CONGRESSUS NUMERANTIUM **105**, (1994), 97–115.

A Necessary Condition for the Existence of Disjoint Bases of a Family of Infinite Matroids

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Abstract. Let $\mathcal{M} = (M_r)_{r \in R}$ be a system of matroids on a set S. Following the ideas of Nash-Williams [7], for every transfinite sequence f of distinct elements of S, we define a number $\eta(f)$. We prove that the condition that $\eta(f) \geq 0$ for every possible choice of f is necessary for $\mathcal{M}$ to have a system of mutually disjoint bases. Further, we show that this condition is sufficient if R is countable and M_r is a rank-finite transversal matroid for every $r \in R$. We also present conjectures about edge-disjoint spanning trees and detachments of countable graphs.

1. Introduction.

Let S be a set and let M be a family of subsets of S. We say that M has *finite character* if a set A belongs to M if and only if any finite subset I of A belongs to M. We say that M is a *matroid* on S if M is a non-empty family of subsets of S satisfying the conditions:

(I-1) If $A \in M$ and $B \subset A$, then $B \in M$.

(I-2) If $I, J \in M$ are finite and $|I| = |J| + 1$, then there is an element $y \in I \setminus J$ such that $J \cup \{y\} \in M$.

(I-3) M has finite character.

A maximal element of M is called a *base* of M, and the cardinality of any base of M is called the *rank* of M. The matroid M is said to be *rank-finite* if M has finite rank, and to be *rank-countable* if it has countable rank ($\rho(M) \leq \aleph_0$).

Let R and S be disjoint sets and for each $r \in R$ let M_r be a matroid on the set S. The system $\mathcal{M} = (M_r)_{r\in R}$ will be called a *system of matroids* on S. $\mathcal{M}$ will be said to be *countable* (*finite*) if R is countable (finite). The system $\mathcal{B} = (B_r)_{r\in R}$ of subsets of S will be called a *system of disjoint bases* for $\mathcal{M}$ if B_r is a base of M_r, $r \in R$, and $B_r \cap B_{r'} = \emptyset$ for every $r, r' \in R$ such that $r \neq r'$.

The problem of finding a necessary and sufficient condition for a system of matroids to have a system of disjoint bases (known as the packing problem) is solved only for some special cases. Tutte [12] and Nash-Williams [6] independently proved a necessary and sufficient condition for a finite graph to have k edge-disjoint spanning trees. These results were later generalized by Edmonds [3] who thus settled the packing problem for finite systems of finite matroids. Brualdi [2] generalized the condition of Edmonds, and thus solved the packing problem for finite systems of rank-finite matroids and for arbitrary systems of finite matroids. Oxley [10] formulated a sufficient condition for a system of matroids to have a system of disjoint bases and gave a counterexample showing that his condition is not necessary.

We are going to formulate a necessary condition for a system of matroids to have a system of disjoint bases which is in the spirit of the condition given by Nash-Williams [7] as a necessary and sufficient condition for a countable family of sets to have a transversal. Then we prove that our condition is sufficient in the case of a countable system of rank-finite transversal matroids. We also conjecture that the condition is sufficient in the case of a countable system of arbitrary matroids.

At the end of this note we shall present a conjecture concerning the existence of k edge-disjoint spanning trees in a countable graph and a conjecture about detachments of countable graphs. We believe that these conjectures can be proved using Conjecture 1.

Let $\mathbb{Z}^* = \mathbb{Z} \cup \{\infty, -\infty\}$ be the set of *quasi-integers*. The arithmetic and inequalities on $\mathbb{Z}^*$ follow the obvious rules with the additional rule that $\infty - \infty = \infty$. If A is a set, then the *size* $\|A\|$ of A is the cardinality $|A|$ if A is finite, and $\|A\| = \infty$ if A is infinite.

If M is a matroid on the set S, and $A \subset S$, then let $M \otimes A$ be the contraction of M to A, *i.e.* let $I \in M \otimes A$ iff $I \subset A$ and $I \cup J \in M$ for every $J \subset S \setminus A$ such that $J \in M$. The following properties are satisfied (see Brualdi [1]):

(P-1) If $I \in M \otimes A$ and $A \subset B \subset S$, then $I \in M \otimes B$.

(P-2) If $A \subset B \subset S$, then $(M \otimes B) \otimes A = M \otimes A$.

(P-3) If B is a base of M and $A \subset S$, then there is $B' \subset B \cap A$ such that B' is a base of $M \otimes A$.

(P-4) Let $B_1 \subset B_2 \subset S$, $B'_1 \subset B'_2 \subset S$, $A \subset S$, B_2, B'_2 be bases of M, and B_1, B'_1 be bases of $M \otimes A$. Then $\|B_2 \setminus B_1\| = \|B'_2 \setminus B'_1\|$.

Assume that ordinals are defined in such a way that an *ordinal* α is the set of all ordinals less than α. A *transfinite sequence* is a function with its domain being an ordinal.

Let us assume now that $\mathcal{M} = (M_r)_{r \in R}$ is a fixed system of matroids on S. If $r \in R$, $A_1, A_2 \in M_r$, $A_1 \subset A_2$, B_1 is a base of $M_r \otimes A_1$ and B_2 is a base of $M_r \otimes A_2$ containing B_1, then let

$$\gamma_r(A_1, A_2) = \|B_2 \setminus B_1\|.$$

It follows from (P-2), (P-3), and (P-4) that γ_r is well-defined (does not depend on the choice of B_1 and B_2).

Suppose that $r \in R$ and that $\mathcal{A} = (A_\alpha)_{\alpha<\lambda}$, $\mathcal{B} = (B_\alpha)_{\alpha<\lambda}$ are transfinite sequences of subsets of S such that

(i) $A_\delta \subset A_\theta$ and $B_\delta \subset B_\theta$, for $\delta \leq \theta < \lambda$,

(ii) B_δ is a base of $M_r \otimes A_\delta$, for $\delta < \lambda$,

and suppose that B be a base of $M_r \otimes (\cup_{\alpha<\lambda} A_\alpha)$ such that

$$\bigcup_{\alpha<\lambda} B_\alpha \subset B.$$

Then we say that the pair $(\mathcal{B}, B)$ is a *proper M_r-choice of bases for $\mathcal{A}$*. If $\mathcal{A} = (A_\alpha)_{\alpha<\lambda}$ is a transfinite sequence of subsets of S, and λ is a limit ordinal, then let

$$\Gamma_r(\mathcal{A}) = \min\left\{\left\| B \setminus \bigcup_{\alpha<\lambda} B_\alpha \right\| : ((B_\alpha)_{\alpha<\lambda}, B) \text{ is a proper } M_r\text{-choice of bases for } \mathcal{A}\right\}.$$

