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Disjoint Bases for a Countable Family of Rank-Finite Matroids

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Abstract. Let $\mathcal{M} = (M_r)_{r\in R}$ be a system of matroids on a set S. For every transfinite sequence f of distinct elements of S, we define a number $\eta(f)$. In [12] we proved that the condition that $\eta(f) \geq 0$ for every possible choice of f is necessary for $\mathcal{M}$ to have a system of mutually disjoint bases. Further, we showed that this condition is sufficient if R is countable and M_r is a rank-finite transversal matroid for every $r \in R$. In this paper, we prove that our condition is also sufficient in the much more general case of countable systems of arbitrary rank-finite matroids.

1. Introduction.

Let S be a set and let M be a family of subsets of S. We say that M has *finite character* if a set A belongs to M if and only if every finite subset I of A belongs to M. We say that M is a *matroid* on S if M is non-empty and satisfies the following conditions:

(1.1) If $A \in M$ and $B \subseteq A$, then $B \in M$.

(1.2) If $I, J \in M$ are finite and $|I| = |J| + 1$, then there is an element $y \in I \smallsetminus J$ such that $J \cup \{y\} \in M$.

(1.3) M has finite character.

Obviously, if M is a matroid on S and $S \subseteq S'$, then M is a matroid on S'. A matroid is *finite* if it is a finite family of sets or, equivalently, if it is a matroid on a finite set. A maximal element of M is called a *base* of M, and the cardinality $\rho(M)$ of any base of M is called the *rank* of M. The matroid M is said to be *rank-finite* if M has finite rank, and to be *rank-countable* if it has countable rank ($\rho(M) \leq \aleph_0$).

Let R and S be disjoint sets and, for each $r \in R$, let M_r be a matroid on the set S. The system $\mathcal{M} = (M_r)_{r \in R}$ will be called a *system of matroids* on S. $\mathcal{M}$ will be said to be *countable* if R is countable ($|R| \leq \aleph_0$), and to be *finite* if R is finite. The system $\mathcal{B} = (B_r)_{r \in R}$ of subsets of S will be called a *system of disjoint bases* for $\mathcal{M}$ if B_r is a base of M_r, $r \in R$, and $B_r \cap B_{r'} = \varnothing$ for every $r, r' \in R$ such that $r \neq r'$.

If M is a matroid on the set S and $A \subseteq S$, then let $M|A$ be the *restriction* of M to A, *i.e.* let $I \in M|A$ iff $I \subseteq A$ and $I \in M$. Further, let $M.A$ be the *contraction* of M to A, *i.e.* let $I \in M.A$ iff $I \subseteq A$ and $I \cup J \in M$ for every $J \in M|(S \smallsetminus A)$.

The problem of finding a necessary and sufficient condition for a system of matroids to have a system of disjoint bases (known as the packing problem) has a long history. The first result, which motivated further development in this area, was a characterization of finite graphs having k edge-disjoint spanning trees proved independently by Tutte [10] and Nash-Williams [6]. Edmonds [3] generalized this result and characterized all finite matroids having k disjoint bases. Edmonds and Fulkerson [4] proved the following basic packing theorem for finite systems of finite matroids.

Theorem 1.1. *If $\mathcal{M}$ is a finite system of finite matroids, then $\mathcal{M}$ has a system of disjoint bases iff for every $A \subseteq S$ we have*

$$|A| \geq \sum_{r \in R} \rho(M_r . A).$$

□

Brualdi [2] used Rado's selection principle to generalize Theorem 1.1. He proved the following theorem.

Theorem 1.2. *If $\mathcal{M}$ is a finite system of rank-finite matroids or an arbitrary system of finite matroids, then $\mathcal{M}$ has a system of disjoint bases iff*

(1.4) *for every finite subset A of S we have*

$$|A| \geq \sum_{r \in R} \rho(M_r . A).$$

□

Unfortunately, condition (1.4) is not sufficient for the existence of a system of disjoint bases of $\mathcal{M}$ when $\mathcal{M}$ is infinite and includes an infinite matroid, even when all the matroids in $\mathcal{M}$ have rank at most one.

Example 1.3. *Let $R = \{i \in \mathbb{Z} : i \geq 1\} \cup \{\infty\}$, $S = \{i \in \mathbb{Z} : i \geq 0\}$, and let $\mathcal{M} = (M_r)_{r \in R}$ be the system of matroids on S such that*

$$M_r = \{\{i\} : i < r\} \cup \{\varnothing\},$$

for every $r \in R$.

In the above example, $\mathcal{M}$ is countable, contains matroids of rank one, and only one matroid in $\mathcal{M}$ is infinite. Clearly condition (1.4) is satisfied, but there is no system of disjoint bases for $\mathcal{M}$.

Oxley [8] formulated a sufficient (but not necessary) condition for a system of matroids to have a system of disjoint bases.

In [12], we formulated a condition which is a generalization of condition (1.4) and is necessary for any system of matroids to have a system of disjoint bases (see Theorem 1.4). Our condition is an adaptation of the necessary and sufficient condition for a countable family of sets to have a transversal that was proved by Nash-Williams [7].

Given a family $\mathcal{A} = (A_r)_{r \in R}$ of nonempty subsets of S, we can form the corresponding family of matroids $\mathcal{M}_{\mathcal{A}} = (M_r)_{r \in R}$ on S by taking

$$M_r = \{\{a\} : a \in A_r\} \cup \{\varnothing\},$$

for every $r \in R$. Since any transversal of $\mathcal{A}$ corresponds to a system of disjoint bases of $\mathcal{M}_{\mathcal{A}}$, the theorem of Nash-Williams [7] can be interpreted as a result about the existence of a system of disjoint bases for a countable system of matroids of rank at most one (see Theorem 1.7).

Before we can formulate our condition, we need to introduce some more terminology. Let $\mathbb{Z}^* = \mathbb{Z} \cup \{\infty, -\infty\}$ be the set of *quasi-integers*. The arithmetic and inequalities on $\mathbb{Z}^*$ follow the obvious rules with the additional rule that $\infty - \infty = \infty$ (see [7] for details). If A is a set, then the *size* $\|A\|$ of A is the cardinality $|A|$ if A is finite, and $\|A\| = \infty$ if A is infinite.

Assume that ordinals are defined in such a way that an *ordinal* α is the set of all ordinals less than α. A *transfinite sequence* is a function with its domain being an ordinal.

Let us assume now that $\mathcal{M} = (M_r)_{r \in R}$ is a fixed system of matroids on S. If $r \in R$, $A_1, A_2 \in M_r$, $A_1 \subseteq A_2$, B_1 is a base of $M_r . A_1$, and B_2 is a base of $M_r . A_2$ containing B_1, then let

$$\gamma_r(A_1, A_2) = \|B_2 \smallsetminus B_1\|.$$

The function γ_r is well-defined, *i.e.* the value of $\gamma_r(A_1, A_2)$ does not depend on the choice of B_1 and B_2 (see [12]).

Suppose that $r \in R$ and that $\mathcal{A} = (A_\alpha)_{\alpha<\lambda}$, $\mathcal{B} = (B_\alpha)_{\alpha<\lambda}$ are transfinite sequences of subsets of S such that

(1.5) $A_\delta \subseteq A_\theta$ and $B_\delta \subseteq B_\theta$, for $\delta \leq \theta < \lambda$,

(1.6) B_δ is a base of $M_r \centerdot A_\delta$, for $\delta < \lambda$,

and suppose that B is a base of $M_r \centerdot (\bigcup_{\alpha<\lambda} A_\alpha)$ such that

$$\bigcup_{\alpha<\lambda} B_\alpha \subseteq B.$$

Then we say that the pair $(\mathcal{B}, B)$ is a *proper* M_r*-choice of bases for* $\mathcal{A}$.

If $\mathcal{A} = (A_\alpha)_{\alpha<\lambda}$ is a transfinite sequence of subsets of S and λ is a limit ordinal, then let

$$\Gamma_r(\mathcal{A}) = \min\left\{ \left\| B \setminus \bigcup_{\alpha<\lambda} B_\alpha \right\| : ((B_\alpha)_{\alpha<\lambda}, B) \text{ is a proper } M_r\text{-choice of bases for } \mathcal{A} \right\}.$$

By a *queue* we mean a countable injective transfinite sequence, *i.e.* an injective transfinite sequence whose domain is a countable ordinal. If f is a queue, then $\operatorname{dom} f$ is the domain of f and $\operatorname{rge} f$ is the range of f. If $\operatorname{dom} f = \lambda$ and $\alpha \leq \lambda$, then f_α will denote the restriction of f to α. A *queue in* S is a queue whose range is a subset of S, and a λ*-queue* is a queue with domain λ.

With a λ-queue f in S we associate a quasi-integer $\eta(f)$, called the $\mathcal{M}$*-margin* of f. Let $\eta(f) = 0$ if $\lambda = 0$. Suppose now that $\lambda > 0$ and that $q(f')$ has been defined for every queue f' in S such that $\operatorname{dom} f' < \lambda$. If $\lambda = \theta + 1$ is a successor ordinal, then let

$$\eta(f) = \eta(f_\theta) + 1 - \sum_{r \in R} \gamma_r(\operatorname{rge} f_\theta, \operatorname{rge} f).$$

If λ is a limit ordinal, then let

$$\eta(f) = \liminf_{\theta \to \lambda} \eta(f_\theta) - \sum_{r \in R} \Gamma_r((\operatorname{rge} f_\alpha)_{\alpha<\lambda}).$$

We say that the system $\mathcal{M}$ is ***good*** if $\eta(f) \geq 0$ for every queue f in S.

