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Covering the Hypercube with a Bounded Number of Disjoint Snakes

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Abstract. We present a construction of an induced cycle in the n-dimensional hypercube $I[n]$ ($n \geq 2$), and a subgroup $\mathcal{H}_n$ of $I[n]$ considered as the group $\mathbb{Z}_2^n$, such that $|\mathcal{H}_n| \leq 16$ and the induced cycle uses exactly one element of every coset of $\mathcal{H}_n$. This proves that for any $n \geq 2$ the vertices of $I[n]$ can be covered using at most 16 vertex-disjoint induced cycles.

1. Introduction

Given a positive integer d, let $I[d]$ be the d-dimensional cube (also called hypercube) *i.e.* the graph with all d-tuples of binary digits as vertices, and all pairs of vertices differing at exactly one coordinate as edges. It will be convenient to think of the set of vertices of $I[d]$ as the set of elements of the group $\mathbb{Z}_2^d = (\mathbb{Z}/2\mathbb{Z})^d$ with the operation $\oplus$ of componentwise addition mod 2. Thus, when we refer to $I[d]$, we assume that it has both a structure of a graph and a structure of a group.

A *snake* is an induced cycle in $I[d]$. For each $d \geq 2$, let $S(d)$ denote the length of the longest snake in $I[d]$. An extensive literature has evolved concerning the problem of estimating $S(d)$. See [1], [3], [5], and the references in these papers. What we know now is that

$$\frac{77}{256}2^d \leq S(d) \leq 2^{d-1} - \frac{2^{d-1}}{20d-41},$$

assuming that $d \geq 12$ in the upper bound. The lower bound was proved by Abbott and Katchalski [2] and the upper bound by Snevily [4].

During the XXIII Southeastern International Conference, Boca Raton 1992, Erdős posed the problem of deciding whether there is a number k such that for every $d \geq 2$ the vertices of $I[d]$ can be covered using at most k snakes, and if the answer to the above problem is positive, then whether it can be done in such a way that the snakes are pairwise vertex-disjoint. In this note we show that the answer to both of the above questions is positive with $k = 16$. Actually, we prove the following theorem, which is a stronger result.

Theorem 1. *For every $n \geq 2$, there is a subgroup $\mathcal{H}_n \subset I[n]$ and a snake $C_n \subset I[n]$ such that $|\mathcal{H}_n| \leq 16$ and C_n uses exactly one element of every coset of $\mathcal{H}_n$.*

2. Basic Definitions

Let $\psi_0, \psi_1 : I[d] \to I[d+1]$ be the embeddings defined by

$$\psi_i(v_1 v_2 \dots v_d) = v_1\, v_2 \dots v_d\, i,$$

for $i = 0, 1$. If F is a subgraph of $I[d]$, let $F^{(i)}$ be the subgraph of $I[d+1]$ obtained as the image of F under the embedding ψ_i.

For each $d \geq 2$ we define a function $H_d : \{1, 2, \dots, 2^d\} \to V(I[d])$ such that $H_d^* = (H_d(1), \dots, H_d(2^d), H_d(1))$ is a Hamiltonian cycle in $I[d]$. We set

$$H_2^* = (00, 01, 11, 10, 00),$$

and

$$H_{d+1}(i) = \begin{cases} \big(H_d(i)\big)^{(0)} & \text{if } 1 \leq i \leq 2^d, \\ \big(H_d \circ R_d(i)\big)^{(1)} & \text{if } 2^d + 1 \leq i \leq 2^{d+1}, \end{cases}$$

where $R_d : \{2^d + 1, 2^d + 2, \dots, 2^{d+1}\} \to \{1, 2, \dots, 2^d\}$ is the order reversing bijection,

$$R_d(i) = 2^{d+1} + 1 - i.$$

In other words, H_{d+1}^* is obtained by taking $H_d^{*(0)}$ and $H_d^{*(1)}$, removing the edges connecting their last vertices with their first vertices, joining the first vertex of $H_d^{*(0)}$ with the first vertex of $H_d^{*(1)}$ and analogously the last with the last.

Let us regard $I[d+6]$ as $I[d] \times I[6]$, that is as the d-dimensional cube $I[d]$ with each vertex being a copy of $I[6]$. Suppose that $P_d = \big\{(v_j^1, \dots, v_j^{r_j})\big\}_{j=1}^{2^d}$ is a sequence of 2^d paths in $I[6]$ such that $v_i^{r_i} = v_j^1$ when $1 \leq i \leq 2^d - 1$ and $j = i + 1$ or $i = 2^d$ and $j = 1$. Such a sequence will be called a 2^d-*chain* of paths in $I[6]$. We can use P_d and H_d^* to construct a cycle C_{d+6} in $I[d+6]$. Let us take the jth path $(v_j^1, \dots, v_j^{r_j})$ in the copy of $I[6]$ corresponding to the vertex $H_d(j)$ in $I[d]$; see Figure 1 for the case $d = 3$.