By *queue* we mean a countable injective transfinite sequence, *i.e.* an injective transfinite sequence whose domain is a countable ordinal. If f is a queue, then $\mathrm{dom}(f)$ is the domain of f and $\mathrm{rge}(f)$ is the range of f. If $\mathrm{dom}(f) = \lambda$ and $\alpha \leq \lambda$, then f_α will denote the restriction of f to α. A *queue in S* is a queue whose range is a subset of S, and *λ-queue* is a queue with domain λ. With a λ-queue f in S we associate a quasi-integer $\eta(f)$, called the *$\mathcal{M}$-margin* of f. Let $\eta(f) = 0$ if $\lambda = 0$. Suppose now that $\lambda > 0$ and that $q(f')$ has been defined for every queue f' in S such that $\mathrm{dom}(f') < \lambda$. If $\lambda = \kappa + 1$ is a successor ordinal, then let

$$\eta(f) = \eta(f_\kappa) + 1 - \sum_{r\in R} \gamma_r(\mathrm{rge}(f_\kappa), \mathrm{rge}(f)).$$

If λ is a limit ordinal, then let

$$\eta(f) = \liminf_{\theta\to\lambda} \eta(f_\theta) - \sum_{r\in R} \Gamma_r((\mathrm{rge}(f_\alpha))_{\alpha<\lambda}).$$

We say that the system $\mathcal{M}$ of matroids is *good* if $\eta(f) \geq 0$ for every queue f in S.

We will show that the condition of being good is a necessary condition for a system of matroids to have a system of disjoint bases.

Theorem 1. *If $\mathcal{M} = (M_r)_{r\in R}$ is a system of matroids on S which have a system of disjoint bases, then $\mathcal{M}$ is good.*

Let $\mathcal{E} = (E_i)_{i\in I}$ be any family of subsets of S. A *transversal* of $\mathcal{E}$ is a subset A of S such that there is a bijection $\theta : I \to A$ satisfying $\theta(i) \in E_i$ for every $i \in I$. $A \subset S$ is a *partial transversal* of $\mathcal{E}$ if A is a transversal of a subfamily of $\mathcal{E}$. Further, the family $\mathcal{E}$ is said to be *restricted* if no element of S belongs to E_i for infinitely many values of $i \in I$. Edmonds and Fulkerson [4] (see also [5] Theorem 6.5.3) proved that if $\mathcal{E}$ is a restricted family of subsets of S, then the collection of partial transversals of $\mathcal{E}$ is a matroid on S. Such a matroid will be called a *transversal matroid*. Using a theorem of Nash-Williams [7] we shall prove the following theorem.

Theorem 2. *If $\mathcal{M}$ is a good countable system of rank-finite transversal matroids on S, then $\mathcal{M}$ has a system of disjoint bases.*

We would like to formulate the following conjecture.

Conjecture 1. *If $\mathcal{M}$ is a good countable system of rank-countable matroids on S, then $\mathcal{M}$ has a system of disjoint bases.*

2. Necessity of the condition.

Theorem 1 follows immediately from the following lemma.

Lemma 1. *If $(B_r)_{r\in R}$ is a system of disjoint bases of the system $(M_r)_{r\in R}$ of matroids on S and f is a queue in S, then*

$$\left\|\operatorname{rge}(f)\setminus\bigcup_{r\in R}B_r\right\|\leq\eta(f). \tag{1}$$

Proof. If a queue f in S satisfies the condition

$$\left\|\operatorname{rge}(f)\setminus\bigcup_{r\in R}B_r^f\right\|\leq\eta(f), \tag{2}$$

where B_r^f is any base of $M_r\otimes\operatorname{rge}(f)$ such that $B_r^f\subset B_r$, $r\in R$, then f satisfies (1). Therefore to prove the lemma, it is sufficient to prove that any queue f in S satisfies (2).

We are going to use transfinite induction. If $\operatorname{dom}(f)=0$, then both sides of (2) are equal to 0, so (2) is true. Now suppose that $\operatorname{dom}(f)=\lambda>0$ and that

$$\left\|\operatorname{rge}(g)\setminus\bigcup_{r\in R}B_r^g\right\|\leq q(g)$$

for any queue g in S such that $\operatorname{dom}(g)<\lambda$ and for any system $(B_r^g)_{r\in R}$ of bases of $(M_r\otimes\operatorname{rge}(g))_{r\in R}$ such that $B_r^g\subset B_r$, $r\in R$.

Suppose first that $\lambda=\kappa+1$ is a successor ordinal. Let B_r' be any base of $M_r\otimes\operatorname{rge}(f)$ contained in B_r, and let B_r'' be any base of $M_r\otimes\operatorname{rge}(f_\kappa)$ contained in

B'_r, $r \in R$. Then

$$\begin{aligned}\left\|\operatorname{rge}(f) \setminus \bigcup_{r\in R} B'_r\right\| &= \left\|\left(\left(\operatorname{rge}(f_\kappa) \setminus \bigcup_{r\in R} B''_r\right) \cup \{f(\kappa)\}\right) \setminus \bigcup_{r\in R}(B'_r \setminus B''_r)\right\| \\ &\le \left\|\operatorname{rge}(f_\kappa) \setminus \bigcup_{r\in R} B''_r\right\| + 1 - \left\|\bigcup_{r\in R} B'_r \setminus B''_r\right\| \\ &\le \eta(f_\kappa) + 1 - \sum_{r\in R} \gamma_r(\operatorname{rge}(f_\kappa), \operatorname{rge}(f)) \\ &= \eta(f).\end{aligned}$$