Note that condition (1.4) is equivalent to saying that $\eta(f) \geq 0$ for every finite queue f in S. Also note that in Example 1.3, we have $\eta(f) = -1$, where f is the ω-queue given by $f(i) = i$. In general, the following theorem was proved in [12].

Theorem 1.4. *If $\mathcal{M}$ has a system of disjoint bases, then $\mathcal{M}$ is good.* □

The proof of Theorem 1.4 is relatively easy. A much more interesting problem is when the converse of it holds. It is not hard to see that the exact converse of Theorem 1.4 is false. Consider the following example of an uncountable family of matroids.

Example 1.5. *Let $R = \omega_1$ be the first uncountable ordinal, $S = \omega$ be the first infinite ordinal, and let $\mathcal{M} = (M_r)_{r \in R}$ be the system of matroids on S such that*

$$M_r = \{\{i\} : i \in S\} \cup \{\varnothing\},$$

for every $r \in R$.

It can be easily seen that in the above example $\eta(f)$ is equal to $\|\operatorname{rge} f\|$ for any finite queue, hence $\eta(f) = \infty$ for any infinite queue. Therefore $\mathcal{M}$ is good but, obviously, there is no system of disjoint bases for $\mathcal{M}$.

The following partial converse of Theorem 1.4 is a corollary of Theorem 1.2 obtained by using the equivalence between condition (1.4) and the condition that $\eta(f) \geq 0$ for every finite queue f in S.

Corollary 1.6. *Let $\mathcal{M}$ be a finite system of rank-finite matroids or an arbitrary system of finite matroids. If $\mathcal{M}$ is good, then $\mathcal{M}$ has a system of disjoint bases.* □

The first partial converse of Theorem 1.4 not restricted by the assumption that either $\mathcal{M}$ or all its components are finite is the following theorem proved implicitly by

Nash-Williams [7].

Theorem 1.7. *If $\mathcal{M}$ is a good countable system of matroids of rank at most one, then $\mathcal{M}$ has a system of disjoint bases.* $\square$

Using Theorem 1.7, we proved in [12] the following theorem.

Theorem 1.8. *If $\mathcal{M}$ is a good countable system of rank-finite transversal matroids on S, then $\mathcal{M}$ has a system of disjoint bases.* $\square$

Since any matroid of rank at most one is a transversal matroid, Theorem 1.8 is a generalization of Theorem 1.7. In general, the assumption that a given matroid is a transversal matroid is quite restrictive. In this paper we show that it can be removed. We shall prove the following theorem.

Theorem 1.9. *If $\mathcal{M}$ is a good countable system of rank-finite matroids on S, then $\mathcal{M}$ has a system of disjoint bases.*

Our proof of Theorem 1.9 (broken down into several lemmas and completed in section 6) is an adaptation of the technique used by Nash-Williams [7]. We strongly believe that a further refinement of this method is possible, so that it can be used to prove the following general conjecture, formulated first in [12].

Conjecture 1.10. *If $\mathcal{M}$ is a good countable system of rank-countable matroids on S, then $\mathcal{M}$ has a system of disjoint bases.*

Furthermore, settling Conjecture 1.10 should allow us to characterize countable graphs with k edge-disjoint spanning trees, and to prove a result about detachments of countable graphs. (See [12] for details.)

2. Preliminaries.

From now on, we assume that all the matroids considered are rank-finite. Let $\{S_1, S_2, \ldots, S_k\}$ be a finite partition of the set S. If M_i is a matroid on S_i, $i = 1, 2, \ldots, k$, then the *direct sum* $M_1 \oplus M_2 \oplus \ldots \oplus M_k$ of M_1, M_2, $\ldots$, M_k is the family M_0 of subsets of S defined by

$$M_0 = \left\{ \bigcup_{i=1}^{k} I_i : I_i \in M_i, i = 1, 2, \ldots, k \right\}.$$

It is easy to verify that M_0 is a matroid on S, and that $B \subseteq S$ is a base of M_0 iff

$$B = \bigcup_{i=1}^{k} B_i,$$

for some bases B_1, B_2, $\ldots$ B_k of M_1, M_2, $\ldots$, M_k, respectively. Therefore

$$\rho(M_0) = \sum_{i=1}^{k} \rho(M_i).$$

If, moreover, M is a matroid on S such that $M_0 \subseteq M$ and

$$\rho(M) = \rho(M_0) = \sum_{i=1}^{k} \rho(M_i),$$

then we say that $(M_1, M_2, \ldots, M_k)$ is a *decomposition of* M.

Let M be a matroid on S. If $X \subseteq S$, then let

$$M \backslash X = M|(S \setminus X).$$

Lemma 2.1. *Let $X \subseteq S$ and B be any base of $M \backslash X$. Then $I \in M.X$ iff $I \subseteq X$ and $I \cup B \in M$.*

Proof. See [1] Lemma 2.2. □

Immediately from Lemma 2.1, we get the following corollary.

Corollary 2.2. *If $X \subseteq S$, then $(M\backslash X, M\mathbf{.}X)$ is a decomposition of M.* $\square$

Lemma 2.3. *If $X \subseteq Y \subseteq S$, then*

$$(M\backslash X)\mathbf{.}Y = (M\mathbf{.}Y)\backslash X.$$

Proof. Let B be a base of $M\backslash Y = (M\backslash X)\backslash Y$. Assume that $I \in (M\backslash X)\mathbf{.}Y$. By Lemma 2.1, we have $I \subseteq Y$ and $I \cup B \in M\backslash X$. Thus $I \subseteq Y \setminus X$ and $I \cup B \in M$. Therefore $I \in (M\mathbf{.}Y)\backslash X$.

Now assume that $I \in (M\mathbf{.}Y)\backslash X$. Then $I \in M\mathbf{.}Y$ and $I \subseteq S \setminus X$. Thus $I \subseteq Y \setminus X$ and $I \cup B \in M$. Therefore $I \cup B \in M\backslash X$, and so $I \in (M\backslash X)\mathbf{.}Y$. $\square$

If $u \in S$, then let $M - u = M\backslash\{u\}$ and $M \sim u = M\mathbf{.}(S \setminus \{u\})$. If f is a queue in S, then let $M\backslash f = M\backslash \operatorname{rge} f$ and $M\mathbf{.}f = M\mathbf{.}\operatorname{rge} f$.

Let $\mathcal{F}$ be the set of all queues in S and let $\overline{\mathcal{F}}$ be the subset of $\mathcal{F}$ such that $f \in \overline{\mathcal{F}}$ if the domain of f is a limit ordinal. Suppose $\zeta : \mathcal{F} \to \mathbb{Z}^*$ and that the limit $\lim_{\theta\to\lambda} \zeta(f_\theta)$ exists for every limit ordinal λ and every λ-queue f. Then the function $\overline{\zeta} : \overline{\mathcal{F}} \to \mathbb{Z}^*$ defined by

$$\overline{\zeta}(f) = \lim_{\theta\to\lambda} \zeta(f_\theta),$$

where $\lambda = \operatorname{dom} f$, will be called the *limit* of ζ.

Define the M-*reserve* μ to be the function from $\mathcal{F}$ to $\mathbb{Z}$ such that $\mu(f) = \rho(M\backslash f)$. Note that if f is a λ-queue in S and $\theta \le \gamma \le \lambda$, then $\mu(f_\theta) \ge \mu(f_\gamma)$. Therefore, the limit $\overline{\mu}$ of μ exists, and

$$\overline{\mu}(f) = \min\{\mu(f_\theta) : \theta < \operatorname{dom} f\},$$

for any $f \in \overline{\mathcal{F}}$.

If $f_1, f_2, \ldots, f_k$ are queues with mutually disjoint ranges, and if $\operatorname{dom} f_i = \lambda_i$, then let $f_1 * \ldots * f_k$ be the queue g such that $\operatorname{dom} g = \lambda_1 + \ldots + \lambda_k$, and

$$g(\lambda_1 + \ldots + \lambda_{i-1} + \theta) = f_i(\theta), \qquad i = 1, 2, \ldots, k, \quad \theta < \lambda_i.$$

Informally, the queue $f_1 * \ldots * f_k$ is the juxtaposition of the queues $f_1, f_2, \ldots, f_k$.

Let the *empty queue* ε be the 0-queue, and if $u \in S$, then let $[u]$ be the 1-queue f such that $f(0) = u$.

