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On Small Graphs with Highly Imperfect Powers

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Abstract. Let an integer $s \geq 1$ and a graph G be given. Let us denote by $\chi_s(G)$ the smallest integer χ for which there exists a vertex-colouring of G with χ colours such that any two distinct vertices of the same colour are at a distance greater than s. Let us denote by $\omega_s(G)$ the maximal cardinality of a subset of the vertices of G with diameter at most s. Clearly $\chi_s(G) \geq \omega_s(G)$. For $s \geq 1$ and $h \geq 0$ set $\gamma_s(G) = \chi_s(G) - \omega_s(G)$ and

$$\nu_s(h) = \max \{n \in \mathbb{N} : \text{for any graph } G,\ |G| < n \text{ implies } \gamma_s(G) < h\}.$$

Gionfriddo [13] has given estimates for $\nu_s(h)$. We improve the recent bound $\nu_2(h) \leq 6h$ $(h \geq 3)$ of Gionfriddo and Milici [14] to $\nu_2(h) \leq 5h$ $(h \geq 3)$. More generally, we give the following tight bounds for arbitrary $s \geq 1$ and large enough h:

$$2h + \frac{1}{3\sqrt{2}}(h \log h)^{1/2} \leq \nu_s(h) \leq 2h + h^{1-\epsilon_s},$$

where $\epsilon_s > 0$ depends only on s. The upper bound is proved entirely by constructive methods.

1. Introduction

Let an integer $s \geq 1$ and a graph G be given. A vertex-colouring of G is said to be an L_s*-colouring* if any two distinct vertices of the same colour are separated by a distance greater than s. Let us denote by $\chi_s(G)$ the smallest integer χ such that there exists an L_s-colouring of G with χ colours. Thus an L_1-colouring is simply a proper vertex-colouring and $\chi_1(G)$ is the ordinary chromatic number of G. The parameters $\chi_s(G)$ have been studied in the literature by several authors. Amongst others, Antonucci [2] has found an upper bound for $\chi_2(G)$ for graphs G of girth at least 5 in terms of the number of vertices and the number of edges of G, a bound that has been shown to be best possible by Kramer and Kramer [18]. In [18], the authors also give upper bounds for $\chi_3(G)$ for bipartite and planar graphs G in terms of the maximal degree $\Delta(G)$ of G. Some authors have also considered the obvious analogue of L_s-colourings for edge-colourings. Indeed, an L_2-colouring of the line graph $L(G)$ of a graph G is known as a *strong edge-colouring of* G and $\chi_2(L(G))$ is usually referred to as the *strong chromatic index* of G. A well-known conjecture of Erdős and Nešetřil states that the strong chromatic index of a graph G is at most $5\Delta^2/4$, where $\Delta = \Delta(G)$ (see Faudree, Gyárfás, Schelp and Tuza [9] and Chung, Gyárfás, Tuza and Trotter [5]). Some more papers dealing with L_s-colourings are [17], [21] and [22].

In this note we shall study a variant of a very well-known extremal problem concerning the chromatic number. A classical result of Tutte (see [3]) says that triangle-free graphs of arbitrarily high chromatic number exist, and Erdős (cf. [8]) has posed the question of estimating the minimal order $f(q)$ of a triangle-free graph with chromatic number q. The best current bounds for $f(q)$ are as follows: for some constants c_1 and c_2 and large enough q

$$q^2(\log q)^{c_1} < f(q) < q^2(\log q)^{c_2},$$

and hence $f(q)$ is known up to a $(\log q)^c$ factor only. A natural variant of this problem is that of determining the smallest possible order of a graph whose chromatic and clique numbers differ by at least a fixed constant. More generally, we can ask the corresponding question for L_s-colourings and in this note we shall see that, rather surprisingly, one can give very precise results concerning this problem.

Let us denote by $\omega_s(G)$ the maximal cardinality of a subset of $V(G)$ of diameter at most s. Clearly $\chi_s(G) \geq \omega_s(G)$. The main question we shall study here is the following problem raised by Gionfriddo and others (cf. [13] and [15]). If G is a graph with $\chi_s(G) - \omega_s(G) \geq h$ how small can the order $|G|$ of G be?

For $s \geq 1$ and $h \geq 0$, let us set $\gamma_s(G) = \chi_s(G) - \omega_s(G)$ and

$$\nu_s(h) = \max\{n \in \mathbb{N} : \text{for any graph } G,\, |G| < n \text{ implies } \gamma_s(G) < h\}.$$

We are then interested in estimating the function ν_s. As one would expect, the exact value of $\nu_s(h)$ is known only for very few s and h. For instance, $\nu_2(1) = 7$ and $\nu_2(2) = 11$ are the only exact results for $s = 2$ (see [11] and [12]). Moreover, having established that $15 \leq \nu_2(3) \leq 18$, Gionfriddo [13] asks what the value of $\nu_2(3)$ is. More generally, Gionfriddo and Milici [14] have proved that $\nu_2(h) \leq 6h$ for $h \geq 3$. As to $\nu_s(h)$ for $s \geq 3$, the following estimates are proved in [10]. For $h \geq 3$,

$$\nu_s(h) \leq \frac{1}{2}(3s+1)(h+1)$$

if s is odd and

$$\nu_s(h) \leq \frac{1}{2}(3s+4)(h+1) - 2$$

if s is even. Our main concern in this note is to improve the bounds above. We first show that $\nu_2(h) \leq 5h$ for $h \geq 3$, proving that $\nu_2(3) = 15$. We then study the growth of $\nu_s(h)$ as a function of h; we shall prove that for fixed $s \geq 1$ and sufficiently large h

$$2h + \frac{1}{3\sqrt{2}}(h \log h)^{1/2} \leq \nu_s(h) \leq 2h + h^{1-\epsilon_s}, \tag{1}$$

where $\epsilon_s > 0$ is a constant which depends only on s; in particular $\nu_s(h) = (2 + o(1))h$ for any fixed $s \geq 1$ and $h \to \infty$.