Now suppose that λ is a limit ordinal. Let B'_r be any base of $M_r \otimes \operatorname{rge}(f)$. Let $(B^\alpha_r)_{\alpha<\lambda}$ be a transfinite sequence of subsets of S such that B^α_r is a base of $M_r \otimes \operatorname{rge}(f_\alpha)$ contained in B'_r, and $B^\alpha_r \subset B^\theta_r$ for $\alpha < \theta < \lambda$. Then $((B^\alpha_r)_{\alpha<\lambda}, B'_r)$ is a proper M_r-choice of bases for $((\operatorname{rge}(f_\alpha))_{\alpha<\lambda})$ and

$$\begin{aligned}\left\|\operatorname{rge}(f) \setminus \bigcup_{r\in R} B'_r\right\| &= \left\|\left(\operatorname{rge}(f) \setminus \bigcup_{r\in R}\bigcup_{\alpha<\lambda} B^\alpha_r\right) \setminus \bigcup_{r\in R}\left(B'_r \setminus \bigcup_{\alpha<\lambda} B^\alpha_r\right)\right\| \\ &\le \left\|\operatorname{rge}(f) \setminus \bigcup_{r\in R}\bigcup_{\alpha<\lambda} B^\alpha_r\right\| - \sum_{r\in R}\left\|B'_r \setminus \bigcup_{\alpha<\lambda} B^\alpha_r\right\| \\ &\le \left\|\operatorname{rge}(f) \setminus \bigcup_{r\in R}\bigcup_{\alpha<\lambda} B^\alpha_r\right\| - \sum_{r\in R}\Gamma_r((\operatorname{rge}(f_\alpha))_{\alpha<\lambda}).\end{aligned}$$

We claim that

$$\left\|\operatorname{rge}(f) \setminus \bigcup_{r\in R}\bigcup_{\alpha<\lambda} B^\alpha_r\right\| \le \liminf_{\theta\to\lambda} \eta(f_\theta).$$

Indeed, we have

$$\liminf_{\theta\to\lambda} \eta(f_\theta) = \sup\{\xi_\delta : \delta < \lambda\},$$

where

$$\xi_\delta = \inf\{\eta(f_\theta) : \delta \le \theta < \lambda\}.$$

By the induction hypothesis we have

$$\eta(f_\theta) \ge \left\|\operatorname{rge}(f_\theta) \setminus \bigcup_{r\in R} B^\theta_r\right\| \ge \left\|\operatorname{rge}(f_\delta) \setminus \bigcup_{r\in R}\bigcup_{\alpha<\lambda} B^\alpha_r\right\|$$

for any θ such that $\delta \le \theta < \lambda$. Hence

$$\xi_\delta \ge \left\|\operatorname{rge}(f_\delta) \setminus \bigcup_{r\in R}\bigcup_{\alpha<\lambda} B^\alpha_r\right\|$$

for any $\delta < \lambda$. Therefore

$$\liminf_{\theta \to \lambda} \eta(f_\theta) \geq \sup \left\{ \left\| \mathrm{rge}(f_\delta) \setminus \bigcup_{r \in R} \bigcup_{\alpha < \lambda} B_r^\alpha \right\| : \delta < \lambda \right\} = \left\| \mathrm{rge}(f) \setminus \bigcup_{r \in R} \bigcup_{\alpha < \lambda} B_r^\alpha \right\|.$$

Thus the proof of the claim is complete.

Hence we have

$$\left\| \mathrm{rge}(f) \setminus \bigcup_{r \in R} B'_r \right\| \leq \liminf_{\theta \to \lambda} \eta(f_\theta) - \sum_{r \in R} \Gamma_r((\mathrm{rge}(f_\alpha))_{\alpha < \lambda}) = \eta(f),$$

and the proof of the lemma is complete. □

3. Proof of Theorem 2.

Let I be a countable set and let $\mathcal{E} = (E_i)_{i \in I}$ be a family of subsets of S. If $X \subset S$, then the $\mathcal{E}$-*demand set* $D(X)$ is defined by

$$D(X) = \{i \in I : E_i \subset X\}.$$

Let f be a λ-queue in S. Denote $\Delta(f) = D(\mathrm{rge}(f))$, and let the $\mathcal{E}$-*margin* $q(f)$ be defined as follows:

(i) $q(f) = -\|D(\emptyset)\|$ if $\lambda = 0$,

(ii) $q(f) = q(f_\kappa) + 1 - \|\Delta(f) \setminus \Delta(f_\kappa)\|$ if $\lambda = \kappa + 1$,

(iii) $q(f) = \liminf_{\theta \to \lambda} q(f_\theta) - \|\Delta(f) \setminus \cup_{\theta < \lambda} \Delta(f_\theta)\|$ if λ is a limit ordinal.

We say that $\mathcal{E}$ is *good* if $q(f) \geq 0$ for every queue f in S. To prove Theorem 2 we shall use the following theorem of Nash-Williams [7].

Theorem 3. *(Nash-Williams [7]) $\mathcal{E}$ has a transversal iff $\mathcal{E}$ is good.* □

Let

$$I = \bigcup_{r \in R} I_r$$

be a partition of I into finite subsets. For each $r \in R$, define $\mathcal{E}_r = (E_i)_{i \in I_r}$ to be the subfamily of $\mathcal{E}$ corresponding to I_r. Let M_r be the family of partial transversals of $\mathcal{E}_r$. Since I_r is finite it follows that $\mathcal{E}_r$ is a restricted family and that M_r is a rank-finite matroid. If f is a queue in S, then let $\mu_r(f)$ be the rank of the matroid $M_r \otimes \mathrm{rge}(f)$. Let $\mathcal{M} = (M_r)_{r \in R}$.

The following lemma holds since $\mathcal{M}$ is a system of rank-finite matroids.

Lemma 2. *If f is a λ-queue in S, then the $\mathcal{M}$-margin $\eta(f)$ satisfies the following conditions:*

(i) $\eta(f) = 0$ *if* $\lambda = 0$,

(ii) $\eta(f) = \eta(f_\kappa) + 1 - \sum_{r\in R} \left(\mu_r(f) - \mu_r(f_\kappa)\right)$ *if* $\lambda = \kappa + 1$,

(iii) $\eta(f) = \liminf_{\theta\to\lambda} \eta(f_\theta) - \sum_{r\in R} \left(\mu_r(f) - \max\left\{\mu_r(f_\theta) : \theta < \lambda\right\}\right)$ *if λ is a limit ordinal.*

Proof. Use transfinite induction. □

Assume that, for every $r \in R$, $\mathcal{E}_r$ has a transversal X_r, and that $\theta_r : I_r \to X_r$ is a bijection such that $\theta_r(i) \in E_i$ for every $i \in I_r$.