Then, let us join $v_i^{r_i}$ from the ith copy of $I[6]$ with v_{i+1}^1 from the $(i+1)$st copy of $I[6]$ for all $i \in \{1, \dots, 2^d\}$, where the indices are understood circularly. Hence we have

$$C_{d+6} = \big((H_d(1), v_1^1), (H_d(1), v_1^2), \dots, (H_d(1), v_1^{r_1}), (H_d(2), v_2^1), \dots, (H_d(2), v_2^{r_2}), \dots, \\ (H_d(2^d), v_{2^d}^1), \dots, (H_d(2^d), v_{2^d}^{r_{2^d}}), (H_d(1), v_1^1)\big).$$

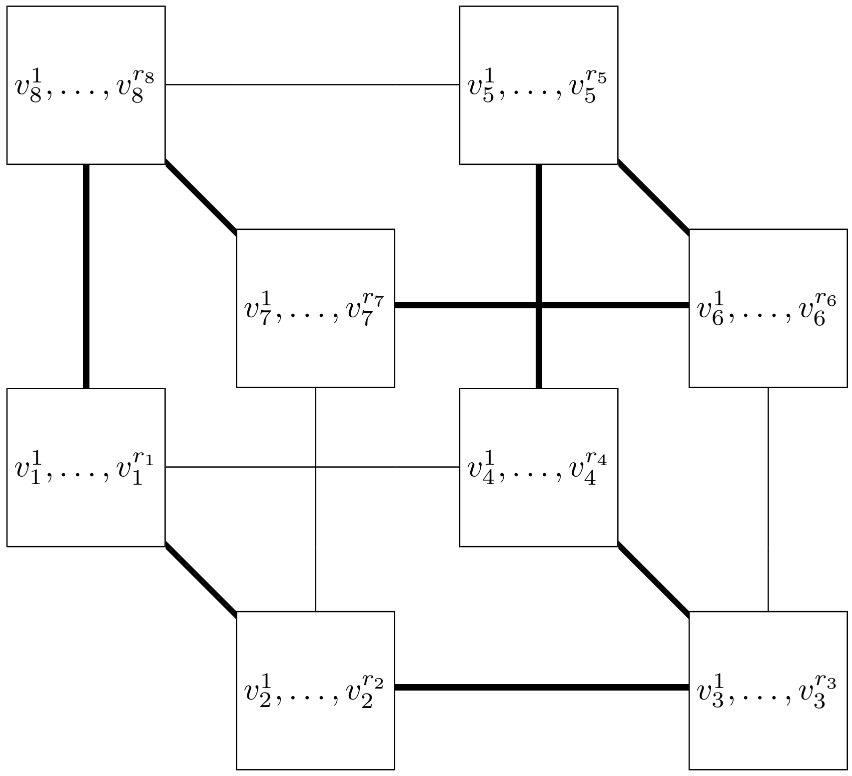

Fig. 1.

It is clear that C_{d+6} is a cycle in $I[d+6]$, we will call it the cycle *generated* by H_d and P_d. We call P_d *well separated* with respect to H_d if the cycle C_{d+6} generated by H_d and P_d is an induced cycle.

Let $\mathcal{P}$ be the set of all 12 paths in $I[6]$ that can be obtained from one of the following two paths

$$p_0 = (000000, 100000, 110000, 111000)$$
$$q_0 = (111000, 111100, 111110, 111111)$$

by a cyclic permutation of coordinates of every vertex of the path. Let $\mathcal{H}$ be the subgroup of $I[6]$ containing all $a_1a_2\ldots a_6 \in I[6]$ such that the following two conditions are satisfied

(i) $a_1 + a_3 + a_5$ is even, and

(ii) $a_2 + a_4 + a_6$ is even.

Note that $\mathcal{H}$ is generated by the set of elements of $I[6]$ that can be obtained from

101000 by cyclic permutations of the coordinates. Also note that the elements of $\mathcal{H}$ are 000000, six cyclic permutations of 101000, six cyclic permutations of 111100, and three cyclic permutations of 110110, so $|\mathcal{H}| = 16$. To prove Theorem 1, we will define a 2^d-chain P_d of paths which is well separated with respect to H_d and the paths in P_d are elements of $\mathcal{P}$. We will need the following lemma.

Lemma 1. *Every path $P \in \mathcal{P}$ uses exactly one element of each coset of $\mathcal{H}$ in $I[6]$.*

Proof. Let $P \in \mathcal{P}$, and let v_1, v_2 be two vertices of P. It easy to observe that if $v_1 \neq v_2$ then $v_1 \oplus v_2$ has an odd number of ones or two consecutive ones (in the cyclic order), where $\oplus$ is the operation of the group $I[6]$. Thus, if $v_1 \oplus v_2 \in \mathcal{H}$ then $v_1 = v_2$.