Let us also mention that the upper bound in (1) improves previous bounds for certain related functions [13]. Let us denote by $m_s(h)$ the smallest number of *edges* in a graph G with $\gamma_s(G) \geq h$. Let us define $\delta_s(h)$ to be the smallest integer n such that there is a graph G of diameter s that can be extended to a graph G' with (i) $\gamma_s(G') \geq h$ and (ii) $|G'| - |G| \leq n$. Upper bounds for $m_s(h)$ and $\delta_s(h)$ trivially follow from (1); it turns out that they are better than those in [13].

Let us introduce some of the definitions we shall need. We generally follow [3] for graph-theoretical terms. In particular, given a graph G, a *walk* in G is a sequence $v_0, v_1, \ldots, v_\ell$ of vertices of G such that $v_{i-1}v_i \in E(G)$ for all $1 \leq i \leq \ell$. The *length* of the walk above is defined to be ℓ; it is said *to connect* v_0 *to* v_ℓ and thus it is referred to as a v_0–v_ℓ *walk*. For convenience, we write $\chi(G) = \chi_1(G)$, $\omega(G) = \omega_1(G)$ and $\gamma(G) = \gamma_1(G)$. The complement of G is denoted by G^c. The independence number of G is denoted by $\alpha(G)$, hence $\alpha(G) = \omega(G^c)$. Given a graph G and $s \geq 1$, we define its sth power G^s to be the graph on $V(G)$ with two distinct vertices joined to each other if and only if the distance between them is at most s. Note that then $\chi_s(G)$ and $\omega_s(G)$ are simply the ordinary chromatic and clique numbers of G^s and thus $\gamma_s(G) = \gamma(G^s)$.

Finally we outline the organisation of this note. We shall prove the new upper bound for $\nu_2(h)$, $h \geq 3$, in Section 2. In the following section we give some preliminary results for the case $s \geq 1$, and draw some easy corollaries concerning $\nu_s(h)$ from estimates on certain Ramsey numbers. In particular, we give the proof of the lower bound in (1). In Section 4 we describe the key result, Theorem 9, and then prove the upper bound in (1) as a corollary. (A more precise statement of this bound is given in Corollary 10.) The proof of Theorem 9 is given in Section 5.

2. A new upper bound for $\nu_2(h)$

Let us start by showing a simple construction that allows us to improve the upper bound on $\nu_2(h)$ of Gionfriddo and Milici [14]. This construction is in fact a special case of a much more general one considered in Sections 4 and 5.

Theorem 1. *For every $h \geq 3$ we have $\nu_2(h) \leq 5h$.*

Proof. Let us fix $h \geq 3$. It is enough to construct a graph G with $|G| = 5h$ and $\gamma_2(G) = h$. Let C^5 be a cycle of order 5 and K^h a complete graph of order h. Let us define the graph G on $V(C^5) \times V(K^h)$ by joining the vertices (c, k) to (c', k') iff $cc' \in E(C^5)$ and $kk' \in E(K^h)$

Obviously $|G| = 5h$. As pointed out in the introduction, $\gamma_2(G) = \gamma(G^2)$ and so we proceed to compute G^2.

We claim that G^2 is the complement of the disjoint union of h pentagons, i.e. cycles of order 5, say $C_1, C_2, \ldots, C_h$. The claim implies that $\gamma(G^2) = h$. Indeed, a maximal clique in G^2 has cardinality $2h$ (two nonconsecutive vertices in each C_i), and the chromatic number of G^2 is $3h$ (three colours for each C_i). Therefore, it only remains to check the claim.

Let $v_1 = (c_1, k_1)$, $v_2 = (c_2, k_2)$ be a pair of distinct vertices of G. We shall show that their distance $d(v_1, v_2)$ in G is greater than 2 if and only if c_1 is adjacent to c_2 in C^5 and $k_1 = k_2$. Note that this proves the claim. Let us consider the following three cases.

Case 1. $c_1c_2 \in E(C^5)$ and $k_1 = k_2$.

By definition (c_1, k_1) is not adjacent to (c_2, k_2) in G since $k_1 = k_2$. No vertex of G is adjacent to both (c_1, k_1) and (c_2, k_2) since no vertex of C^5 is adjacent to both c_1 and c_2. Therefore $d((c_1, k_1), (c_2, k_2)) \geq 3$ in G.

Case 2. $c_1c_2 \in E(C^5)$ and $k_1 \neq k_2$.

By definition the vertices (c_1, k_1) and (c_2, k_2) are adjacent in G.

Case 3. $c_1c_2 \notin E(C^5)$ (including the case $c_1 = c_2$).

There is a vertex adjacent to both c_1 and c_2 in C^5, and there is a vertex adjacent to both k_1 and k_2 in K^h (since $h \geq 3$). So $d((c_1, k_1), (c_2, k_2)) \leq 2$ in G concluding the proof of Theorem 1.

□

The theorem above solves the question about the determination of $\nu_2(3)$, posed by Gionfriddo in [13]. He has proved that $\nu_2(3) \geq 15$ and from Theorem 1 it follows that $\nu_2(3) \leq 15$, so we obtain that 15 is the exact value of $\nu_2(3)$.

3. Bounds arising from estimates on Ramsey numbers

In this section we start a more systematic study of $\nu_s(h)$ for arbitrary $s \geq 1$. Let us first consider the case $s = 1$ and remark that certain bounds for Ramsey numbers give us rather good information about $\nu_1(h)$. As usual, let us denote by $R(s,t)$ the smallest positive integer n such that any graph of order at least n has either a clique of order at least s or an independent set of order at least t. Erdős [6], with an ingenious probabilistic proof, established that

$$R(s,3) \geq c(s/\log s)^2 \tag{2}$$

for some $c > 0$. In fact (2) holds for any $0 < c < 1/27$ and large enough s, cf. [4], Chapter XII, §2. The following result is an immediate corollary of (2).