Assume that f is a queue in S. For each $r \in R$ let $\Delta_r(f)$ be the $\mathcal{E}_r$-demand set $D_r(\mathrm{rge}(f))$, and let $\Psi_r(f) = \theta_r(\Delta_r(f))$. In other words, $\Psi_r(f)$ is the part of the transversal X_r which corresponds to the subset $\Delta_r(f)$ of I_r. Clearly $\Psi_r(f) \in M_r \otimes \mathrm{rge}(f)$. Let $\psi_r(f) = \|\Psi_r(f)\|$, and let $\zeta_r(f) = \mu_r(f) - \psi_r(f)$. Set

$$\zeta(f) = \sum_{r\in R} \zeta_r(f).$$

The following lemmas are satisfied.

Lemma 3. *If f is a λ-queue in S and $\theta < \lambda$, then*

$$\|\Delta(f) \setminus \Delta(f_\theta)\| = \sum_{r\in R} (\psi(f) - \psi(f_\theta)).$$

Proof. We have

$$\begin{aligned}
\|\Delta(f) \setminus \Delta(f_\theta)\| &= \left\| \bigcup_{r\in R} (\Delta_r(f) \setminus \Delta_r(f_\theta)) \right\| \\
&= \sum_{r\in R} (\|\Delta_r(f)\| - \|\Delta_r(f_\theta)\|) \\
&= \sum_{r\in R} (\psi_r(f) - \psi_r(f_\theta)).
\end{aligned}$$

□

Lemma 4. *If λ is a limit ordinal, if f is a λ-queue in S, and if, for every $r \in R$, $\delta_r < \lambda$ is an ordinal such that*

$$\zeta_r(f_\gamma) = \zeta_r(f_{\delta_r})$$

for every ordinal γ satisfying $\delta_r \leq \gamma < \lambda$, then

$$\sum_{r \in R} \zeta_r(f_{\delta_r}) \leq \liminf_{\theta \to \lambda} \zeta(f_\theta).$$

Proof. If $\theta < \lambda$, then let

$$R_\theta = \big\{ r \in R : \zeta_r(f_\gamma) = \zeta_r(f_\theta) \text{ for } \theta \leq \gamma < \lambda \big\}.$$

Clearly, we have $r \in R_{\delta_r}$ so

$$\bigcup_{\theta < \lambda} R_\theta = R.$$

Moreover, we have $R_\theta \subset R_\gamma$ for $\theta \leq \gamma < \lambda$, and $\zeta_r(\theta) \geq 0$ for any $\theta < \lambda$ and $r \in R$. Therefore

$$\begin{aligned}
\sum_{r \in R} \zeta_r(f_{\delta_r}) &= \sup \left\{ \sum_{r \in R_\theta} \zeta_r(f_{\delta_r}) : \theta < \lambda \right\} \\
&= \liminf_{\theta \to \lambda} \sum_{r \in R_\theta} \zeta_r(f_{\delta_r}) \\
&= \liminf_{\theta \to \lambda} \sum_{r \in R_\theta} \zeta_r(f_\theta) \\
&\leq \liminf_{\theta \to \lambda} \sum_{r \in R} \zeta_r(f_\theta) \\
&= \liminf_{\theta \to \lambda} \zeta(f_\theta).
\end{aligned}$$

□

The following lemma, relating the $\mathcal{E}$-margin and the $\mathcal{M}$-margin, is the key step in the proof of Theorem 2.

Lemma 5. *For any queue f in S, $\eta(f) \leq q(f) - \zeta(f)$.*

Proof. Let $\mathrm{dom}(f) = \lambda$. We shall use transfinite induction. If $\lambda = 0$, then $\eta(f) = q(f) = \zeta(f) = 0$ so the inequality stated in the lemma is satisfied.

Now assume that $\lambda > 0$ and that we have $\eta(g) \leq q(g) - \zeta(g)$ for any queue g in S such that $\mathrm{dom}(g) < \lambda$. If $\lambda = \kappa + 1$, then by the inductive assumption and

Lemma 3, we have:

$$\begin{aligned}
\eta(f) &= \eta(f_\kappa) + 1 - \sum_{r\in R}(\mu_r(f) - \mu_r(f_\kappa)) \\
&\le q(f_\kappa) - \sum_{r\in R}\zeta_r(f_\kappa) + 1 - \sum_{r\in R}(\mu_r(f) - \mu_r(f_\kappa)) \\
&= q(f_\kappa) + 1 - \sum_{r\in R}\big(\zeta_r(f_\kappa) + \mu_r(f) - \mu_r(f_\kappa)\big) \\
&= q(f_\kappa) + 1 - \sum_{r\in R}(\mu_r(f) - \psi_r(f_\kappa)) \\
&= q(f_\kappa) + 1 - \sum_{r\in R}(\zeta_r(f) + \psi_r(f) - \psi_r(f_\kappa)) \\
&= q(f_\kappa) + 1 - \sum_{r\in R}(\psi_r(f) - \psi_r(f_\kappa)) - \sum_{r\in R}\zeta_r(f) \\
&= q(f_\kappa) + 1 - \|\Delta(f) \setminus \Delta(f_\kappa)\| - \zeta(f) \\
&= q(f) - \zeta(f).
\end{aligned}$$

Assume now that λ is a limit ordinal. Since I_r is a finite set and M_r is a rank-finite matroid, for each $r \in R$, there is an ordinal $\delta_r < \lambda$ such that

$$\mu_r(f_{\delta_r}) = \max\{\mu_r(f_\theta) : \theta < \lambda\}, \tag{3}$$

and

$$\Delta_r(f_{\delta_r}) = \bigcup_{\theta<\lambda} \Delta_r(f_\theta).$$

Then we also have

$$\zeta_r(f_\gamma) = \zeta_r(f_{\delta_r}),$$

for any ordinal γ such that $\delta_r \le \gamma < \lambda$.