Lemma 2.4. *Let $u \in S$, h be a λ-queue in $S \smallsetminus \{u\}$, and $g = [u] * h$. If $\theta \leq \lambda$, $h' = h_\theta$, and $g' = [u] * h'$, then*

$$\mu(g') - \mu(g) \geq \mu(h') - \mu(h).$$

Proof. Let B be a base of $M \backslash g$, let D be a base of $M \backslash h$ such that $B \subseteq D$, and let D' be a base of $M \backslash h'$ such that $D \subseteq D'$. If $u \notin D$, then $D \cup \{u\} \notin M$ and so $u \notin D'$. Therefore $u \notin D' \smallsetminus D$, and hence $B \cup (D' \smallsetminus D) \subseteq S \smallsetminus \operatorname{rge} g'$. Since obviously $B \cup (D' \smallsetminus D) \subseteq D' \in M$, we conclude that

$$B \cup (D' \smallsetminus D) \in M \backslash g'.$$

Let B' be a base of $M \backslash g'$ such that $B \cup (D' \smallsetminus D) \subseteq B'$. Since $B \cap (D' \smallsetminus D) = \varnothing$, we have $D' \smallsetminus D \subseteq B' \smallsetminus B$. Therefore

$$\mu(g') - \mu(g) = \|B' \smallsetminus B\| \geq \|D' \smallsetminus D\| = \mu(h') - \mu(h),$$

and the proof is complete. □

Lemma 2.5. *Let $u \in S$, $M' = M \sim u$, and let μ' be the M'-reserve. If $\{u\} \in M$, and f is a queue in $S \smallsetminus \{u\}$, then*

$$\mu'(f) = \mu(f) - 1.$$

Proof. Let B' be a base of $M' \backslash f$. Then $B' \in M'$ and since $\{u\} \in M$, we conclude that $B' \cup \{u\} \in M$, and so there is a base B of $M \backslash f$ such that $B' \cup \{u\} \subseteq B$. Clearly, $B = B' \cup \{u\}$ and $u \notin B'$. Therefore

$$\mu'(f) = \|B'\| = \|B\| - 1 = \mu(f) - 1,$$

and the proof is complete. □

Let ν be the function from $\mathcal{F}$ to $\mathbb{Z}$ such that $\nu(\varepsilon) = 0$ and if f is a nonempty queue in S with $u = f(0)$, then let

$$\nu(f) = \begin{cases} 0 & \text{if } \{u\} \in M \smallsetminus M_{\bullet}f, \\ 1 & \text{otherwise.} \end{cases}$$

The function ν will be called the M-*character*. Note that if f is a λ-queue in S and $\theta \leq \gamma \leq \lambda$, then $\nu(f_\theta) \leq \nu(f_\gamma)$. Therefore the limit $\overline{\nu}$ of ν exists, and

$$\overline{\nu}(f) = \max\{\nu(f_\theta) : \theta < \operatorname{dom} f\},$$

for any $f \in \overline{\mathcal{F}}$.

Lemma 2.6. *Let $u \in S$, $M' = M \sim u$, and let μ' be the M'-reserve. If f is a nonempty queue in S such that $u = f(0)$, then*

$$\mu'(f) = \mu(f) + \nu(f) - 1.$$

Proof. Assume first that $\nu(f) = 1$. If $\{u\} \notin M$, then $M' = M$ and so $\mu' = \mu$. If $\{u\} \in M_{\bullet}f$ and if B is a base of $M \backslash f$, then $B \cup \{u\} \in M$ and so $B \in M'$. Thus B is a base of $M' \backslash f$, and hence $\mu'(f) = \mu(f)$.

Now assume that $\nu(f) = 0$. Let B be a base of $M \backslash f$. Since $\{u\} \notin M_{\bullet}f$, we have $B \cup \{u\} \notin M$. Let B' be a base of $M' \backslash f$ such that $B' \subseteq B$. Then $B' \in M'$ and since $\{u\} \in M$, we conclude that $B' \cup \{u\} \in M$. Therefore B' is a proper subset of B. Clearly $\|B'\| = \|B\| - 1$, so $\mu'(f) = \mu(f) - 1$, and the proof is complete. □

We say that an element u of S is M-*essential* if there is a countable set $X \subseteq S$ such that $\{u\} \in M_{\bullet}X$.

Lemma 2.7. *Let $u \in S$, $M'' = M - u$, and let μ'' be the M''-reserve. If u is not M-essential, and f is a queue in $S \smallsetminus \{u\}$, then*

$$\mu''(f) = \mu(f).$$

Proof. Let $X = \operatorname{rge} f \cup \{u\}$. We have

$$\mu''(f) = \rho(M''\backslash f) = \rho(M\backslash X).$$

Let B be a base of $M\backslash X$. Since u is not M-essential and X is countable, we have

$$\{u\} \notin M\mathbf{.}X.$$

Therefore $B \cup \{u\} \notin M$ and so B is a base of $M\backslash f$. It follows that $\mu''(f) = \mu(f)$, and the proof is complete. □

Lemma 2.8. *The set of M-essential elements in S is countable.*

Proof. We shall use induction on the rank of M. If $\rho(M) = 0$, then the set of M-essential elements in S is empty hence countable.

Suppose that $\rho(M) > 0$ and that u is an M-essential element in S. Let X be a countable subset of S such that $\{u\} \in M\mathbf{.}X$, and let $M_0 = M\backslash X$. Then $\rho(M_0) < \rho(M)$ and so, by the inductive hypothesis, there are only countably many M_0-essential elements.

We claim that if $y \in S \setminus X$ is M-essential, then y is also M_0-essential. Indeed, let y be an M-essential element in $S \setminus X$ and let $Y \subseteq S$ be a countable set such that $\{y\} \in M\mathbf{.}Y$. Let B be a base of $M_0\backslash Y = M\backslash(X \cup Y)$ and let B' be a base of $M\backslash Y$ such that $B \subseteq B'$. Since $\{y\} \in M\mathbf{.}Y$, we have $B' \cup \{y\} \in M$. Therefore $B \cup \{y\} \in M$ and so $B \cup \{y\} \in M_0$ since $B \cup \{y\} \subseteq S \setminus X$. Thus $\{y\} \in M_0\mathbf{.}Y$, and the claim is proved.

From our claim it follows that there are only countably many M-essential elements in $S \setminus X$. Since X is countable, there are only countably many M-essential elements in S, and the proof is complete. □

Lemma 2.9. *Let $X \subseteq S$ and μ^X be the $M\mathbf{.}X$-reserve. If f is a λ-queue in X, and $\theta \leq \lambda$, then*

$$\mu(f_\theta) - \mu(f) = \mu^X(f_\theta) - \mu^X(f).$$

Proof. Let D be a base of $M\backslash X = (M\backslash f)\backslash X$. Let B be a base of $(M\textbf{.}X)\backslash f$, and B' be a base of $(M\textbf{.}X)\backslash f_\theta$ such that $B \subseteq B'$. By Lemma 2.3, we have $(M\textbf{.}X)\backslash f = (M\backslash f)\textbf{.}X$ and $(M\textbf{.}X)\backslash f_\theta = (M\backslash f_\theta)\textbf{.}X$. By Lemma 2.1, the set $B \cup D$ is a base of $M\backslash f$ and $B' \cup D$ is a base of $M\backslash f_\theta$. Therefore

$$\mu(f_\theta) - \mu(f) = \|B' \smallsetminus B\| = \mu^X(f_\theta) - \mu^X(f),$$

and the proof is complete. □

Lemma 2.10. *If g is a queue in S, f is a queue in $S \smallsetminus \operatorname{rge} g$, and $\hat{\mu}$ is the $M\backslash g$-reserve, then*

$$\mu(g * f) = \hat{\mu}(f).$$

Proof. We have

$$\begin{aligned} \hat{\mu}(f) &= \rho((M\backslash g)\backslash f) \\ &= \rho(M\backslash g * f) \\ &= \mu(g * f), \end{aligned}$$

and the proof is complete. □

3. Weighted matroids.

For technical reasons we need to generalize the notion of a rank-finite matroid. We define a *weighted matroid* on S to be a pair $N = (M, m)$, where M is a rank-finite matroid on S and m is an integer such that $m \geq \rho(M)$. If $m = \rho(M)$, then N will be called *normal*. If M is a rank-finite matroid on S, then $\overline{M}$ will denote the normal weighted matroid $(M, \rho(M))$.

Let $N = (M, m)$ be a weighted matroid on S and let $X \subseteq S$. Set

$$\zeta(N, X) = m - \rho(M \backslash X).$$

Note that if N is normal, then $\zeta(N, X) = \rho(M_{\bullet}X)$.

Let

$$N \backslash X = \overline{M \backslash X},$$

and

$$N_{\bullet}X = (M_{\bullet}X, \zeta(N, X)).$$

Clearly, $N \backslash X$ is a weighted matroid on $S \setminus X$, and $N_{\bullet}X$ is a weighted matroid on X.

Assume now that the set R is countable and that, for each $r \in R$, $N_r = (M_r, m_r)$ is a weighted matroid on S. The system $\mathcal{N} = (N_r)_{r \in R}$ will be called a *system of weighted matroids* on S. We say that $\mathcal{N}$ is *normal* if N_r is normal for every $r \in R$.

If $X \subseteq S$, then let $\mathcal{N}_{\bullet}X = (N_{r\bullet}X)_{r \in R}$ and $\mathcal{N} \backslash X = (N_r \backslash X)_{r \in R}$, and if f is a queue in S, then let $\mathcal{N}_{\bullet}f = \mathcal{N}_{\bullet}\mathrm{rge}\, f$ and $\mathcal{N} \backslash f = \mathcal{N} \backslash \mathrm{rge}\, f$.

Let μ_r be the M_r-reserve and $\overline{\mu_r}$ be the limit of μ_r, $r \in R$. Define the $\mathcal{N}$-*margin* $\xi(f)$ of f in the following way:

(3.1) $\xi(f) = \sum_{r \in R}(\rho(M_r) - m_r)$ if $\lambda = 0$,

(3.2) $\xi(f) = \xi(f_\theta) + 1 - \sum_{r \in R}(\mu_r(f_\theta) - \mu_r(f))$ if $\lambda = \theta + 1$,

(3.3) $\xi(f) = \liminf_{\theta \to \lambda} \xi(f_\theta) - \sum_{r \in R}(\overline{\mu_r}(f) - \mu_r(f))$ if λ is a limit ordinal.

We say that $\mathcal{N}$ is *good* if $\xi(f) \geq 0$ for every queue f in S.

Let $\mathcal{M} = (M_r)_{r \in R}$ be the system of matroids corresponding to the system $\mathcal{N}$, and let η be the $\mathcal{M}$-margin.

Lemma 3.1. *If $\mathcal{N}$ is normal, then*

$$\eta(f) = \xi(f),$$

for every queue f in S.

Proof. Let $\lambda = \operatorname{dom} f$. We shall use transfinite induction on λ. If $\lambda = 0$, then since $\mathcal{N}$ is normal, we have $\eta(f) = \xi(f) = 0$.