Theorem 2. *For sufficiently large h,*

$$\nu_1(h) < 2h + 20h^{1/2}\log h.$$

Proof. By taking $s = \lfloor (n/c)^{1/2}\log n \rfloor$, it can be easily checked that Erdős's lower bound for $R(s,3)$ tells us the following: for any $0 < c < 1/27$ there is an integer $n_0 = n_0(c)$ such that, for any $n \geq n_0$, there is a graph of order n with clique number less than $(n/c)^{1/2}\log n$ and independence number at most 2. Let us fix $c = 1/28$ and a large enough h (it will be clear that our inequalities hold if $h \geq h_0$, where h_0 is an absolute constant). Let n satisfy

$$\frac{n}{2} - \left(\frac{n}{c}\right)^{1/2}\log n \geq h > \frac{n-1}{2} - \left(\frac{n-1}{c}\right)^{1/2}\log(n-1) \geq \frac{n}{3}$$

and $n \geq n_0(c)$. Let G be a graph of order n with $\omega(G) < (n/c)^{1/2}\log n$ and $\alpha(G) \leq 2$. Clearly $\chi(G) \geq n/2$ and so

$$\gamma(G) \geq \frac{n}{2} - \left(\frac{n}{c}\right)^{1/2}\log n \geq h.$$

Moreover, by the choice of n,

$$\begin{aligned} |G| &= n \\ &\leq 2h + 2\left(\frac{n-1}{c}\right)^{1/2}\log(n-1) + 1 \\ &< 2h + 20h^{1/2}\log h. \end{aligned}$$

Hence this G proves the bound in the theorem. □

We now turn our attention to arbitrary $s \geq 1$. An obvious way of generalising Theorem 2 is to prove the existence of graphs with large order and small clique and independence numbers which are, furthermore, powers. Neither the probabilistic approach of Erdős in [6] nor a more recent one by Spencer [20] based on the Erdős–Lovász sieve seems to be directly applicable; we shall use instead an explicit construction of Erdős [7] which proves that $R(s,3)$ grows at least as fast as a power of s.

In order to describe Erdős's construction, let us define the *n-dimensional cube* Q^n as the graph whose vertices are the 0–1 sequences of length n, two of them being adjacent iff they differ in exactly one coordinate. The graph Q^n induces a natural metric on its set of vertices; let us denote this metric by d. Hence $d(x, y)$, which is usually called the *Hamming distance* between x and y, is simply the number of coordinates in which x and y differ. Erdős's graph J_r, $r \geq 1$, has as its set of vertices the 0–1 sequences of length $3r + 1$, two distinct vertices being adjacent iff their distance is at most $2r$. Thus J_r is the $2r$th power of Q^{3r+1}.

It is easy to check that in J_r any three distinct vertices span at least one edge. The fact that it has only small cliques is a consequence of the following theorem conjectured by Erdős and proved by Kleitman [16].

Theorem 3. *Let n and $r \geq 1$ be integers with $n \geq 2r$. Let $S \subset Q^n$ be a set vertices of the n-dimensional cube Q^n with diameter at most $2r$. Then*

$$|S| \leq \sum_{i=0}^{r} \binom{n}{i}. \qquad \square$$

We thus have the following.

Theorem 4.

(i) The independence number of J_r is 2 for all $r \geq 1$.

(ii) Set $c = (5 \log 2 - 3 \log 3)/(3 \log 2) = \cdot 0817\ldots$ and let $0 < \epsilon < c$. Then, for $r \geq r_0(\epsilon)$,

$$\omega(J_r) < |J_r|^{1-\epsilon}/2. \qquad \square$$

In particular, we conclude that

$$\gamma(J_r) > \frac{1}{2}|J_r| \left(1 - |J_r|^{-c+o(1)}\right) \tag{3}$$

as $r \to \infty$. Since J_r has an sth root when s divides r, we immediately notice the following.

Corollary 5. *For all $s \geq 1$, we have that $\liminf_h \nu_s(h)/h \leq 2$.*

Proof. For all s and $t \geq 1$, let us define the graph $J(s, t)$ on 0–1 sequences of length $3st + 1$ by joining two distinct sequences iff their distance is at most $2t$. Clearly $J(s, t)^s = J_{st}$ for all s and t. This remark coupled with inequality (3) completes the proof. $\square$

A moment's thought reveals that the drawback of using the J_r only is that the set $\{|J_r| : r \geq 1\}$ is much too sparse. Indeed, with such an approach we can merely conclude that $\limsup_h \nu_1(h)/h \leq 16$. In the next section, we introduce a technique to generate more graphs F with large $\gamma(F^s)$ and thus improve Corollary 5.

Let us now turn to the problem of bounding $\nu_1(h)$ from below. Trivially, $\nu_s(h) \geq \nu_t(h)$ for all h if t divides s; hence the lower bound we shall prove for $\nu_1(h)$ bounds $\nu_s(h)$ for arbitrary $s \geq 1$ as well. We shall need the following simple lemma.

Lemma 6. *Let G be a graph. Then for any induced subgraph H of G*

$$|G| \geq 2\gamma(G) + \omega(G) + |H| - 2\chi(H).$$

Proof. Let us set $G' = G - V(H)$. Note that

$$\chi(G') + \chi(H) \geq \chi(G) = \gamma(G) + \omega(G),$$

and so

$$\chi(G') \geq \gamma(G) + \omega(G) - \chi(H). \tag{4}$$

Clearly, in a proper minimal colouring of a graph the union of any two colour classes must span an edge. Hence, in such a colouring, the set of vertices which are assigned colours which occur only once must span a complete graph. Thus

$$|G'| \geq 2\chi(G') - \omega(G').$$

By (4) we conclude that

$$\begin{aligned}|G'| &\geq 2(\gamma(G) + \omega(G) - \chi(H)) - \omega(G')\\ &\geq 2\gamma(G) + \omega(G) - 2\chi(H).\end{aligned}$$

As $|G| = |G'| + |H|$, the proof is complete. $\square$

A way of applying the lemma above is to take $V(H)$ to be an independent set of order $\alpha(G)$. Doing so, we conclude that

$$\begin{aligned}|G| &\geq 2\gamma(G) + \omega(G) + \alpha(G) - 2\\ &> 2\gamma(G) + (\log|G|)/\log 4,\end{aligned} \tag{5}$$

where the second inequality follows from the well-known bound of Erdős and Szekeres

$$R(s,s) \leq \binom{2s-2}{s-1} < \frac{1}{6}4^s s^{-1/2},$$

for $s \geq 4$. We can in fact improve the log term in (5) by choosing a better subgraph H; we shall make use of an upper bound for off-diagonal Ramsey numbers to find a suitable H.