Moreover, by Lemma 3, we have:

$$\begin{aligned}
\left\|\Delta(f) \setminus \bigcup_{\theta<\lambda}\Delta(f_\theta)\right\| &= \left\|\bigcup_{r\in R}\left(\Delta_r(f) \setminus \bigcup_{\theta<\lambda}\Delta_r(f_\theta)\right)\right\| \\
&= \left\|\bigcup_{r\in R}\left(\Delta_r(f) \setminus \Delta_r(f_{\delta_r})\right)\right\| \\
&= \sum_{r\in R}\|\Delta_r(f) \setminus \Delta_r(f_{\delta_r})\| \\
&= \sum_{r\in R}(\psi_r(f) - \psi_r(f_{\delta_r})).
\end{aligned} \tag{4}$$

Using the inductive hypothesis, Lemma 4, equations (3) and (4), and the inequality

$$\liminf_{\theta\to\lambda} q(f_\theta) - \liminf_{\theta\to\lambda} \zeta(f_\theta) \geq \liminf_{\theta\to\lambda}(q(f_\theta) - \zeta(f_\theta)),$$

we get

$$\begin{aligned}
q(f) - \zeta(f) &= \liminf_{\theta\to\lambda} q(f_\theta) - \left\| \Delta(f) \setminus \bigcup_{\theta<\lambda} \Delta(f_\theta) \right\| - \sum_{r\in R}(\mu_r(f) - \psi_r(f)) \\
&= \liminf_{\theta\to\lambda} q(f_\theta) - \sum_{r\in R}(\psi_r(f) - \psi_r(f_{\delta_r})) - \sum_{r\in R}(\mu_r(f) - \psi_r(f)) \\
&= \liminf_{\theta\to\lambda} q(f_\theta) - \sum_{r\in R}(\mu_r(f) - \psi_r(f_{\delta_r})) \\
&= \liminf_{\theta\to\lambda} q(f_\theta) - \sum_{r\in R}(\mu_r(f) + \zeta_r(f_{\delta_r}) - \mu_r(f_{\delta_r})) \\
&= \liminf_{\theta\to\lambda} q(f_\theta) - \sum_{r\in R}\zeta_r(f_{\delta_r}) - \sum_{r\in R}(\mu_r(f) - \mu_r(f_{\delta_r})) \\
&\geq \liminf_{\theta\to\lambda} q(f_\theta) - \liminf_{\theta\to\lambda} \zeta(f_\theta) - \sum_{r\in R}(\mu_r(f) - \mu_r(f_{\delta_r})) \\
&\geq \liminf_{\theta\to\lambda}(q(f_\theta) - \zeta(f_\theta)) - \sum_{r\in R}(\mu_r(f) - \mu_r(f_{\delta_r})) \\
&\geq \liminf_{\theta\to\lambda} \eta(f_\theta) - \sum_{r\in R}(\mu_r(f) - \max\{\mu_r(f_\theta) : \theta < \lambda\}) \\
&= \eta(f)
\end{aligned}$$

□

The following theorem, which is a countable version of König's theorem, was proved by Podewski and Steffens ([11], section 3.).

Theorem 4. *Any countable bipartite graph has a matching F and a cover C such that C contains exactly one vertex from each edge in F.* □

The following lemma is implied by Theorem 4.

Lemma 6. *Let J and T be countable sets. If $\mathcal{A} = (A_i)_{i\in J}$ is a family of sets on T, then there is a subset $J' \subset J$ such that*

(i) the family $\mathcal{A}' = (A_i)_{i\in J'}$ has a transversal, and

(ii) if $X \subset T$, then X is a partial transversal of $\mathcal{A}$ iff X is a partial transversal of $\mathcal{A}'$.

Proof. Let G be the bipartite graph with bipartition (J, T) and such that $jt \in E(G)$

iff $t \in A_j$. Let F be a matching and C be a cover in G such that C contains exactly one vertex from each edge in F. Let J' be the subset of J defined as follows:

$$J' = \{j \in J : j \text{ is a vertex of an edge of } F\}.$$

Then (i) is clearly satisfied. We are going to show that (ii) is satisfied. Of course, if X is a partial transversal of $\mathcal{A}' = (A_i)_{i \in J'}$, then X is a partial transversal of $\mathcal{A}$. Suppose now that X is a partial transversal of $\mathcal{A}$. Let $Y \subset J$ and let $\theta : Y \to X$ be a bijection. Set

$$\begin{aligned} C_1 &= C \cap J, \\ C_2 &= C \cap T, \\ X_2 &= X \cap C_2, \\ X_1 &= X \setminus X_2, \\ Y_1 &= Y \cap C_1, \\ Y_2 &= Y \setminus Y_1. \end{aligned}$$

Since C is a cover of G we have

$$\theta(Y_2) \subset X_2$$

and

$$\theta^{-1}(X_1) \subset Y_1.$$

Let $\xi : X \to J'$ be defined by

$$\xi(x) = \begin{cases} \theta^{-1}(x) & \text{if } x \in X_1, \\ y_x & \text{if } x \in X_2, \end{cases}$$

where y_x is the vertex of G such that $y_x x$ is and edge of the matching F. It is easy to see that ξ is an injection. Hence X is a partial transversal of $\mathcal{A}'$. □

We can now conclude the proof of Theorem 2.

Proof of Theorem 2. Let S be any set and let $\mathcal{M} = (M_r)_{r \in R}$ be a good countable system of rank-finite transversal matroids on S. For each $r \in R$, let $\mathcal{E}_r = (E_i)_{i \in I_r}$ be a family of subsets of S such that M_r is the the family of partial transversals of $\mathcal{E}_r$. We can assume that $I_{r_1} \cap I_{r_2} = \emptyset$ for $r_1 \neq r_2$ and, by Lemma 6, that $\mathcal{E}_r$ has a transversal for any $r \in R$. Let

$$I = \bigcup_{r \in R} I_i,$$

and

$$\mathcal{E} = (E_i)_{i \in I}.$$

Since $\mathcal{M}$ is good it follows from Lemma 5 that $\mathcal{E}$ is good. Thus, by Theorem 3, $\mathcal{E}$ has a transversal X. Let $\theta : I \to X$ be a bijection such that $\theta(i) \in E_i$, and let $B_r = \theta(I_r)$, $r \in R$. Then it is easy to see that $(B_r)_{r \in R}$ is a system of disjoint bases for $\mathcal{M}$. $\square$

4. Edge-disjoint spanning trees.

Let $G = (V, E)$ be a countable graph, *i.e.* a graph with countably many vertices and edges (loops, multiple edges, and vertices of infinite degrees are allowed). Let $\mathcal{W}_G$ be the set of all patitions of V. If P_1 and P_2 are partitions of V, then we say that P_1 *precedes* P_2 ($P_1 \preceq P_2$) if for every $A \in P_2$ there is $B \in P_1$ such that $A \subset B$. It is well known that $(\mathcal{W}_G, \preceq)$ is a partially ordered set, and that if the subset $\mathcal{W}'$ of $\mathcal{W}_G$ is a chain, then $\mathcal{W}'$ has the least upper bound (denoted by l.u.b.$(\mathcal{W}')$).