If $\lambda = \theta + 1$ for some ordinal θ, then

$$\eta(f) = \eta(f_\theta) + 1 - \sum_{r \in R} \gamma_r(\operatorname{rge} f_\theta, \operatorname{rge} f),$$

and

$$\xi(f) = \xi(f_\theta) + 1 - \sum_{r \in R} (\mu_r(f_\theta) - \mu_r(f)).$$

We claim that

$$\gamma_r(\operatorname{rge} f_\theta, \operatorname{rge} f) = \mu_r(f_\theta) - \mu_r(f), \tag{3.4}$$

for every $r \in R$. Indeed, let $r \in R$, B_1 be a base of $M_r . f_\theta$, and B_2 be a base of $M_r . f$ containing B_1. Further, let D_1 be a base of $M_r \backslash f_\theta$ and D_2 be a base of $M_r \backslash f$ such that $D_2 \subseteq D_1$. Then $B_1 \cup D_1$ and $B_2 \cup D_2$ are bases of M_r and hence

$$\gamma_r(\operatorname{rge} f_\theta, \operatorname{rge} f) = \|B_2 \smallsetminus B_1\| = \|D_1 \smallsetminus D_2\| = \mu_r(f_\theta) - \mu_r(f).$$

Therefore our claim is proved. From (3.4) and the inductive hypothesis it follows that $\eta(f) = \xi(f)$.

Now assume that λ is a limit ordinal. Let $\mathcal{A} = (\operatorname{rge} f_\alpha)_{\alpha < \lambda}$. Then

$$\eta(f) = \liminf_{\theta \to \lambda} \eta(f_\theta) - \sum_{r \in R} \Gamma_r(\mathcal{A}),$$

and

$$\xi(f) = \liminf_{\theta \to \lambda} \xi(f_\theta) - \sum_{r \in R} (\overline{\mu_r}(f) - \mu_r(f)).$$

We claim that

$$\Gamma_r(\mathcal{A}) = \overline{\mu_r}(f) - \mu_r(f), \tag{3.5}$$

for every $r \in R$. Indeed, let $r \in R$, and $(\mathcal{B}, B)$ be a proper M_r-choice of bases for $\mathcal{A}$, where $\mathcal{B} = (B_\alpha)_{\alpha<\lambda}$. Since M_r is a rank-finite matroid, there is an ordinal $\delta < \lambda$ such that $B_\alpha = B_\delta$ for every α satisfying $\delta \leq \alpha < \lambda$. Then $\Gamma_r(\mathcal{A}) = \|B \setminus B_\delta\|$. Let D be a base of $M_r \backslash f$ and let D_δ be the base of $M_r \backslash f_\delta$ such that $D_\delta \subseteq D$. Clearly, we have $\|D_\delta\| = \overline{\mu_r}(f)$. Since $B \cup D$ and $B_\delta \cup D_\delta$ are bases of M_r, we get

$$\Gamma_r(\mathcal{A}) = \|B \setminus B_\delta\| = \|D_\delta \setminus D\| = \overline{\mu_r}(f) - \mu_r(f),$$

and so our claim is proved. From (3.5) and the inductive hypothesis it follows that $\eta(f) = \xi(f)$ so the proof is complete. □

The following corollary follows immediately from Lemma 3.1.

Corollary 3.2. *If $\mathcal{N}$ is normal, then $\mathcal{N}$ is good if and only if $\mathcal{M}$ is good.* □

The following lemmas will be needed later.

Lemma 3.3. *If $u \in S$, then*

$$\xi([u]) = \sum_{r \in R} (\mu_r([u]) - m_r) + 1.$$

Proof. Since $m_r - \rho(M_r) \geq 0$ and $\rho(M_r) - \mu_r([u]) \geq 0$ for every $r \in R$, we have

$$\sum_{r \in R} (m_r - \rho(M_r)) + \sum_{r \in R} (\rho(M_r) - \mu_r([u])) = \sum_{r \in R} (m_r - \mu_r([u])).$$

Therefore

$$\begin{aligned} \xi([u]) &= \xi(\varepsilon) + 1 - \sum_{r \in R} (\mu_r(\varepsilon) - \mu_r([u])) \\ &= \sum_{r \in R} (\rho(M_r) - m_r) + 1 - \sum_{r \in R} (\rho(M_r) - \mu_r([u])) \\ &= \sum_{r \in R} (\mu_r([u]) - m_r) + 1. \end{aligned}$$

□

Lemma 3.4. *If $u \in S$, h is a queue in $S \setminus \{u\}$, and $g = [u] * h$, then*

$$\xi(g) \leq \xi(h) + 1.$$

Proof. Let $\lambda = \operatorname{dom} h$ and $\gamma = \operatorname{dom} g$. We shall use transfinite induction on λ. If $\lambda = 0$, then $\gamma = 1$ and $h = \varepsilon$. Since $\mu_r(\varepsilon) - \mu_r(g) \geq 0$ for every $r \in R$, we have

$$\xi(g) = \xi(\varepsilon) + 1 - \sum_{r \in R} (\mu_r(\varepsilon) - \mu_r(g)) \leq \xi(h) + 1.$$

If $\lambda = \theta + 1$ for some ordinal θ, then $\gamma = \delta + 1$ for some ordinal δ and

$$g_\theta = [u] * h_\delta.$$

By Lemma 2.4, we have

$$\mu_r(g_\theta) - \mu_r(g) \geq \mu_r(h_\delta) - \mu_r(h),$$

for any $r \in R$. Thus, by the inductive hypothesis, we get

$$\begin{aligned} \xi(g) &= \xi(g_\theta) + 1 - \sum_{r \in R} (\mu_r(g_\theta) - \mu_r(g)) \\ &\leq \xi(h_\delta) + 1 + 1 - \sum_{r \in R} (\mu_r(h_\delta) - \mu_r(h)) \\ &= \xi(h) + 1. \end{aligned}$$

If λ is a limit ordinal, then $\gamma = \lambda$. We have

$$\xi(g) = \liminf_{\theta \to \lambda} \eta(g_\theta) - \sum_{r \in R} (\overline{\mu_r}(g) - \mu_r(g)),$$

and

$$\xi(h) = \liminf_{\theta \to \lambda} \eta(h_\theta) - \sum_{r \in R} (\overline{\mu_r}(h) - \mu_r(h)).$$

By the inductive hypothesis, we have

$$\liminf_{\theta \to \lambda} \xi(g_\theta) \leq \liminf_{\theta \to \lambda} \xi(h_\theta) + 1.$$

To conclude the proof of the lemma it suffices to show that

$$\overline{\mu_r}(g) - \mu_r(g) \geq \overline{\mu_r}(h) - \mu_r(h), \tag{3.6}$$

for every $r \in R$. But for every $r \in R$, since M_r is rank-finite, there is an ordinal $\theta_r < \lambda$ such that

$$\overline{\mu_r}(h) = \mu_r(h_{\theta_r}),$$

and

$$\overline{\mu_r}(g) = \mu_r([u] * h_{\theta_r}).$$

Therefore (3.6) follows from Lemma 2.4 and the proof is complete. □

Lemma 3.5. *If $X \subseteq S$ and ξ^X is the $\mathcal{N}_{\bullet}X$-margin, then*

$$\xi(f) = \xi^X(f),$$

for any queue f in X.

Proof. For every $r \in R$, let μ_r^X be the $M_r{\bullet}X$-reserve. Let $\lambda = \operatorname{dom} f$. We shall use transfinite induction on λ. If $\lambda = 0$, then

$$\begin{aligned}
\xi^X(f) &= \sum_{r \in R} (\rho(M_r{\bullet}X) - \zeta(N_r, X)) \\
&= \sum_{r \in R} (\rho(M_r{\bullet}X) + \rho(M_r \backslash X) - m_r) \\
&= \sum_{r \in R} (\rho(M_r) - m_r) \\
&= \xi(f).
\end{aligned}$$

If $\lambda = \theta + 1$ for some ordinal θ, then by the inductive hypothesis and by Lemma 2.9, we have

$$\begin{aligned}\xi^X(f) &= \xi^X(f_\theta) + 1 - \sum_{r \in R} (\mu_r^X(f_\theta) - \mu_r^X(f)) \\ &= \xi(f_\theta) + 1 - \sum_{r \in R} (\mu_r(f_\theta) - \mu_r(f)) \\ &= \xi(f).\end{aligned}$$

If λ is a limit ordinal, then for every $r \in R$, there is an ordinal $\theta_r < \lambda$ such that

$$\overline{\mu_r}(f) = \mu_r(f_{\theta_r}),$$

and

$$\overline{\mu_r^X}(f) = \mu_r^X(f_{\theta_r}).$$

Thus, by the inductive hypothesis and by Lemma 2.9, we have

$$\begin{aligned}\xi^X(f) &= \liminf_{\theta \to \lambda} \xi^X(f_\theta) - \sum_{r \in R} (\overline{\mu_r^X}(f) - \mu_r^X(f)) \\ &= \liminf_{\theta \to \lambda} \xi(f_\theta) - \sum_{r \in R} (\mu_r^X(f_{\theta_r}) - \mu_r^X(f)) \\ &= \liminf_{\theta \to \lambda} \xi(f_\theta) - \sum_{r \in R} (\mu_r(f_{\theta_r}) - \mu_r(f)) \\ &= \liminf_{\theta \to \lambda} \xi(f_\theta) - \sum_{r \in R} (\overline{\mu_r}(f) - \mu_r(f)) \\ &= \xi(f),\end{aligned}$$

and the proof is complete. □

Corollary 3.6. *If $\mathcal{N}$ is good and g is a queue in S, then $\mathcal{N}.g$ is good.*

Proof. Let f be any queue in $\operatorname{rge} g$. Then $\xi(f) \geq 0$ since $\mathcal{N}$ is good. Taking $X = \operatorname{rge} g$ in Lemma 3.5 gives

$$\xi^X(f) = \xi(f) \geq 0.$$

Therefore $\mathcal{N}.g$ is good. □

Lemma 3.7. *If g is a queue in S such that $\xi(g) = 0$, and f is a queue in $S \setminus \operatorname{rge} g$, then*

$$\xi(g * f) = \xi_g(f),$$

where ξ_g is the $\mathcal{N}\backslash g$-margin.