Ajtai, Komlós and Szemerédi [1] were the first to prove that

$$R(s,3) = O(s^2/\log s),$$

and Shearer [19] a little later gave a simple and elegant proof of a slightly stronger result (see also [4], Chapter XII, §3). The following bound is sufficient for our purposes:

$$R(s,3) \leq \frac{(s-1)(s-2)^2}{(s-1)\log(s-1) - s + 2} + 1 \leq \frac{2s^2}{\log s} \tag{6}$$

for s large enough. It follows immediately from this bound that any graph of order n has either three independet vertices or a clique of order at least $(n\log n)^{1/2}/3$, provided n is sufficiently large.

Theorem 7. *For all graphs G of sufficiently large order,*

$$|G| > 2\gamma(G) + \frac{1}{6}(|G|\log|G|)^{1/2}. \tag{7}$$

In particular, for all $s \geq 1$ and large enough h,

$$\nu_s(h) > 2h + \frac{1}{3\sqrt{2}}(h\log h)^{1/2}. \tag{8}$$

Proof. Throughout the proof of (7) we assume that n is a large enough integer. Let G be a graph of order n, which we may trivially assume not complete. We may furthermore assume that

$$\omega(G) < \frac{1}{6}(n\log n)^{1/2},$$

since otherwise Lemma 6 completes the proof: we simply choose H to be two independent vertices. By the remark following (6), we can find an independent 3-set $W_0 \subset V(G)$ in G. Define $G_1 = G - W_0$ and $n_1 = |G_1| = n - 3$. We have that

$$\omega(G_1) \leq \omega(G) < \frac{1}{6}(n\log n)^{1/2} < \frac{1}{3}(n_1\log n_1)^{1/2}.$$

Hence we can find an independent 3-set W_1 in G_1. Define $G_2 = G_1 - W_1$ and $n_2 = |G_2| = n - 6$. In this fashion we obtain $G = G_0 \supset G_1 \supset \cdots \supset G_t$ with $W_i = V(G_i) \setminus V(G_{i+1})$ an independent 3-set in G_i, $0 \leq i \leq t-1$, and $n_i = |G_i| = n - 3i$ for all i. We claim that if $t < n/5$, and hence $n_t > 2n/5$, we can still continue the process. Indeed

$$\omega(G_t) \leq \omega(G) < \frac{1}{6}(n\log n)^{1/2} < \frac{1}{6}((5n_t/2)\log n)^{1/2} < \frac{1}{3}(n_t\log n_t)^{1/2},$$

and again we know that there is an independent 3-set in G_t. Thus we find $s = \lceil n/5 \rceil$ pairwise disjoint independent 3-sets $W_0, \ldots, W_{s-1}$ in G. Set H to be the subgraph of G induced by the union of these W_i. Then $|H| = 3s$ and $\chi(H) \leq s$ and hence $|H| - 2\chi(H) \geq s = \lceil n/5 \rceil$. Therefore an application of Lemma 6 with this H completes the proof of (7).

Finally, given a large enough h, if G is a graph with $\gamma(G) \geq h$ then (7) tells us that

$$\begin{aligned} |G| &\geq 2h + \frac{1}{6}(|G|\log|G|)^{1/2} \\ &> 2h + \frac{1}{3\sqrt{2}}(h\log h)^{1/2}, \end{aligned}$$

which completes the proof of (8), since trivially $\nu_s(h) \geq \nu_1(h)$ for all s and h. □

We conclude this section by remarking the following. In Theorem 2, our approach in the search for graphs G with large $\gamma(G)$ is rather crude in the sense that we guarantee a large $\chi(G)$ simply by taking a G with $\alpha(G) = 2$. Indeed, by (6), we must have a large clique in such a G and this forces $\gamma(G)$ down. However, Theorem 7 tells us that this simple approach gives us in fact a reasonable bound.

4. The main construction and the asymptotic upper bound

Our aim in this section is to introduce a new class of graphs in order to prove our upper bound (1) for $\nu_s(h)$. We shall make use of the following two operations. Given two graphs G and H, let us define their (*categorical*) *product* $G \times H$ as the graph on $V(G) \times V(H)$ whose edges are

$$E(G \times H) = \{(g_1, h_1)(g_2, h_2) : g_1 g_2 \in E(G) \text{ and } h_1 h_2 \in E(H)\}.$$

Also, we define their $*$-*product* $G * H$ as the graph on $V(G) \times V(H)$ whose edges are

$$\begin{aligned} E(G * H) = \{(g_1, h_1)(g_2, h_2) : \text{either } g_1 = g_2 \text{ and } h_1 h_2 \in E(H) \\ \text{or } g_1 g_2 \in E(G) \text{ and } h_1 = h_2\}. \end{aligned}$$

In the last section we considered the graphs J_r, as their chromatic numbers are large and their clique numbers small. The reason $\chi(J_r)$ is large is that $\alpha(J_r) = 2$ or, in other words, their complement $G_r = J_r^{\mathrm{c}}$ is triangle-free. The point of considering the $*$-product is that $G * H$ is triangle-free if both G and H are. Moreover, the independence number of $G * H$ is trivially at most $|H|\alpha(G)$. Thus, if H is triangle-free,

$$\begin{aligned} \gamma\left[(G_r * H)^{\mathrm{c}}\right] &= \chi\left[(G_r * H)^{\mathrm{c}}\right] - \omega\left[(G_r * H)^{\mathrm{c}}\right] \\ &\geq |H||G_r|/2 - |H|\alpha(G_r) \\ &= (1/2 - o(1))|G_r * H|, \end{aligned}$$

as $r \to \infty$, by (3). Thus, if we can find a triangle-free H for which $G_r * H$ is the complement of a square, say of F^2, then we shall have a good upper bound for $\nu_2(\gamma(F^2))$, namely, $|F^2| = (2 + o(1))\gamma(F^2)$.