If $\mathcal{P}$ is a λ-queue in $\mathcal{W}_G$, then we say that $\mathcal{P}$ is *proper* if either $\lambda = 0$ or $\lambda > 0$ and the following conditions are satisfied:

(i) $\mathcal{P}(0) = \{V\}$,

(ii) $\mathcal{P}(\theta + 1) = (\mathcal{P}(\theta) \setminus \{V_0\}) \cup \{V_0', V_0''\}$, where $V_0 \in \mathcal{P}(\theta)$ and $\{V_0', V_0''\}$ is a partition of V_0, for $\theta + 1 < \lambda$,

(iii) $\mathcal{P}(\gamma) = \text{l.u.b.}(\text{rge}(\mathcal{P}_\gamma))$, if $\gamma < \lambda$ is a limit ordinal.

Roughly speaking, $\mathcal{P}$ is proper if $\mathcal{P}(\theta+1)$ is obtained from $\mathcal{P}(\theta)$ by splitting one set into two sets, and for a limit ordinal γ the partition $\mathcal{P}(\gamma)$ is the least upper bound of all partitions which are before it in the queue $\mathcal{P}$.

If $P \in \mathcal{W}_G$, then let $E(P)$ be the set of edges of G whose end-vertices are in different sets of the partition P.

Let $\mathcal{P}$ be a proper λ-queue in $\mathcal{W}_G$. Clearly $\mathcal{P}_\theta$ is proper for any $\theta \leq \lambda$. If k is a positive integer, then the k-*margin* $\xi_k(\mathcal{P})$ is defined as follows:

(i) $\xi_k(\mathcal{P}) = 0$ if $\lambda = 0$ or $\lambda = 1$,

(ii) $\xi_k(\mathcal{P}) = \xi_k(\mathcal{P}_\theta)$ if $\lambda = \theta + 1$ and θ is a limit ordinal,

(iii) $\xi_k(\mathcal{P}) = \xi_k(\mathcal{P}_{\theta+1}) + \|E(\mathcal{P}(\theta + 1) \setminus E(\mathcal{P}(\theta))\| - k$ if $\lambda = \theta + 2$, and

(iv) $\xi_k(\mathcal{P}) = \liminf_{\theta \to \lambda} \xi_k(\theta)$ if λ is a limit ordinal.

We say that G is k-*good* if $\xi_k(\mathcal{P}) \geq 0$ for every proper queue $\mathcal{P}$ in $\mathcal{W}_G$.

We would like to formulate the following conjecture.

Conjecture 2. *Let G be a countable graph, and k be a positive integer. Then G has k edge-disjoint spanning trees if and only if G is k-good.*

The "only if" part of Conjecture 2 is easy to prove by transfinite induction, and we believe that the "if" part follows from Conjecture 1.

Oxley [10] gave an example of a countable graph G which has no 2 edge-disjoint spanning trees, and which satitsfies the condition

$$\|E(P)\| \geq 2(\|P\| - 1)$$

for every finite partition P of its set of vertices $V(G)$ (see Fig. 1, where vertices are denoted by small circles).

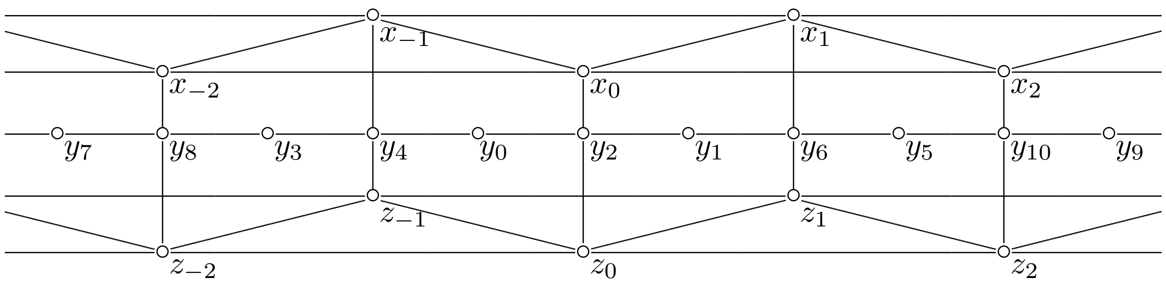

Fig. 1.

As an illustration of Conjecture 2, we will prove that the graph G given by Oxley is not 2-good. Let $\lambda = \omega + 2$, where ω is the first infinite ordinal, and let $\mathcal{P}$ be the proper λ-queue in $\mathcal{W}_G$ defined as follows:

$$\begin{aligned}
\mathcal{P}(i) &= \{\{y_0\}, \{y_1\}, \dots, \{y_{i-1}\}\} \cup \{V(G) \setminus \{y_0, y_1, \dots, y_{i-1}\}\}, \qquad i < \omega, \\
\mathcal{P}(\omega) &= \{\{y_0\}, \{y_1\}, \{y_2\}, \dots\} \cup \{\{x_i : i \in \mathbb{Z}\} \cup \{z_i : i \in \mathbb{Z}\}\}, \\
\mathcal{P}(\omega + 1) &= \{\{y_0\}, \{y_1\}, \{y_2\}, \dots\} \cup \{\{x_i : i \in \mathbb{Z}\}, \{z_i : i \in \mathbb{Z}\}\}.
\end{aligned}$$

It is easy to see that

$$\xi_2(\mathcal{P}_i) = 0, \qquad i < \omega,$$

so

$$\xi_2(\mathcal{P}_\omega) = \xi_2(\mathcal{P}_{\omega+1}) = 0,$$

and hence

$$\xi_2(\mathcal{P}) = -2.$$

Therefore G is not 2-good.

It may appear to some readers that it should not be necessary to deal with arbitrary transfinite sequences in Conjecture 2, *i.e.* that perhaps some modification of Conjecture 2 involving sequences of length at most ω would be true (at least for locally finite graphs). We believe that, even in the case of locally finite graphs, it is essential to consider arbitrary sequences in Conjecture 2. To illustrate this point let us consider the graph G from Fig. 2. (small circles denote vertices and double lines denote edges of multiplicity 2).