Proof. Let $\hat{\mu}_r$ be the $M_r\backslash g$-reserve, and let $\overline{\overline{\mu_r}}$ be the limit of $\hat{\mu}_r$, $r \in R$. We shall use transfinite induction on $\lambda = \operatorname{dom} f$. If $\lambda = 0$, then

$$\xi(g * f) = \xi(g) = 0,$$

and, since $\mathcal{N}\backslash g$ is normal, we have

$$\xi_g(f) = 0 = \xi(g * f).$$

If $\lambda = \theta + 1$ for some ordinal θ, then by the inductive hypothesis and by Lemma 2.10, we have

$$\begin{aligned}
\xi(g * f) &= \xi(g * f_\theta) + 1 - \sum_{r \in R} (\mu_r(g * f_\theta) - \mu_r(g * f)) \\
&= \xi_g(f_\theta) + 1 - \sum_{r \in R} (\hat{\mu}_r(f_\theta) - \hat{\mu}_r(f)) \\
&= \xi_g(f).
\end{aligned}$$

If λ is a limit ordinal, then by Lemma 2.10, we have

$$\mu_r(g * f_\theta) = \hat{\mu}_r(f_\theta),$$

for every $\theta \leq \lambda$, so

$$\overline{\mu_r}(g * f) = \lim_{\theta \to \lambda} \mu_r(g * f_\theta) = \lim_{\theta \to \lambda} \hat{\mu}_r(f_\theta) = \overline{\overline{\mu_r}}(f).$$

Moreover, by the inductive hypothesis, we have

$$\liminf_{\theta \to \lambda} \xi(g * f_\theta) = \liminf_{\theta \to \lambda} \xi_g(f_\theta),$$

hence if $\gamma = \operatorname{dom}(g * f)$, then

$$\begin{aligned}\xi(g * f) &= \liminf_{\delta \to \gamma} \xi((g * f)_\delta) - \sum_{r \in R} (\overline{\mu_r}(g * f) - \mu_r(g * f)) \\ &= \liminf_{\theta \to \lambda} \xi(g * f_\theta) - \sum_{r \in R} (\overline{\mu_r}(g * f) - \mu_r(g * f)) \\ &= \liminf_{\theta \to \lambda} \xi_g(f_\theta) - \sum_{r \in R} (\overline{\overline{\mu_r}}(f) - \hat{\mu}_r(f)) \\ &= \xi_g(f),\end{aligned}$$

and the proof is complete. □

Corollary 3.8. *If $\mathcal{N}$ is good and g is a queue in S such that $\xi(g) = 0$, then $\mathcal{N} \backslash g$ is good.*

Proof. Let f be a queue in $S \setminus \operatorname{rge} g$. Since $\mathcal{N}$ is good, we have

$$\xi(g * f) \geq 0,$$

and by Lemma 3.7, we have

$$\xi_g(f) = \xi(g * f) \geq 0.$$

Therefore $\mathcal{N} \backslash g$ is good. □

4. The reduction procedure.

To find a system of disjoint bases for $\mathcal{M}$, we might take an element $a \in R$ and start constructing a base for M_a by selecting an element $u \in S$ such that $\{u\} \in M_a$, and forming a new system $\mathcal{M}'$ of matroids representing the rest of the work to be done. For $r = a$, we need to consider the matroid $M_a \sim u$ since B is a base of $M_a \sim u$ iff $B \cup \{u\}$ is a base of M_a. For the remaining values of r, we should consider the matroid $M'_r = M_r - u$ and include the demand that a base of M'_r is also a base of M_r. We can easily express all these requirements using weighted matroids instead of matroids.

If $N = (M, m)$ is a weighted matroid on S and $u \in S$, then let

$$N - u = (M - u, m),$$

and if moreover $\{u\} \in M$, then let

$$N \sim u = (M \sim u, m - 1).$$

Clearly, $N - u$ and $N \sim u$ are weighted matroids on $S \setminus \{u\}$. Note that $N \sim u = N.(S \setminus \{u\})$, and that $N - u = N \backslash \{u\}$ if and only if $\{u\} \notin M.\{u\}$.

If $a \in R$ and $\{u\} \in M_a$, then for every $r \in R$, let $N'_r = (M'_r, m'_r)$ be the weighted matroid on $S \setminus \{u\}$ defined by

$$N'_r = \begin{cases} N_r - u & \text{if } r \neq a, \\ N_a \sim u & \text{if } r = a. \end{cases} \tag{4.1}$$

The system $(N'_r)_{r \in R}$ defined by (4.1) will be denoted by $\mathcal{N}(a, u)$. Note that it is possible that $\mathcal{N}$ is normal and $\mathcal{N}(a, u)$ is not normal.

From now to the end of this section, let us assume that $a \in R$ and $u \in S$ are fixed elements such that $\{u\} \in M_a$. Let $\mathcal{N}' = (N'_r)_{r \in R} = \mathcal{N}(a, u)$ and let ξ' be the $\mathcal{N}'$-margin. For every $r \in R$, let $N'_r = (M'_r, m'_r)$, μ'_r be the M'_r-reserve, $\overline{\mu'_r}$ be the limit of μ'_r, and ν_r be the M_r-character.

Lemma 4.1. *If u is not M_r-essential for any $r \in R \smallsetminus \{a\}$, and f is a queue in $S \smallsetminus \{u\}$, then*

$$\xi'(f) = \xi(f).$$

Proof. Let $\lambda = \operatorname{dom} f$. If $\delta \leq \lambda$, then by Lemma 2.5, we have

$$\mu'_a(f_\delta) = \mu_a(f_\delta) - 1, \tag{4.2}$$

and by Lemma 2.7, we have

$$\mu'_r(f_\delta) = \mu_r(f_\delta), \tag{4.3}$$

for every $r \in R \smallsetminus \{a\}$. Therefore

$$\mu'_r(f_\theta) - \mu'_r(f) = \mu_r(f_\theta) - \mu_r(f), \tag{4.4}$$

for any $\theta < \lambda$ and $r \in R$.

Now we shall use transfinite induction on λ. Assume that $\lambda = 0$. Taking $\delta = 0$ in the equations (4.2) and (4.3), we get

$$\rho(M'_a) = \rho(M_a) - 1$$

and

$$\rho(M'_r) = \rho(M_r),$$

for $r \in R \smallsetminus \{a\}$. Since $m'_a = m_a - 1$ and $m'_r = m_r$ for every $r \in R \smallsetminus \{a\}$, we conclude that

$$\xi(f) = \sum_{r \in R} (\rho(M_r) - m_r) = \sum_{r \in R} (\rho(M'_r) - m'_r) = \xi'(f).$$

If $\lambda = \theta + 1$ for some ordinal θ, then using the inductive hypothesis and the equation (4.4) we get

$$\begin{aligned}
\xi'(f) &= \xi'(f_\theta) + 1 - \sum_{r \in R} (\mu'_r(f_\theta) - \mu'_r(f)) \\
&= \xi(f_\theta) + 1 - \sum_{r \in R} (\mu_r(f_\theta) - \mu_r(f)) \\
&= \xi(f)
\end{aligned}$$

Assume now that λ is a limit ordinal. After taking limits of both sides of the equation (4.4), as $\theta \to \lambda$, we get

$$\overline{\mu'_r}(f) - \mu'_r(f) = \overline{\mu_r}(f) - \mu_r(f), \tag{4.5}$$

for any $r \in R$. By the inductive hypothesis, we have

$$\liminf_{\theta \to \lambda} \xi(f_\theta) = \liminf_{\theta \to \lambda} \xi'(f_\theta).$$

Therefore, using the equation (4.5), we get

$$\begin{aligned} \xi'(f) &= \liminf_{\theta \to \lambda} \xi'(f_\theta) - \sum_{r \in R} (\overline{\mu'_r}(f) - \mu'_r(f)) \\ &= \liminf_{\theta \to \lambda} \xi(f_\theta) - \sum_{r \in R} (\overline{\mu_r}(f) - \mu_r(f)) \\ &= \xi(f). \end{aligned}$$

Thus the proof is complete. □

Lemma 4.2. *Suppose that* $\mathcal{N}$ *is good and that the set* $\{x \in S : \{x\} \in M_a\}$ *is uncountable. Then there is* $y \in S$ *such that* $\{y\} \in M_a$ *and* $\mathcal{N}(a, y)$ *is good.*

Proof. Since R is countable and, by Lemma 2.8, the set of M_r-essential elements is countable for every $r \in R$, we conclude that there is $y \in S$ such that $\{y\} \in M_a$ and y is not M_r-essential for any $r \in R \setminus \{a\}$.