Let us define two families of graphs. First, for each q and $r \geq 1$, we denote by $G_{r,q}$ the graph whose vertices are the 0–1 sequences of length $(2q+1)r + 1$, two of them being adjacent iff they differ in at least $2qr + 1$ coordinates. Thus, for instance, we have $G_{r,1} = G_r = J_r^{\mathrm{c}}$. Secondly, for each $k \geq 1$ and $\ell \geq 0$, set $m = (2\ell + 1)k + 2$ and denote the cycle of order m by C^m; we define $H_{k,\ell}$ as the graph whose vertices are the vertices of C^m, two of them being adjacent in our $H_{k,\ell}$ iff their distance in C^m is at least $\ell k + 1$. Note that in $H_{k,\ell}$ the neighbours of a vertex h are the farthest $k + 1$ points from h in C^m.

It is easy to check that $G_{r,q}$ is triangle-free for all q and $r \geq 1$. Moreover, Theorem 3 gives us the following upper bound for $\alpha(G_{r,q}) = \omega(G_{r,q}^{\mathrm{c}})$.

Lemma 8. *Let $q \geq 1$ be fixed and set*

$$\eta_q = \left(\frac{(2q+1)^{2q+1}}{q^q (q+1)^{q+1}}\right)^{1/(2q+1)}.$$

Then, for sufficiently large r,

$$\omega(G_{r,q}^{\mathrm{c}}) < \frac{1}{2}\eta_q^{(2q+1)r+1}.$$ □

For all r and $k \geq 1$ and $s \geq 2$, let us set

$$F_{r,k,s} = G_{r,\lfloor s/2 \rfloor} \times H_{k,\lfloor (s-1)/2 \rfloor}.$$

As usual, a graph with no edges is said to be *empty*; we denote the empty graph of order m by E^m. For any graph G, we note that $G * E^m$ is simply the disjoint union of m copies of G. We are now ready to state our key result.

Theorem 9. *Let r, $k \geq 1$ and $s \geq 2$. Set $q = \lfloor s/2 \rfloor$, $\ell = \lfloor (s-1)/2 \rfloor$ and $m = |H_{k,\ell}| = (2\ell+1)k+2$. Then*

$$(F_{r,k,s})^s = \begin{cases} (G_{r,q} * E^m)^{\mathrm{c}} & \text{if } s \text{ is even} \\ (G_{r,q} * H_{k,\ell})^{\mathrm{c}} & \text{if } s \text{ is odd.} \end{cases}$$

Theorem 9, whose proof is given in the next section, implies the promised upper bound for $\nu_s(h)$.

Corollary 10. *Let $s \geq 2$ be fixed, $q = \lfloor s/2 \rfloor$ and η_q as defined in Lemma 8. Moreover, set $\epsilon_0 = \epsilon_0(s) = 1 - (\log \eta_q)/\log 2 > 0$ and $C_s = 4 + s2^{s+1}$. Then for sufficiently large h*

$$\nu_s(h) < 2h + C_s h^{1/(1+\epsilon_0)}. \tag{9}$$

Proof. Fix an h and $s \geq 2$. We shall assume throughout the proof that h is large enough; it will be clear that our inequalities hold if $h \geq h_0$ for some constant $h_0 = h_0(s)$. We shall choose suitable parameters r and k for which $F = F_{r,k,s}$ shows that (9) holds.

First, let $r \geq 1$ be the minimal integer such that setting $n = (2q+1)r+1$ we have

$$2^n \geq h^{1/(1+\epsilon_0)}. \tag{10}$$

Now put $\ell = \lfloor (s-1)/2 \rfloor$ and let $k \geq 1$ be the minimal integer such that setting $m = (2\ell+1)k+2$ we have

$$m \geq 2^{1-n} h \left(1 + 2(2h)^{-\epsilon_0/(1+\epsilon_0)}\right). \tag{11}$$

Claim. We have

$$\gamma\left[(F_{r,k,s})^s\right] \geq \left|(F_{r,k,s})^s\right|/2 - m\omega(G_{r,q}^{\mathrm{c}}) > h \tag{12}$$

and

$$\left|(F_{r,k,s})^s\right| = 2^n m < 2h + C_s h^{1/(1+\epsilon_0)}. \tag{13}$$

Note that the claim above proves (9); it now remains to check (12) and (13). Let us start with (12). We first note that $G_{r,q}$ (r, $q \geq 1$) and $H_{k,\ell}$ (k, $\ell \geq 1$) are triangle-free (see Lemmas 12(i) and 13(i)), and hence so are $G_{r,q} * E^m$ and $G_{r,q} * H_{k,\ell}$. Therefore, by Theorem 9, we have that $\alpha\left[(F_{r,k,s})^s\right] = 2$ and so $\chi\left[(F_{r,k,s})^s\right] \geq \left|(F_{r,k,s})^s\right|/2$. Secondly, since $G_{r,q} * E^m$ is a spanning subgraph of $G_{r,q} * H_{k,\ell}$, we trivially have that $\omega\left[(G_{r,q} * E^m)^{\mathrm{c}}\right] = \alpha\left(G_{r,q} * E^m\right) \geq \alpha\left(G_{r,q} * H_{k,\ell}\right) = \omega\left[(G_{r,q} * H_{k,\ell})^{\mathrm{c}}\right]$. Theorem 9 then tells us that

$$\begin{aligned} \omega\left[(F_{r,k,s})^s\right] &\leq \omega\left[(G_{r,q} * E^m)^{\mathrm{c}}\right] \\ &\leq m\omega\left(G_{r,q}^{\mathrm{c}}\right). \end{aligned}$$

Furthermore, by the definition of ϵ_0 and Lemma 8, we know that

$$\omega\left(G_{r,q}^{\mathrm{c}}\right) < \eta_q^n/2 = 2^{n-1}(2^n)^{-\epsilon_0}.$$