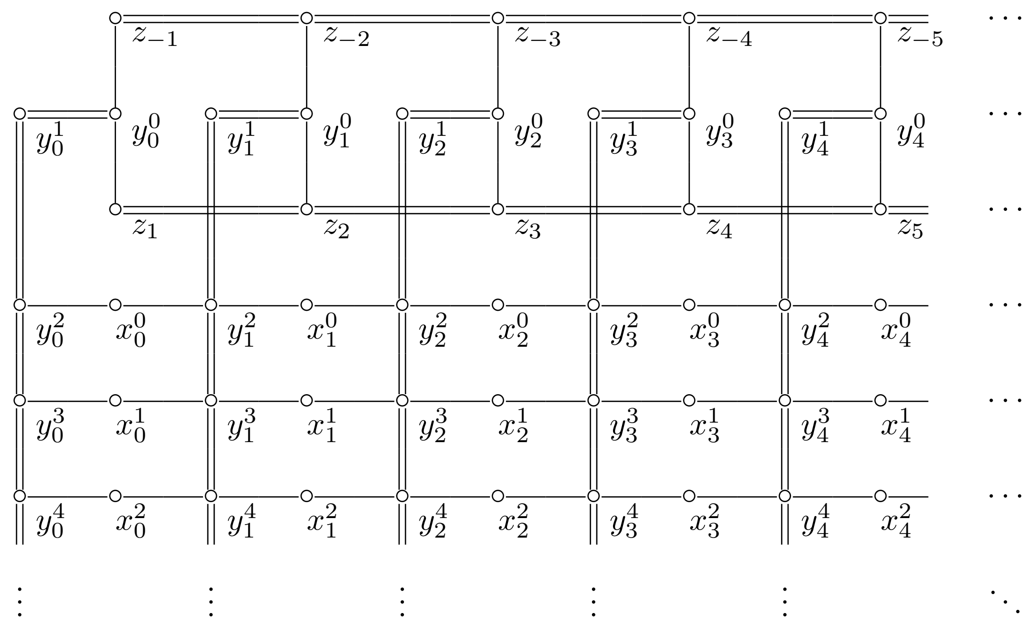

Fig. 2.

We claim that G is not 2-good. Let $\lambda = 2\omega + 2$. We shall define a λ-queue $\mathcal{P}$ in $\mathcal{W}_G$ such that $\xi_2(\mathcal{P}) < 0$. Let f be an ω-queue in $V(G)$ such that

$$\mathrm{rge}(f) = \{x_i^j : i, j = 0, 1, \dots\}.$$

For every $i = 0, 1, 2, \dots$, define

$$\begin{aligned}
X_i &= \{f(i)\}, \\
X_i' &= V(G) \setminus \bigcup_{j=0}^{i-1} X_j, \\
Y_i &= \{y_i^j : j = 0, 1, \dots\}, \\
Y_i' &= \left(V(G) \setminus \bigcup_{j=0}^{\infty} X_j\right) \setminus \bigcup_{j=0}^{i-1} Y_j.
\end{aligned}$$

Further, let

$$\begin{aligned}
Z_0 &= \{z_j : j \in \mathbb{Z} \setminus \{0\}\}, \\
Z_1 &= \{z_j : j > 0\}, \\
Z_{-1} &= \{z_j : j < 0\}.
\end{aligned}$$

Note that $X_0' = V(G)$, and

$$Y_0' = V(G) \setminus \bigcup_{j=0}^{\infty} X_j.$$

Let $\mathcal{P}$ be the λ-queue in $\mathcal{W}_G$ defined by:

$$\begin{aligned}
\mathcal{P}(i) &= \{X_0, X_1, \dots, X_{i-1}, X_i'\}, \qquad i = 0, 1, \dots\ , \\
\mathcal{P}(\omega + i) &= \{X_j : j = 0, 1, \dots\} \cup \{Y_0, Y_1, \dots, Y_{i-1}, Y_i'\}, \qquad i = 0, 1, \dots\ , \\
\mathcal{P}(2\omega) &= \{X_j : j = 0, 1, \dots\} \cup \{Y_j : j = 0, 1, \dots\} \cup \{Z_0\}, \\
\mathcal{P}(2\omega + 1) &= \{X_j : j = 0, 1, \dots\} \cup \{Y_j : j = 0, 1, \dots\} \cup \{Z_{-1}, Z_1\}.
\end{aligned}$$

It is easy to see that $\xi_2(\mathcal{P}_\alpha) = 0$, for $\alpha \leq 2\omega + 1$ and $\xi_2(\mathcal{P}) = -2$. Therefore G is not 2-good.

It is not hard to see that there is no λ-queue $\mathcal{P}$ in $\mathcal{W}_G$ such that $\lambda < 2\omega + 2$ and $\xi_2(\mathcal{P}) < 0$.

5. Detachments of countable graphs.

Let $G = (V, E)$ be a countable graph, and let $b : V(G) \to \{1, 2, \dots\} \cup \{\infty\}$ be a function. We say that the graph D is a b-detachment of G if

(i) $V(D) = \bigcup_{v \in V(G)} \Omega_v$, where $(\Omega_v)_{v \in V(G)}$ is a family of mutually disjoint sets such that $\|\Omega_v\| = b(v)$,

(ii) $E(D) = E(G)$, and

(iii) if $e \in E(G)$ joins vertices v and w in G, then e joins a vertex of Ω_v to a vertex of Ω_w in the graph D.

Nash-Williams [8] [9] proved that if G is finite, then there is a connected b-detachment D of G if and only if for every partition $\{V_1, V_2\}$ of $V(G)$ we have

$$|E_{V_1}| \geq \sum_{v \in V_1} b(v) + c(G[V_2]) - 1,$$

where E_{V_1} is the set of edges adjacent to a vertex in V_1 and $c(G[V_2])$ is the number of components of the graph spanned by V_2 in G.

The existence of a connected b-detachment of a finite graph G is equivalent to the existence of a family of disjoint bases of a particular family of matroids on $E(G)$ (see [9]). We will formulate a conjecture about b-detachments of countable graphs which is analogous to Conjecture 1.

Let g be a λ-queue in $V(G)$. Set $P(g)$ to be the partition of $V(G)$ such that the elements of $P(g)$ are the vertex sets of the components of the graph obtained from G by removing all the edges incident with a vertex in $\mathrm{rge}(g)$. Let

$$\overline{P}(g) = \mathrm{l.u.b.}(\{P(g_\theta) : \theta < \lambda\}).$$

If $v \in V(G)$ and $A \subset V(G)$, then let $v \bigtriangledown A$ be the set of edges of G which are incident to v and to a vertex in $V(G)$. Let the *b-margin* $\xi_b(g)$ be defined as follows:

(i) $\xi_b(g) = 1 - \|P(g)\|$ if $\lambda = 0$,

(ii) $\xi_b(g) = \xi_b(g_\theta) + \|g(\theta) \bigtriangledown (V(G) \setminus \mathrm{rge}(g_\theta))\| + \|P(g_\theta) \setminus P(g)\| + 1 - b(\theta) - \|P(g) \setminus P(g_\theta)\|$ if $\lambda = \theta + 1$,

(iii) $\xi_b(g) = \liminf_{\theta \to \lambda} \xi_b(g_\theta) + \|\overline{P}(g) \setminus P(g)\| - \|P(g) \setminus \overline{P}(g)\|$ if λ is a limit ordinal.