Since $\mathcal{N}$ is good, we have $\xi(f) \geq 0$ for every queue f in S. Therefore, if $u = y$, then by Lemma 4.1, we have $\xi'(f) \geq 0$ for any queue f in $S \setminus \{u\}$ and so $\mathcal{N}' = \mathcal{N}(a, y)$ is good. □

Lemma 4.3. *If* h *is a queue in* $S \setminus \{u\}$ *and* $g = [u] * h$, *then*

$$\xi'(g) = \xi(g) + \nu_a(g).$$

Proof. Let $\lambda = \operatorname{dom} g$. If $0 < \delta \leq \lambda$, then $u \in \operatorname{rge} g_\delta$ and so

$$\mu'_r(g_\delta) = \mu_r(g_\delta), \tag{4.6}$$

for every $r \in R \smallsetminus \{a\}$. Moreover, by Lemma 2.6, we have

$$\mu'_a(g_\delta) = \mu_a(g_\delta) + \nu_a(g_\delta) - 1. \tag{4.7}$$

We shall now use transfinite induction to complete the proof. If $\lambda = 1$, then $h = \varepsilon$ and $g = [u]$. By Lemma 3.3, we have

$$\xi(g) = \sum_{r \in R} (\mu_r(g) - m_r) + 1, \tag{4.8}$$

and

$$\xi'(g) = \sum_{r \in R} (\mu'_r(g) - m'_r) + 1. \tag{4.9}$$

Since $m'_r = m_r$ for $r \in R \smallsetminus \{a\}$, and $m'_a = m_a - 1$, using the equations (4.6) and (4.7) with $\delta = \lambda$, and the equations (4.8) and (4.9), we get

$$\xi'(g) = \sum_{r \in R} (\mu_r(g) - m_r) + \nu_a(g) + 1 = \xi(g) + \nu_a(g).$$

If $\lambda = \theta + 1$ for some ordinal θ, then by the inductive hypothesis we have

$$\begin{aligned} \xi'(g) &= \xi'(g_\theta) + 1 - \sum_{r \in R} (\mu'_r(g_\theta) - \mu'_r(g)) \\ &= \xi(g_\theta) + 1 + \nu_a(g_\theta) - \sum_{r \in R} (\mu'_r(g_\theta) - \mu'_r(g)). \end{aligned}$$

Using the equations (4.6) and (4.7) with $\delta = \lambda$ and with $\delta = \theta$, we get

$$\xi'(g) = \xi(g_\theta) + 1 + \nu_a(g) - \sum_{r \in R} (\mu_r(g_\theta) - \mu_r(g)) = \xi(g) + \nu_a(g).$$

If λ is a limit ordinal, then by the inductive hypothesis we have

$$\xi'(g_\theta) = \xi(g_\theta) + \nu_a(g_\theta),$$

for every $\theta < \lambda$. Therefore

$$\liminf_{\theta\to\lambda} \xi'(g_\theta) = \liminf_{\theta\to\lambda} \xi(g_\theta) + \overline{\nu_a}(g). \tag{4.10}$$

Moreover, taking limits of both sides of (4.6) and (4.7), as $\delta \to \lambda$, we get

$$\overline{\mu_r'}(g) = \overline{\mu_r}(g), \tag{4.11}$$

for $r \in R \setminus \{a\}$, and

$$\overline{\mu_a'}(g) = \overline{\mu_a}(g) + \overline{\nu_a}(g) - 1. \tag{4.12}$$

Now using the equations (4.6) and (4.7) with $\delta = \lambda$, and the equations (4.10), (4.11) and (4.12), we conclude that

$$\begin{aligned}
\xi'(g) &= \liminf_{\theta\to\lambda} \xi'(g_\theta) - \sum_{r\in R}(\overline{\mu_r'}(g) - \mu_r'(g)) \\
&= \liminf_{\theta\to\lambda} \xi(g_\theta) + \overline{\nu_a}(g) - \sum_{r\in R}(\overline{\mu_r'}(g) - \mu_r'(g)) \\
&= \liminf_{\theta\to\lambda} \xi(g_\theta) + \nu_a(g) - \sum_{r\in R}(\overline{\mu_r}(g) - \mu_r(g)) \\
&= \xi(g) + \nu_a(g),
\end{aligned}$$

and so the proof is complete. □

Lemma 4.4. *If $\mathcal{N}$ is good and $\mathcal{N}'$ is not good, then there is a queue g in S such that $u \in \operatorname{rge} g$, $\mu_a(g) > 0$, and $\xi(g) = 0$.*

Proof. Since $\mathcal{N}'$ is not good, there is a queue h in $S \setminus \{u\}$ such that $\xi'(h) < 0$. Let $g = [u] * h$. By Lemma 3.4, we have $\xi'(g) \leq \xi'(h) + 1$, and by Lemma 4.3, we have $\xi'(g) = \xi(g) + \nu_a(g)$. Therefore

$$\xi(g) + \nu_a(g) \leq 0.$$

Since $\xi(g) \geq 0$ and $\nu_a(g) \geq 0$, we must have $\xi(g) = 0$ and $\nu_a(g) = 0$.

To conclude the proof it remains to show that $\mu_a(g) > 0$. Suppose that $\mu_a(g) = \rho(M_a \backslash g) = 0$. Then $M_a = M_a \centerdot g$. Since $\nu_a(g) = 0$ and $g \neq \varepsilon$, we have $\{u\} \in M_a$ and

$\{u\} \notin M_a \cdot g$ in contradiction with the equality of M_a and $M_a \cdot g$. Therefore the proof is complete. $\square$

5. Piecewise good systems.

In order to prove Theorem 1.9, we need to generalize the notion of a good system of weighted matroids.

Let $\{S_1, S_2, \ldots, S_k\}$ be a partition of S. If $N_i = (M_i, m_i)$ is a weighted matroid on S_i, $i = 1, 2, \ldots, k$, such that

$$M_1 \oplus M_2 \oplus \ldots \oplus M_k \subseteq M,$$

and $m = m_1 + m_2 + \ldots + m_k$, then we will say that the sequence $(N_1, N_2, \ldots, N_k)$ is a *decomposition* of N. Note that it is possible that $(N_1, N_2, \ldots, N_k)$ is a decomposition of N but $(M_1, M_2, \ldots, M_k)$ is not a decomposition of M since M may have a larger rank than $M_1 \oplus M_2 \oplus \ldots \oplus M_k$ does. However, if N_i is normal for every i, $1 \leq i \leq k$, then $(N_1, N_2, \ldots, N_k)$ is a decomposition of N if and only if $(M_1, M_2, \ldots, M_k)$ is a decomposition of M.

Let $\mathcal{N}_i = (N_r^{(i)})_{r \in R}$ be a system of weighted matroids on S_i, $i = 1, 2, \ldots, k$, such that $(N_r^{(1)}, N_r^{(2)}, \ldots, N_r^{(k)})$ is a decomposition of N_r for every $r \in R$. Then we will say that $(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k)$ is a *decomposition* of $\mathcal{N}$, and we will write

$$\mathcal{N} \trianglerighteq \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \ldots \oplus \mathcal{N}_k.$$

Assume that $N_r^{(i)} = (M_r^{(i)}, m_r^{(i)})$, for every $i = 1, 2, \ldots, k$ and $r \in R$.

Lemma 5.1. *If $\mathcal{N} \trianglerighteq \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \ldots \oplus \mathcal{N}_k$, $a \in R$, and $\{u\} \in M_a^{(j)}$ for some j, $1 \leq j \leq k$, then*

$$\mathcal{N}(a, u) \trianglerighteq \mathcal{N}_1 \oplus \ldots \oplus \mathcal{N}_{j-1} \oplus \mathcal{N}_j(a, u) \oplus \mathcal{N}_{j+1} \oplus \ldots \oplus \mathcal{N}_k.$$

Proof. Because of symmetry, we can assume that $j = 1$. Let $\mathcal{N}(a,u) = (N'_r)_{r\in R}$ and $\mathcal{N}_1(a,u) = (N''_r)_{r\in R}$, where $N'_r = (M'_r, m'_r)$ and $N''_r = (M''_r, m''_r)$, $r \in R$. Clearly, we have

$$m'_r = m''_r + \sum_{i=2}^{k} m_r^{(i)},$$

for every $r \in R$. To complete the proof, we have to show that

$$M''_r \oplus M_r^{(2)} \oplus M_r^{(3)} \oplus \ldots \oplus M_r^{(k)} \subseteq M'_r, \tag{5.1}$$

for every $r \in R$.

Assume first that $r \in R \setminus \{a\}$. Then $M'_r = m_r - u$ and $M''_r = M_r^{(1)} - u$. Let $I_1 \in M''_r \subseteq M_r^{(1)}$ and let $I_i \in M_r^{(i)}$ for $i = 2, 3, \ldots, k$. Then

$$I = \bigcup_{i=1}^{k} I_i \in \bigoplus_{i=1}^{k} M_r^{(i)} \subseteq M_r.$$

Since $I_1 \in M_r^{(1)} - u$, we have $u \notin I_1$. Clearly $u \notin I_i$ for $i = 2, 3, \ldots, k$, hence $u \notin I$. Therefore $I \in M_r - u = M'_r$, and so the condition (5.1) is satisfied for every $r \in R \setminus \{a\}$.

It remains to show that (5.1) holds for $r = a$. We have $M'_a = M_a \sim u$ and $M''_a = M_a^{(1)} \sim u$. Let $I'_1 \in M''_a$ and let $I_i \in M_a^{(i)}$ for $i = 2, 3, \ldots, k$. Then

$$I_1 = I'_1 \cup \{u\} \in M_a^{(1)},$$

and so

$$I = \bigcup_{i=1}^{k} I_i \in \bigoplus_{i=1}^{k} M_a^{(i)} \subseteq M_a.$$

Let

$$I' = I'_1 \cup \bigcup_{i=2}^{k} I_i.$$

Clearly $u \notin I'$. Since $I' \cup \{u\} = I \in M_a$, we conclude that $I' \in M_a \sim u = M'_a$. Therefore (5.1) holds for $r = a$ and so the proof is complete. □

Corollary 5.2. *If $a \in R$, $X \subseteq S$, and $u \in S \setminus X$ is such that $\{u\} \in M_a \backslash X$, then*

$$\mathcal{N}(a,u) \trianglerighteq (\mathcal{N}\backslash X)(a,u) \oplus (\mathcal{N}.X).$$

Proof. It follows from Corollary 2.2 that

$$\mathcal{N} \trianglerighteq (\mathcal{N}\backslash X) \oplus (\mathcal{N}.X).$$

Thus, using Lemma 5.1, we get the desired result. □

We say that $\mathcal{N}$ is *piecewise good* if $\mathcal{N}$ can be decomposed into finitely many good weighted matroids, *i.e.* if there is a finite partition $\{S_1, S_2, \ldots, S_k\}$ of S and good systems of weighted matroids $\mathcal{N}_i$ on S_i, $i = 1, 2, \ldots, k$, such that $(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k)$ is a decomposition of $\mathcal{N}$.