Hence, by (10) and (11),

$$\begin{aligned} \gamma\left[(F_{r,k,s})^s\right] &= \chi\left[(F_{r,k,s})^s\right] - \omega\left[(F_{r,k,s})^s\right] \\ &\geq \left|(F_{r,k,s})^s\right|/2 - m\omega(G_{r,q}^{\mathrm{c}}) \\ &> m2^{n-1}\left(1 - (2^n)^{-\epsilon_0}\right) \\ &\geq h\left(1 + 2(2h)^{-\epsilon_0/(1+\epsilon_0)}\right)\left(1 - (2^n)^{-\epsilon_0}\right) \\ &\geq h\left(1 + \frac{2^{1/(1+\epsilon_0)}}{h^{\epsilon_0/(1+\epsilon_0)}}\right)\left(1 - \frac{1}{h^{\epsilon_0/(1+\epsilon_0)}}\right) \\ &> h, \end{aligned}$$

proving (12).

Inequality (13) follows from the choices of r and k. Indeed, we first note that by the minimality of r

$$2^n < 2^{2q+1} h^{1/(1+\epsilon_0)}.$$

By the minimality of k, we have that

$$(m-(2\ell+1))2^n < 2h\left(1+2(2h)^{-\epsilon_0/(1+\epsilon_0)}\right).$$

Thus

$$\begin{aligned} m2^n &< 2h + 2(2h)^{1/(1+\epsilon_0)} + (2\ell+1)2^{2q+1}h^{1/(1+\epsilon_0)} \\ &< 2h + 4h^{1/(1+\epsilon_0)} + s2^{s+1}h^{1/(1+\epsilon_0)} \\ &= 2h + C_s h^{1/(1+\epsilon_0)}, \end{aligned}$$

completing the proof of the claim and hence establishing our result. □

We now remark that (9) trivially improves some upper bounds for certain functions mentioned in [13]. Let us recall the following two definitions given in the introduction. Given $s \geq 1$ and $h \geq 0$, set

$$m_s(h) = \max\{m \in \mathbb{N} : \text{for any graph } G,\ |E(G)| < m \text{ implies } \gamma_s(G) < h\}$$

and

$$\delta_s(h) = \max\{n \in \mathbb{N} : \text{for any graph } G,\ |G| < n + \omega_s(G) \text{ implies } \gamma_s(G) < h\}.$$

It has been known [13] that for $h \geq 3$ one has $m_2(h) \leq 13h^2$ and $\delta_2(h) \leq 3h$. Moreover, for $s \geq 3$,

$$m_s(h) \leq \begin{cases} (3s+1)h^2 - 2 & \text{if } s \text{ is odd} \\ (3s+4)h^2 & \text{if } s \text{ is even,} \end{cases}$$

and

$$\delta_s(h) \leq \begin{cases} s(h+1) & \text{if } s \text{ is odd} \\ (s+1)(h+1)-1 & \text{if } s \text{ is even.} \end{cases}$$

Corollary 10 immediately gives us the following bounds.

Corollary 11. *Let s be fixed and C_s and $\epsilon_0 = \epsilon_0(s)$ as in Corollary 10. Then for sufficiently large h*

$$\begin{aligned} m_s(h) &\leq \binom{\nu_s(h)}{2} \\ &< 2h^2 + 3C_s h^{(2+\epsilon_0)/(1+\epsilon_0)} \end{aligned}$$

and

$$\delta_s(h) \leq \nu_s(h) < 2h + C_s h^{1/(1+\epsilon_0)}$$

□

5. Proof of the key result

In this section we prove Theorem 9 . We shall need the following two lemmas about walks in the graphs $G_{r,q}$ and $H_{k,\ell}$.

Lemma 12. *For any $r \geq 1$ and $q \geq 1$ the following conditions hold.*

(i) Any odd closed walk in $G_{r,q}$ has length at least $2q+3$.

(ii) Let g and g' be nonadjacent vertices in $G_{r,q}$. Then they are connected by a walk of length $2q$. If they are furthermore distinct then they are also connected by a walk of length $2q+1$.

Proof. (i) Assume $g_1, g_2, \ldots, g_{2j+1}$ is a walk in $G_{r,q}$ with $j \leq q$. We claim that g_1 is not adjacent to g_{2j+1}. Indeed, for $i = 1, 2, \ldots, 2j$ we have $d(g_i, g_{i+1}) \geq 2qr+1$, so g_i and g_{i+1} agree at no more than r coordinates. Therefore, for $i = 1, 2, \ldots, 2j-1$,

$$d(g_i, g_{i+2}) \leq 2r,$$

since if g_i and g_{i+2} disagree at a coordinate j, say, then g_{i+1} agrees at j either with g_i or else with g_{i+2}. Hence

$$d\left(g_1, g_{2j+1}\right) \leq 2jr \leq 2qr,$$

and g_1 is not adjacent to g_{2j+1} concluding the proof of (i).

(ii) If g and g' are two nonadjacent vertices in $G_{r,q}$ then $d(g, g') \leq 2qr$ by definition. Let us construct a walk of length $2q$ from g to g'. Let C be the set of coordinates on which g and g' disagree. Since the cardinality of C is at most $2qr$ we can write

$$C = C_1 \cup C_2 \cup \cdots \cup C_{2q},$$

where the C_i are pairwise disjoint and satisfy

$$0 \leq |C_i| \leq r$$

for all i. Let us consider the walk $g = g_0, g_1, \ldots, g_{2q}$ in $G_{r,q}$ defined by the condition that C_i is the set of coordinates on which g_{i-1} and g_i agree. It is easy to check that $g_{2q} = g'$, and so we have found the required g–g' walk.

We now assume that $g \neq g'$. To find a walk of length $2q+1$ joining g to g' it is enough to find g'' adjacent to g' but not adjacent to g. In order to construct such a sequence g'' put D to be a set of coordinates of cardinality $r+1$ containing at least one coordinate at which g and g' disagree. Now let g'' be equal to g at each coordinate in D and different from g' at each coordinate outside D.