We say that G is *b-good* if $\xi_b(\mathcal{P}) \geq 0$ for every queue $\mathcal{P}$ in $V(G)$.

Conjecture 3. *Let G be a countable graph, and let $b : V(G) \to \{1, 2, \dots\} \cup \{\infty\}$ be a function. Then G has a connected b-detachment if and only if G is b-good.*

The "only if" part of Conjecture 3 is easy to prove by transfinite induction, and we believe that the "if" part follows from Conjecture 1.

As an illustration of Conjecture 3, let us consider the graph G shown at Fig. 3, and let the function $b : V(G) \to \{1, 2, \dots\} \cup \{\infty\}$ be defined by

$$b(v) = \begin{cases} 2 & \text{if } v = x_i^j \text{ or } v = y_i, \qquad i, j = 0, 1, 2, \dots, \\ 1 & \text{otherwise.} \end{cases}$$

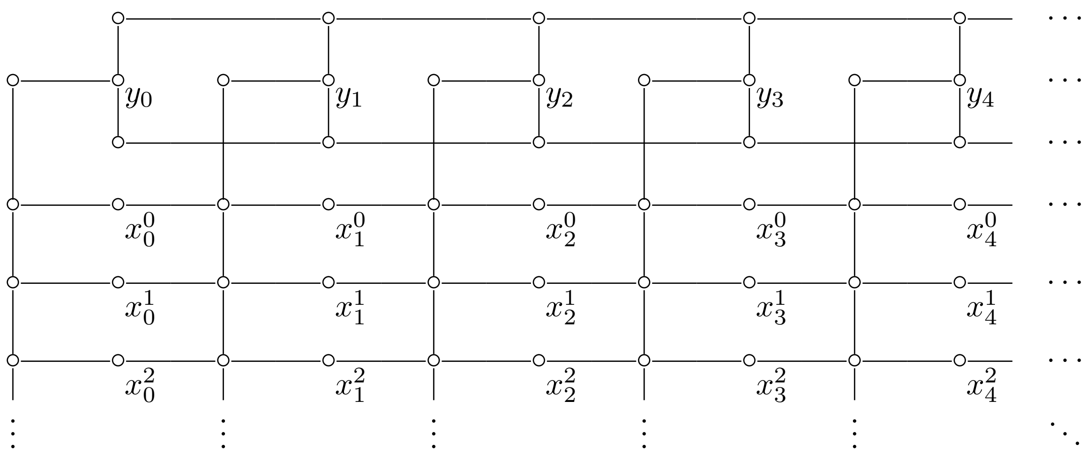

Fig. 3.

It is easy to see that G has no connected b-detachments. Let g be a 2ω-queue in $V(G)$ such that

$$\operatorname{rge}(g_\omega) = \{x_i^j : i, j = 0, 1, 2, \dots\},$$

and

$$g(\omega + i) = y_i, \qquad i = 0, 1, 2, \dots .$$

It is easy to see that

$$\xi_b(g_\omega) = 0,$$

hence

$$\xi_b(g_{\omega+i}) = 0, \qquad \text{for any } i < \omega,$$

and thus

$$\xi_b(g) = -1.$$

Therefore G is not b-good. Note that if f is a λ-queue in $V(G)$ and $\{y_i : i = 0, 1, \dots\}$ is not contained in $\mathrm{rge}(f)$, then $\xi_b(f) \geq 0$. Moreover, if $\lambda \leq \omega$, then

$$\xi_b(f_n) = \|\mathrm{rge}(f_n) \cap \{y_i : i = 0, 1, \dots\}\|,$$

for any $n < \lambda$ so $\xi_b(f) \geq 0$. Therefore, if $\lambda \leq \omega$, then $\xi_b(f) \geq 0$ for every λ-queue in $V(G)$. This observation shows that even when G is locally finite we cannot restrict our attention to ω-queues only.

Acknowledgements. This work was inspired by the seminars and lectures given by Dr Crispin Nash-Williams during his visit at West Virginia University in the academic year 92/93. The author would like to thank him for his encouragement and his extremely clear style of communicating complex ideas.

References

[1] Brualdi, R.A., Admissible mappings between dependence spaces, *Proc. London Math. Soc.* **21** (1970), 296–312.

[2] Brualdi, R.A., On families of finite independence structures, *Proc. London Math. Soc.* **22** (1971), 265–293.

[3] Edmonds, J., Lehman's switching game and a theorem of Tutte and Nash-Wiliams, *J. Res. Nat. Bur. Standards Sect. B* **69** (1965), 73–77.

[4] Edmonds, J., Fulkerson, D.R., Transversals and matroid partition, *J. Res. Nat. Bur. Standards* **69B** (1965), 147–153.

[5] Mirsky, L., Transversal Theory. An Account of Some Aspects of Combinatorial Mathematics, Academic Press, New York and London 1971.

[6] Nash-Williams, C.St.J.A., Edge-disjoint spanning trees of finite graphs, *J. London Math. Soc.* **36** (1961), 445-450.

[7] Nash-Williams, C.St.J.A., Another criterion for marriage in denumerable societies, *Annals of Discrete Mathematics* **3** (1978), 165–179.

[8] Nash-Williams, C.St.J.A., Connected detachments of graphs and generalized Euler trails, *J. London Math. Soc.(2),* **31** (1985), 17–29.

[9] Nash-Williams, C.St.J.A., Another proof of a theorem concerning detachments of graphs, *Europ. J. Comb.* **12** (1991), 245–247.

[10] Oxley J.G., On a packing problem for infinite graphs and independence spaces, *J. Comb. Theory Ser. B* **26** (1979), 123–130.

[11] Podewski, K.-P., Steffens, K.S., Injective choice functions for countable families, *J. Comb. Theory Ser. B* **21** (1976), 40-46.

[12] Tutte W.T., On the problem of decomposing a graph into n connected factors, *J. London Math. Soc.* **36** (1961), 221–230.

[13] Welsh, D.J.A., Matroid Theory (London Math. Soc. Monographs No. 8), Academic Press, New York and London 1976.