Recall that R and S are fixed sets with R being countable, $N_r = (M_r, m_r)$ is a weighted matroid on S for each $r \in R$, and $\mathcal{N} = (N_r)_{r\in R}$. Let $a \in R$ be a fixed element such that $\rho(M_a) > 0$. We shall apply now the reduction procedure described in section 4. Assuming that $\mathcal{N}$ is piecewise good, our aim is to show that we can select an element $u \in S$ such that $\{u\} \in M_a$ and the system $\mathcal{N}(a, u)$ is also piecewise good. Repeating such selection $\rho(M_r)$ times for every $r \in R$, we will get a set $B_r \subseteq S$ consisting of $\rho(M_r)$ elements which will be a base of M_r. Moreover, since any selected element is eliminated from consideration during further selections, the bases obtained will be disjoint.

We will need the following lemmas.

Lemma 5.3. *If $\mathcal{N}$ is piecewise good, then $\mathcal{N}$ is normal.*

Proof. Let $(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k)$ be a decomposition of $\mathcal{N}$ into good weighted matroids such that $\mathcal{N}_i = (N_r^{(i)})_{r\in R}$, $i = 0, 1, \ldots, k$, and $N_r^{(i)} = (M_r^{(i)}, m_r^{(i)})$, $r \in R$.

Since for each $i = 1, 2, \ldots, k$, the system $\mathcal{N}_i$ is good, the value of the $\mathcal{N}_i$-margin on the empty queue is nonnegative and hence

$$\rho(M_r^{(i)}) = m_r^{(i)},$$

for every $r \in R$.

Let $r \in R$. Since $(N_r^{(1)}, N_r^{(2)}, \dots, N_r^{(k)})$ is a decomposition of N_r, we have

$$\rho(M_r) \geq \rho(M_r^{(1)}) + \rho(M_r^{(2)}) + \ldots + \rho(M_r^{(k)})$$
$$= m_r^{(1)} + m_r^{(2)} + \ldots + m_r^{(k)},$$

and

$$m_r = m_r^{(1)} + m_r^{(2)} + \ldots + m_r^{(k)}.$$

Thus $\rho(M_r) \geq m_r$. Since N_r is a weighted matroid, we have $\rho(M_r) \leq m_r$, hence $\rho(M_r) = m_r$, and so N_r is normal. Therefore $\mathcal{N}$ is normal and the proof is complete. □

Lemma 5.4. *If the system $\mathcal{N}$ is good and the set $\{x \in S : \{x\} \in M_a\}$ is countable, then there is $u \in S$ such that $\{u\} \in M_a$ and $\mathcal{N}(a, u)$ is piecewise good.*

Proof. Let

$$E = \{x \in S : \{x\} \in M_a\}.$$

Note that $E \neq \varnothing$ since $\rho(M_a) > 0$.

Suppose that $\mathcal{N}(a, u)$ is not piecewise good for any $u \in E$. Let t be an α-queue in E such that $\operatorname{rge} t = E$ and $\alpha \leq \omega$. (ω is the first infinite ordinal.) Let $x_i = t(i)$, $i < \alpha$. We claim that for every nonnegative integer j, there is a countable ordinal λ_j and a λ_j-queue $f^{(j)}$ in S such that

(5.2) $\{x_i : i < j\} \subseteq \operatorname{rge} f^{(j)}$,

(5.3) $\mu_a(f^{(j)}) > 0$,

(5.4) $\xi(f^{(j)}) = 0$, and

(5.5) if $j \geq 1$, then $\lambda_j > \lambda_{j-1}$ and $f^{(j)}_{\lambda_{j-1}} = f^{(j-1)}$.

Note that after the claim is proved it will follow that $\alpha = \omega$.

To prove the claim we shall use induction on j. If $j = 0$, then $f^{(0)} = \varepsilon$ satisfies the conditions (5.2)–(5.5).

Suppose now that $j = k+1$, and $f^{(k)}$ is a λ_k-queue in S satisfying the requirements. Since $\mu_a(f^{(k)}) > 0$, so $E \smallsetminus \operatorname{rge} f^{(k)}$ is a nonempty set. Let p be the minimal integer such that $x_p \in E \setminus \operatorname{rge} f^{(k)}$. By Corollary 5.2, we have

$$\mathcal{N}(a, x_p) \trianglerighteq (\mathcal{N} \backslash f^{(k)})(a, x_p) \oplus (\mathcal{N}_{\bullet} f^{(k)}).$$

Since $\mathcal{N}_{\bullet} f^{(k)}$ is good by Corollary 3.6, and $\mathcal{N}(a, x_p)$ is not piecewise good, we conclude that $(\mathcal{N} \backslash f^{(k)})(a, x_p)$ is not good. Since $\mathcal{N} \backslash f^{(k)}$ is good by Corollary 3.8, it follows from Lemma 4.4 that there is a queue g in $S \setminus \operatorname{rge} f^{(k)}$ such that $x_p \in \operatorname{rge} g$, $\hat{\mu}_a(g) > 0$, and $\xi_k(g) = 0$, where $\hat{\mu}_a$ is the $M_a \backslash f^{(k)}$-reserve and ξ_k is the $\mathcal{N} \backslash f^{(k)}$-margin.

Let $f^{(j)} = f^{(k)} * g$. Then

$$\{x_i : i < j\} \subseteq \operatorname{rge} f^{(j)}$$

since by the inductive hypothesis, we have

$$\{x_i : i < k\} \subseteq \operatorname{rge} f^{(k)} \subseteq \operatorname{rge} f^{(j)},$$

and either $x_k \in \operatorname{rge} f^{(k)}$ or else $x_k = x_p \in \operatorname{rge} g$. Moreover, we have

$$\mu_a(f^{(j)}) = \rho(M_a \backslash (f^{(k)} * g)) = \rho((M_a \backslash f^{(k)}) \backslash g) = \hat{\mu}_a(g) > 0,$$

and by Lemma 3.7, we have $\xi(f^{(j)}) = \xi_k(g) = 0$. Finally, if $\lambda_k = \operatorname{dom} f^{(k)}$ and $\lambda_j = \operatorname{dom} f^{(j)}$, then of course $\lambda_j > \lambda_k$ and

$$f^{(j)}_{\lambda_k} = f^{(k)}.$$

Thus, $f^{(j)}$ satisfies the conditions (5.2)–(5.5) and our claim is proved.

Let

$$\lambda = \sup\{\lambda_j : j = 1, 2, \dots\},$$

and let f be the λ-queue in S such that if $\theta < \lambda$, then $f(\theta) = f^{(j)}(\theta)$, where j satisfies $\theta < \lambda_j$. By the condition (5.5), the queue f is well-defined and λ is a limit ordinal. It follows from (5.2) that $E \subseteq \operatorname{rge} f$ and thus

$$\mu_a(f) = 0,$$

it follows from (5.3) that

$$\overline{\mu_a}(f) > 0,$$

and it follows from (5.4) that

$$\liminf_{\theta\to\lambda} \xi(f_\theta) \le 0.$$

Since obviously $\overline{\mu_r}(f) \ge \mu_r(f)$ for every $r \in R$, we conclude that

$$\xi(f) = \liminf_{\theta\to\lambda} \xi(f_\theta) - \sum_{r\in R}(\overline{\mu_r}(f) - \mu_r(f)) < 0,$$

which is in contradiction with our assumption that $\mathcal{N}$ is good. Therefore, there is $u \in S$ such that $\mathcal{N}(a, u)$ is piecewise good, and so the proof is complete. □

Corollary 5.5. *If $\mathcal{N}$ is piecewise good, then there is $u \in S$ such that $\{u\} \in M_a$ and $\mathcal{N}(a, u)$ is piecewise good.*

Proof. Let $(\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k)$ be a decomposition of $\mathcal{N}$ into good weighted matroids. Let $\{S_1, S_2, \dots, S_k\}$ be a partition of S such that $\mathcal{N}_i = (M_r^{(i)}, m_r^{(i)})$ is a weighted matroid on S_i, $1 \le i \le k$. Since $m_a \ge \rho(M_a) > 0$, there is an integer j, $1 \le j \le k$, such that $m_a^{(j)} > 0$. Since N_j is good, it is normal and hence

$$\rho(M_a^{(j)}) = m_a^{(j)} > 0.$$

By Lemma 4.2 and Lemma 5.4, there is $u \in S_j$ such that $\mathcal{N}_j(a, u)$ is piecewise good. Let $(\mathcal{N}_j^{(1)}, \mathcal{N}_j^{(2)}, \dots, \mathcal{N}_j^{(\ell)})$ be a decomposition of $\mathcal{N}_j(a, u)$ into good weighted matroids. By Lemma 5.1, we have

$$\mathcal{N}(a, u) \trianglerighteq \mathcal{N}_1 \oplus \dots \oplus \mathcal{N}_{j-1} \oplus \mathcal{N}_j(a, u) \oplus \mathcal{N}_{j+1} \oplus \dots \oplus \mathcal{N}_k,$$

which implies that

$$\mathcal{N}(a, u) \trianglerighteq \mathcal{N}_1 \oplus \dots \oplus \mathcal{N}_{j-1} \oplus \mathcal{N}_j^{(1)} \oplus \dots \oplus \mathcal{N}_j^{(\ell)} \oplus \mathcal{N}_{j+1} \oplus \dots \oplus \mathcal{N}_k,$$

and so the proof is complete. □

6. The main result.

Let r_0, r_1, $\dots$ be a (possibly finite) enumeration of R. Let $\rho_i = \rho(M_{r_i})$, $i < \|R\|$. We say that $\mathcal{N}$ is *weedless* if $\rho_i > 0$ for every $i < \|R\|$.