The sequence g'' is not adjacent to g since they can only differ on coordinates not in D, and so $d(g, g'') \leq 2qr$. On the other hand g'' differs from g' on each coordinate outside D and on at least one coordinate in D, hence $d(g', g'') \geq 2qr+1$ and so g'' is adjacent to g'. □

Lemma 13. *For any $k \geq 1$ and $\ell \geq 0$ the following conditions hold.*

(i) Any odd closed walk in $H_{k,\ell}$ has length at least $2\ell+3$.

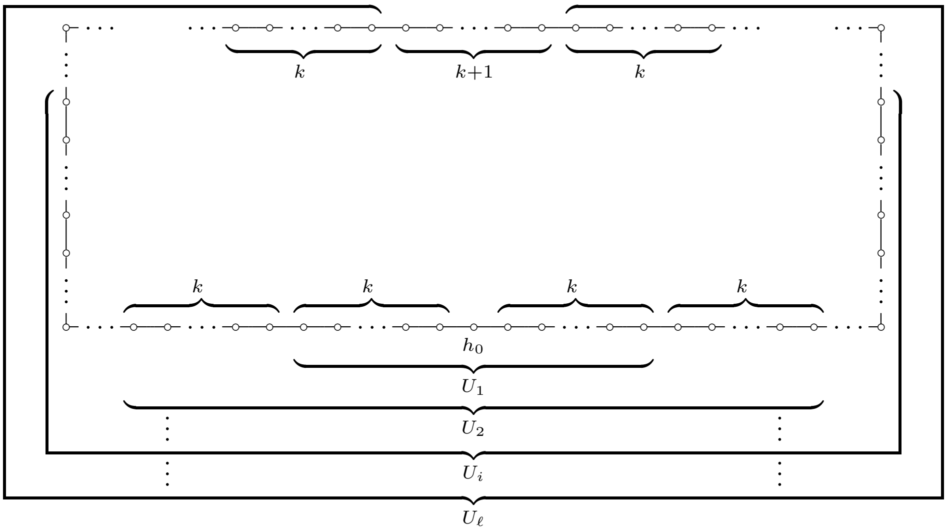

Figure 1. The sets U_i in C^m

(ii) Let h and h' be two distinct vertices in $H_{k,\ell}$. Then they are connected both by a walk of length $2\ell+1$ and by a walk of length $2\ell+2$. If they are furthermore nonadjacent, then they are also connected by a walk of length 2ℓ.

Proof. Let h_0 be a fixed vertex of $H_{k,\ell}$. Let U_i be the set of the $2ik+1$ nearest vertices to h_0 in C^m, $m=(2\ell+1)k+2$, $i=1,\dots,\ell$. Note that the complement of U_ℓ is the set of vertices adjacent to h_0. See Figure 1.

It is easy to check that U_1 is the set of vertices h of $H_{k,\ell}$ such that there is a walk of length 2 from h_0 to h. By induction, U_i is the set of vertices connected to h_0 by a walk of length $2i$. So U_ℓ is the set of vertices h for which there is a walk of length 2ℓ from h_0 to h.

Since h_0 is not adjacent to any vertex of U_ℓ there are no walks of length $2\ell+1$ from h_0 to itself. This concludes the proof of (i).

It can be easily seen that h_0 is the only vertex of $H_{k,\ell}$ not adjacent to any vertex in U_ℓ. Hence there is a walk of length $2\ell+1$ from h_0 to any other vertex of $H_{k,\ell}$. To show that there is a walk of length $2\ell+2$ from h_0 to any other vertex h_1 of $H_{k,\ell}$ let us consider any vertex h_2 adjacent to h_1 and different from h_0 (clearly h_2 exists since the degree of each vertex in $H_{k,\ell}$ is at least 2). We know that there is a walk of length $2\ell+1$ from h_0 to h_2 and, since h_2 is adjacent to h_1, there is a walk of length $2\ell+2$ from h_0 to h_1.

To finish our proof, it is enough to show that if h_1 is not adjacent to h_0, then there is a walk of length 2ℓ between them. But this follows from the fact that the set of vertices nonadjacent to h_0 is U_ℓ. Indeed, as remarked above, U_ℓ is precisely the set of vertices connected to h_0 by walks of length 2ℓ. □

We are now ready to prove Theorem 9. Let us once and for all fix r and $k\geq 1$. We shall analyse the cases s even and s odd separately. For $s\geq 2$ even, we have to prove that

$$(F_{r,k,s})^s = (G_{r,q} * E^m)^{\mathrm{c}}, \tag{14}$$

where $s = 2q = 2\ell + 2$ and $m = (2\ell + 1)k + 2$. On the other hand, for $s \geq 3$ odd we have to prove that

$$(F_{r,k,s})^s = (G_{r,q} * H_{k,\ell})^{\mathrm{c}}\,, \tag{15}$$

where $s = 2q + 1 = 2\ell + 1$.

Proof of (14). Let us fix an even $s \geq 2$ and let q and ℓ satisfy $s = 2q = 2\ell + 2$. By definition we have

$$F_{r,k,s} = G_{r,q} \times H_{k,\ell}.$$

Let (g_1, h_1), (g_2, h_2) be any pair of distinct vertices of $F_{r,k,s}$. To prove (14), we have to show that if $g_1g_2 \in E(G_{r,q})$ and $h_1 = h_2$ then there are no (g_1, h_1)–(g_2, h_2) walks of length at most s in $F_{r,k,s}$. Furthermore, we have to show that there is such a walk otherwise.

Let us consider the following three cases. We want to show the nonexistence of our short (g_1, h_1)–(g_2, h_2) walk in the first case, and its existence in the last two cases.

Case 1. $g_1g_2 \in E(G_{r,q})$ and $h_1 = h_2$.