This proves that P uses at most one element of every coset of $\mathcal{H}$. Since $|P| \times |\mathcal{H}| = |I[6]|$, the path P uses exactly one element of every coset of $\mathcal{H}$, and the proof is complete. □

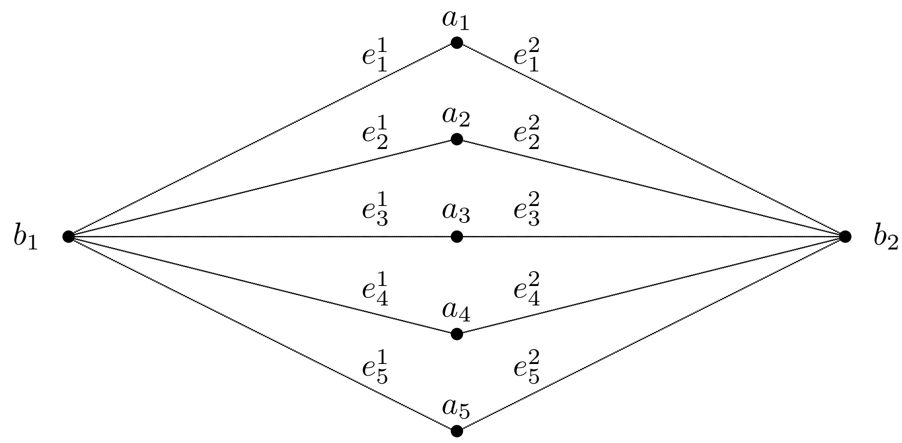

Fig. 2. The graph G

3. The Main Result

Let $G = K_{2,5}$ be the graph shown in Figure 2.

The following lemma is proved in Wojciechowski [5] (Lemma 3).

Lemma 2. *For every $d \geq 2$ there is a function $\Phi_d : \{1, 2, \dots, 2^d\} \to E(G)$ such that*

(i) if $1 \leq i \leq 2^d - 1$ and $j = i+1$, or else $i = 2^d$, $j = 1$, then $\Phi_d(i)$ and $\Phi_d(j)$ have exactly one vertex v_i in common, such that $v_i \in \{a_1, \dots, a_5\}$ for i even, and $v_i \in \{b_1, b_2\}$ for i odd, and

(ii) if $\big(H_d(i), H_d(j)\big) \in E(I[d]) \setminus E(H_d^*)$, then $\Phi_d(i)$ and $\Phi_d(j)$ are vertex-disjoint. $\square$

Let $\mathcal{P}$ be the set of paths defined in Section 2. The following lemma will be used in the proof of the main result.

Lemma 3. *For every $d \geq 2$ there is a 2^d-chain P_d of paths in $I[6]$ such that every path in P_d belongs to $\mathcal{P}$, and P_d is well separated with respect to H_d.*

Proof. Let G' be the subdivision of G obtained by subdividing e_k^1 $(1 \leq k \leq 5)$ with two new vertices c_k^1 and c_k^2 in such a way that we get the path (b_1, c_k^1, c_k^2, a_k), and subdividing e_k^2 with two new vertices c_k^4, c_k^5, giving rise to the path (a_k, c_k^4, c_k^5, b_2). Let $c_k^3 = a_k$ and

$\xi : V[G'] \to V(I[6])$ be defined as follows. Set

$$\xi(b_1) = 000000,$$

$$\xi(c_1^1) = 100000,$$

$$\xi(c_1^2) = 110000,$$

$$\xi(c_1^3) = 111000,$$

$$\xi(c_1^4) = 111100,$$

$$\xi(c_1^5) = 111110,$$

$$\xi(b_2) = 111111,$$

and if $\xi(c_1^i) = \alpha_1 \ldots \alpha_6$, then let

$$\xi(c_k^i) = \alpha_k \ldots \alpha_6 \alpha_1 \ldots \alpha_{k-1},$$

for $2 \leq k \leq 5$.

It is clear that the function ξ defines an embedding of G' into $I[6]$ such that the image of the subdivision of any edge of G is an induced path in $I[6]$. Let $\Phi_d : \{1, 2, \ldots, 2^d\} \to E(G)$ be a function satisfying conditions (i) and (ii) of Lemma 2. Let $P_d = \{(v_i^1, \ldots, v_i^4)\}_{i=1}^{2^d}$ be a 2^d-chain of paths in $I[6]$ such that $(v_i^1, \ldots, v_i^4)$ is the image under ξ of the subdivision of the edge $\Phi_d(i)$.