Assume that $\mathcal{N}$ is weedless. Let

$$\Xi = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq i < \|R\|, 1 \leq j \leq \rho_i\},$$

and let $\prec$ be the linear ordering of Ξ such that $(i,j) \prec (k,\ell)$ if either $i < k$ or else $i = k$ and $j < \ell$. If $\beta, \gamma \in \Xi$, then let $\beta \preceq \gamma$ mean that $\beta \prec \gamma$ or $\beta = \gamma$. If $\beta \in \Xi$, then let

$$\Xi_\beta = \{\gamma \in \Xi : \gamma \prec \beta\}.$$

We say that $\beta = (i,j) \in \Xi$ is a *starter* if $j = 1$. If $\beta \in \Xi$, then let β^- be the largest starter in $\Xi_\beta \cup \{(0,1)\}$. If $\beta \in \Xi$ and $f : \Xi_\beta \to S$, then let

$$X_f = \{f(\gamma) : \beta^- \preceq \gamma \prec \beta\},$$

and let

$$Y_f = S \setminus \{f(\gamma) : \gamma \prec \beta\}.$$

If $f : \Xi \to S$, then let

$$X_f^\beta = X_{f_\beta},$$

and

$$Y_f^\beta = Y_{f_\beta},$$

where f_β is the restriction of f to Ξ_β. Let the *successor* β^+ of β be the smallest $\gamma \in \Xi$ such that $\beta \prec \gamma$.

An injective function $f : \Xi_\beta \to S$, where $\beta = (k,\ell) \in \Xi$, is a *partial foundation* of $\mathcal{N}$ if there is a piecewise good system of weighted matroids $\mathcal{N}_\beta = (N_r^\beta)_{r\in R}$ on Y_f such that if $N_r^\beta = (M_r^\beta, m_r^\beta)$, $r \in R$, then

(6.1) $M_r^\beta \subseteq M_r$ for any $r \in R$, (6.2) $m^{\beta}_{r_k} = \rho_k - \ell + 1$,

(6.3) $m^{\beta}_{r_q} = \rho_q$ for $q > k$, and

(6.4) if $\beta = (i,j)^+$, then for every $I \in M^{\beta}_{r_i}$ we have

$$I \cup X_f \in M_{r_i}.$$

An injective function $f : \Xi \to S$ is a *foundation* of $\mathcal{N}$ if for every $\beta \in \Xi$, the restriction f_β of f to Ξ_β is a partial foundation of $\mathcal{N}$.

Lemma 6.1. *If $\mathcal{N}$ is piecewise good and weedless, then there is a foundation of $\mathcal{N}$.*

Proof. Clearly, it is enough to show that for every $\beta \in \Xi$, there is a partial foundation $f_\beta : \Xi_\beta \to S$ of $\mathcal{N}$ such that f_β is the restriction of f_{β^+} to Ξ_β.

We shall use induction on β. If $\beta = (0,1)$, and f_β is the empty function, then f_β is a partial foundation of $\mathcal{N}$ since $\mathcal{N}_\beta = \mathcal{N}$ is a piecewise good system of weighted matroids on $Y_{f_\beta} = S$ and the conditions (6.1)–(6.4) are satisfied. Indeed, the conditions (6.1) and (6.4) are satisfied trivially and the conditions (6.2) and (6.3) hold since, by Lemma 5.3, the system $\mathcal{N}_\beta$ is normal.

Assume now that $\beta = \gamma^+$, where $\gamma = (i,j)$, and that f_γ is a partial foundation of $\mathcal{N}$. Thus, there is a piecewise good system of weighted matroids $\mathcal{N}_\gamma$ on Y_{f_γ} such that:

(6.1′) $M^{\gamma}_r \subseteq M_r$ for any $r \in R$,

(6.2′) $m^{\gamma}_{r_i} = \rho_i - j + 1$,

(6.3′) $m^{\gamma}_{r_q} = \rho_q$ for $q > i$, and

(6.4′) if $\gamma = (a,b)^+$, then for every $I \in M^{\gamma}_{r_a}$ we have

$$I \cup X_{f_\gamma} \in M_{r_a}.$$

We will show that there is a partial foundation f_β of $\mathcal{N}$ such that f_β restricted to Ξ_γ is equal to f_γ.

Since the system $\mathcal{N}_\gamma$ is piecewise good, it follows from Lemma 5.3 that it is normal. Thus by (6.2′), we have

$$\rho(M^\gamma_{r_i}) = \rho_i - j + 1 > 0.$$

Therefore, by Corollary 5.5, there is $u \in Y_{f_\gamma}$ such that $\{u\} \in M^\gamma_{r_i}$ and $\mathcal{N}_\gamma(r_i, u)$ is piecewise good.

Let $f_\beta : \Xi_\beta \to S$ be such that

$$f_\beta(\delta) = \begin{cases} f_\gamma(\delta) & \text{if } \delta \prec \gamma, \\ u & \text{if } \delta = \gamma. \end{cases}$$

Clearly, f_β restricted to Ξ_γ is equal to f_γ, and f_β is an injection since f_γ is an injection and $u \in Y_{f_\gamma}$. Let

$$\mathcal{N}_\beta = \mathcal{N}_\gamma(r_i, u).$$

By the choice of u, $\mathcal{N}_\beta$ is a piecewise good system of weighted matroids on $Y_{f_\beta} = Y_{f_\gamma} \setminus \{u\}$. Clearly, we have $M^\beta_r \subseteq M^\gamma_r$ for every $r \in R$, so (6.1) follows from (6.1′).

If β is a starter, then $\ell = 1$ and $k = i + 1$. By (6.3′), we have $m^\gamma_{r_k} = \rho_k$. Since $m^\beta_{r_k} = m^\gamma_{r_k}$, the condition (6.2) is satisfied in this case.

If β is not a starter, then $\ell = j + 1$ and $k = i$. Thus by (6.2′), we have

$$m^\beta_{r_k} = m^\beta_{r_i} = m^\gamma_{r_i} - 1 = \rho_i - j = \rho_k - \ell + 1,$$

and so (6.2) is satisfied.

Since $k \geq i$ and since we have $m^\beta_{r_q} = m^\gamma_{r_q}$ for any $q > i$, the condition (6.3) follows from (6.3′).

To complete the proof it remains to show that (6.4) is satisfied. Assume first that γ is a starter. Then $\beta^- = \gamma$ and so $X_{f_\beta} = \{u\}$. Since

$$M^\beta_{r_i} = M^\gamma_{r_i} \sim u,$$

and $\{u\} \in M^\gamma_{r_i}$, we conclude that

$$I \cup X_{f_\beta} \in M^\gamma_{r_i},$$

for any $I \in M_{r_i}^{\beta}$. Since by (6.1′) we have $M_{r_i}^{\gamma} \subseteq M_{r_i}$, it follows that (6.4) is satisfied in this case.

Now assume that γ is not a starter. Then $\gamma = (i, j-1)^+$ and

$$X_{f_\beta} = X_{f_\gamma} \cup \{u\}.$$

Let $I \in M_{r_i}^{\beta}$. Since

$$M_{r_i}^{\beta} = M_{r_i}^{\gamma} \sim u,$$

and $\{u\} \in M_{r_i}^{\gamma}$, it follows that $I \cup \{u\} \in M_{r_i}^{\gamma}$. By (6.4′), we have

$$I \cup X_{f_\beta} = I \cup \{u\} \cup X_{f_\gamma} \in M_{r_i},$$

and so (6.4) is satisfied. Thus the proof is complete. □

Proof of **Theorem 1.9.** Let $\mathcal{N} = (\overline{M_r})_{r\in R}$ be the normal system of weighted matroids corresponding to $\mathcal{M}$. By Corollary 3.2, the system $\mathcal{N}$ is good, hence it is piecewise good. Clearly, without loss of generality, we may assume that $\mathcal{N}$ is weedless.

By Lemma 6.1, there is a foundation g of $\mathcal{N}$. For every i, $0 \le i < \|R\|$, let $\beta_i = (i+1, 1) = (i, \rho_i)^+$ and let

$$B_{r_i} = X_g^{\beta_i} = \{g(\gamma) : (i,1) \preceq \gamma \preceq (i, \rho_i)\}.$$

Using the condition (6.4), with $I = \varnothing$, for the partial foundation $f = g_{\beta_i}$, we get

$$B_{r_i} = X_f \in M_{r_i},$$

$0 \le i < \|R\|$. Since g is an injection, we have

$$\|B_{r_i}\| = \rho_i,$$

$0 \le i < \|R\|$, and

$$B_{r_i} \cap B_{r_j} = \varnothing,$$

for any i and j such that $0 \le i, j < \|R\|$ and $i \ne j$. Therefore $(B_r)_{r\in R}$ is a system of disjoint bases for $\mathcal{M}$, and so the proof is complete. □

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