Let us assume that there is a (g_1, h_1)–(g_2, h_2) walk W of length $t \leq s$ in $F_{r,k,s}$. If t is odd then, by projecting W onto the second coordinate, we get an odd closed walk of length $t \leq 2\ell+1$ in $H_{k,\ell}$, contradicting Lemma 13(i). On the other hand, if t is even then, by projecting W onto the first coordinate, we get an even g_1–g_2 walk of length $t \leq 2q$ in $G_{r,q}$. Since $g_1g_2 \in E(G_{r,q})$ we obtain an odd closed walk of length $t + 1 \leq 2q + 1$ in $G_{r,q}$, contradicting Lemma 12(i).

Case 2. $g_1g_2 \in E(G_{r,q})$ and $h_1 \neq h_2$.

By Lemma 13(ii), there is a h_1–h_2 walk of length $2\ell + 1 \leq s$ in $H_{k,\ell}$. Since $g_1g_2 \in E(G_{r,q})$ there clearly is a g_1–g_2 walk of length $2\ell + 1$ in $G_{r,q}$ (in fact of any odd length). Let W be the sequence of vertices of $F_{r,k,s}$ whose projection onto the first and the second coordinates are the above walks in $G_{r,q}$ and in $H_{k,\ell}$. Clearly W is a (g_1, h_1)–(g_2, h_2) walk in $F_{r,k,s}$ and, since its length is $2\ell + 1 \leq s$, the proof of this case is finished.

Case 3. $g_1g_2 \notin E(G_{r,q})$.

As we have seen above, it is enough to show the existence of two suitable walks of the same length $t \leq s$, say, one connecting g_1 to g_2 in $G_{r,q}$ and the other h_1 to h_2 in $H_{k,\ell}$. Here we can take $t = s = 2q = 2\ell + 2$. Indeed, the existence of the required walk in $G_{r,q}$ follows from Lemma 12(ii). To get a suitable walk in $H_{k,\ell}$ we apply Lemma 13(ii) if $h_1 \neq h_2$ and if, on the other hand, $h_1 = h_2$ then we simply note that s is even and that $H_{k,\ell}$ has no isolated vertices.

□

Proof of (15). Let us fix an odd $s \geq 3$ and let q and ℓ satisfy $s = 2q+1 = 2\ell+1$. Let (g_1, h_1) and (g_2, h_2) be any pair of distinct vertices of $F_{r,k,s}$. To prove (15) we have to show that if either $h_1 = h_2$ and $g_1g_2 \in E(G_{r,q})$ or else $g_1 = g_2$ and $h_1h_2 \in E(H_{k,\ell})$, then there are no (g_1, h_1)–(g_2, h_2) walks in $F_{r,k,s}$ of length at most s. Moreover we also need to show that otherwise there is such a walk.

Let us consider four cases. We shall prove the nonexistence of the appropriate walks in the first two cases and their existence in the last two.

Case 1. $g_1g_2 \in E(G_{r,q})$ and $h_1 = h_2$.

This is similar to the Case 1 of the proof of (14). The existence of a (g_1, h_1)–(g_2, h_2) walk of length at most s in $F_{r,k,s}$ requires either that there should be an odd closed walk of length at most $s = 2\ell + 1$ in $H_{k,\ell}$ or else that there should be an odd closed walk of length at most $s + 1 = 2q + 2$ in $G_{r,q}$. By Lemmas 12(i) and 13(i), neither of the above walks can exist.

Case 2. $h_1h_2 \in E(H_{k,\ell})$ and $g_1 = g_2$.

If there is a (g_1, h_1)–(g_2, h_2) walk of length $t \leq s$ in $F_{r,k,s}$, then either there is an odd closed walk of length at most $s = 2q + 1$ in $G_{r,q}$ or else there is an odd closed walk of length at most $s + 1 = 2\ell + 2$ in $H_{k,\ell}$, contradicting either Lemma 12(i) or 13(i).

Case 3. $g_1 \neq g_2$ and $h_1 \neq h_2$.

We can get a (g_1, h_1)–(g_2, h_2) walk of length $s = 2q + 1 = 2\ell + 1$ in $F_{r,k,s}$ by combining apropriate walks in $G_{r,q}$ and in $H_{k,\ell}$. If g_1 is not adjacent to g_2 then the required walk in $G_{r,q}$ exists by Lemma 12(ii), otherwise its existence is obvious (since s is odd). The existence of a suitable walk in $H_{k,\ell}$ follows from Lemma 13(ii).

Case 4. Either $g_1 = g_2$ and $h_1h_2 \notin E(H_{k,\ell})$ or else $h_1 = h_2$ and $g_1g_2 \notin E(G_{r,q})$.

Now we combine appropriate walks of length $s - 1 = 2q = 2\ell$ from $G_{r,q}$ and from $H_{k,\ell}$. Their existence is either obvious (in the case their endpoints are equal) or follows from Lemmas 12(ii) and 13(ii).

$\square$

6. Concluding remarks

Although we have managed to estimate $\nu_s(h)$ quite accurately, some interesting questions concerning the function $e_s(h) = \nu_s(h) - 2h$ remain. Our results show that for large enough h

$$\frac{1}{3\sqrt{2}}(h \log h)^{1/2} < e_s(h) < h^{1-\epsilon_s} \tag{16}$$

where $\epsilon_s > 0$ depends only on s. What is clearly unsatisfactory is that the lower bound does not depend on s. Also, the exponent of h in the upper bound is rather close to one, and in fact by our methods $\epsilon_s \to 0$ as $s \to \infty$. It is natural to ask whether $e_s(h) = O(h^{1-\epsilon})$ for some $\epsilon > 0$ independent of s.

Our proof of the upper bound in (16) is entirely constructive, and the question whether one can do better by probabilistic techniques naturally arises. Let us make the following remark, where for the sake of simplicity we restrict our attention to the case $s = 2$. It turns out that ϵ_2 in (16) can be taken close to 1/2, provided there exists a triangle-free graph G of order n, diameter 2, and with $\alpha(G) = O(n^c)$ for some c close to 1/2. Indeed, the proof of (14) (or of the claim in the proof of Theorem 1) implies that $G * E^{k+2}$, $k \geq 1$, is the complement of a square. By straightforward computations as in the proof of Corollary 10, one then gets an improvement of the upper bound in (16), if c is not much larger than 1/2.

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