Let C_{d+6} be the cycle generated by H_d and P_d. Since each pair of vertex-disjoint edges in G corresponds to a pair of vertex-disjoint paths in P_d, and since each pair of edges having exactly one vertex in common corresponds to a pair of paths in P_d having exactly one vertex in common, it follows from (i) and (ii) of Lemma 2 that C_{d+6} is an induced cycle. So P_d is well separated with respect to H_d, and the proof is complete. □

We can now prove our main result.

Proof of **Theorem 1**. Let us assume first that $n \geq 8$. Set $d = n - 6$. Let P_d be a 2^d-chain of paths in $I[6]$ such that every path in P_d belongs to $\mathcal{P}$ and P_d is well separated with respect to H_d, and let C_n be the snake generated by P_d and H_d. Let $\mathcal{H}_n$ be the subgroup of $I[n]$ defined to be the set of all $(a_1, a_2, \ldots, a_d, b_1, \ldots, b_6) \in I[n]$ such that the following conditions are satisfied.

(*i*) $a_1 = a_2 = \ldots = a_d = 0$,

(*ii*) $b_1 + b_3 + b_5$ is even, and

(*iii*) $b_2 + b_4 + b_6$ is even.

We will show that C_n contains exactly one element of every coset of $\mathcal{H}_n$. Since $|C_n| = 4 \times 2^d$ and $|\mathcal{H}_n| = 16$, it is enough to prove that C_n contains at most one element of every coset of $\mathcal{H}_n$. Suppose $w_1, w_2 \in C_n$ and the cosets of w_1 and w_2 are equal. Then $w_1 \oplus w_2 \in \mathcal{H}_n$. Let $v_1 \in I[6]$ be the sequence of the last six digits of w_1, and similarly let $v_2 \in I[6]$ consist of the last six digits of w_2. Since $w_1 \oplus w_2 \in \mathcal{H}_n$, the first d digits of w_1 are the same as the first d digits of w_2, and $v_1 \oplus v_2 \in \mathcal{H}$. Thus v_1 and v_2 are vertices of the same path in $\mathcal{P}$, and so it follows from Lemma 1 that $v_1 = v_2$. Hence $w_1 = w_2$, and the proof of the case $n \geq 8$ is complete.

If $2 \leq n \leq 6$, then we can take $C_n = (000\ldots0, 100\ldots0, 110\ldots0, 010\ldots0, 000\ldots0)$, and $\mathcal{H}_n$ to be the set of all elements of $I[n]$ with first two coordinates equal 0. It is clear that C_n uses exactly one element of every coset of $\mathcal{H}_n$. In the remaining case $n = 7$, let $C_7 = (0000000, 1000000, 1100000, 1110000, 1111000, 0111000, 0011000, 0001000, 0000000)$, and let $\mathcal{H}_7$ be the set of all elements of $I[7]$ with the first four coordinates being either 0000 or 1010. It is straightforward to check that in this case the conclusion is true as well. Thus the proof is complete. □

The following corollary gives the answer to the problem of Erdős.

Corollary 1. *For every $n \geq 2$ the vertices of $I[n]$ can be covered using at most 16 pairwise disjoint snakes.*

Proof. Let us take a subgroup $\mathcal{H}_n$ and a snake C_n in $I[n]$ such that $|\mathcal{H}_n| \leq 16$ and C_n

uses exactly one element of every coset of $\mathcal{H}_n$. The family of snakes $\{C_n \oplus h : h \in \mathcal{H}_n\}$ contains 16 vertex-disjoint snakes covering $I[n]$. □

4. Concluding remarks

Let k_0 be the smallest integer such that for every $n \geq 2$, the cube $I[n]$ can be vertex covered by at most k_0 snakes. Let k_1 and k_2 be defined in a similar way taking pairwise vertex-disjoint snakes and pairwise vertex-disjoint snakes of equal length, respectively. Set k_3 to be the smallest integer such that for every $n \geq 2$ there is a subgroup $\mathcal{H}_n$ and a snake C_n in $I[n]$ such that $|\mathcal{H}_n| \leq k_3$ and C_n uses exactly one element of every coset of $\mathcal{H}_n$. As a corollary of Theorem 1 and the upper bound for the length of snakes we get the following theorem.

Theorem 2. *We have* $3 \leq k_0 \leq k_1 \leq k_2 \leq k_3 \leq 16$ *and* $k_2, k_3 \in \{4, 8, 16\}$. □

The question of determining the exact values of k_0, k_1, k_2, and k_3 remains open. It might be possible to modify the technique used in this note to improve Theorem 2 to make the upper bound equal 8 or perhaps even 4. The possible approach may involve finding a more sophisticated embedding of a subdivision of $K_{2,5}$ into $I[c]$, where c is a small integer constant (in our proof we used $c = 6$), or else replacing $K_{2,5}$ by another graph, perhaps $K_{2,4}$.

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