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A New Lower Bound for Snake-in-the-Box Codes

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Abstract. In this paper we give a new lower bound on the length of Snake-in-the-Box Codes, i.e., induced cycles in the d-dimensional cube. The bound is a linear function of the number of vertices of the cube and improves the bound $c \cdot 2^d/d$, where c is a constant, proved by Danzer and Klee.

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1. Introduction

A *snake-in-the-box code* is an induced cycle in the d-dimensional cube $I[d]$, i.e., the graph with all d-tuples of binary digits as vertices, and all pairs of vertices differing in exactly one coordinate as edges. For each $d \in \mathbb{N}$, let $S(d)$ denote the length of the longest induced cycle in $I[d]$. The problem of determining $S(d)$ was first met by Kautz [10] in constructing a type of error-checking code for certain analog-to-digital conversion systems. He showed that

$$S(d) \geq \lambda\sqrt{2^d}.$$

This bound was later improved by Ramanujacharyulu and Menon [11], who proved that

$$S(d) \geq (3/2)^d,$$

Brown (unpublished, quoted from Danzer and Klee [3]) and Singleton [12] got

$$S(d) \geq \lambda(\sqrt[4]{6})^d,$$

Abbott [1] obtained

$$\lambda(\sqrt{5/2})^d,$$

Vasil'ev [14] showed that

$$S(d) \geq \frac{2^d}{d} \qquad \text{when } d \text{ is a power of 2,}$$

and that

$$S(d) \geq (1 - \varepsilon(d))\frac{2^{d-1}}{d} \qquad \text{with } \varepsilon(d) \to 0 \text{ as } d \to \infty,$$

and finally Danzer and Klee [3] proved that

$$S(d) \geq \frac{2^{d+1}}{d} \qquad \text{when } d \text{ is a power of 2,}$$

and

$$S(d) \geq \frac{7}{4}\frac{2^d}{d-1} \qquad \text{for all } d \geq 5.$$

In this note we give a new lower bound for $S(d)$, namely

$$S(d) \geq \frac{9}{64}2^d.$$

2. The Main Lemma

Our aim in this section is to state and prove Lemma 3, which will provide a construction of long snakes, leading to the proof of the lower bound stated in the introduction.

Let us first introduce some notions and examine their properties. If F is a subgraph of $I[d]$, then let us denote by $F^{(0)}$ the subgraph of $I[d+1]$ obtained as the image of F in the embedding

$$\psi_0 : I[d] \to I[d+1]$$

such that

$$\psi_0((v^1, \ldots, v^d)) = (v^1, \ldots, v^d, 0).$$

Analogously, let $F^{(1)}$ be the image of F in

$$\psi_1 : I[d] \to I[d+1]$$

such that

$$\psi_1((v^1, \ldots, v^d)) = (v^1, \ldots, v^d, 1).$$

For each $d \geq 2$, let

$$R_d : [2^d + 1, 2^{d+1}] \to [2^d]$$

be the order reversing bijection, i.e., such that $R_d(i) = 2^{d+1} + 1 - i$.

Now, for each $d \geq 2$, we shall define a function

$$H_d : [2^d] \to V(I[d])$$

such that

$$\overline{H_d} = (H_d(1), \ldots, H_d(2^d), H_d(1))$$

is a Hamiltonian cycle in $I[d]$. Set

$$\overline{H_2} = ((0,0), (0,1), (1,1), (1,0), (0,0)),$$

and

$$H_{d+1}(i) = \begin{cases} (H_d(i))^{(0)} & \text{if } 1 \leq i \leq 2^d, \\ (H_d \circ R_d(i))^{(1)} & \text{if } 2^d + 1 \leq i \leq 2^{d+1}. \end{cases}$$

In other words, $\overline{H_{d+1}}$ is obtained by taking $\overline{H_d}^{(0)}$ and $\overline{H_d}^{(1)}$, removing the edges connecting their last vertices with their first vertices, joining the first vertex of $\overline{H_d}^{(0)}$ with the first vertex of $\overline{H_d}^{(1)}$ and analogously the last with the last.

The following lemma can be proved simply by checking all the possible cases.

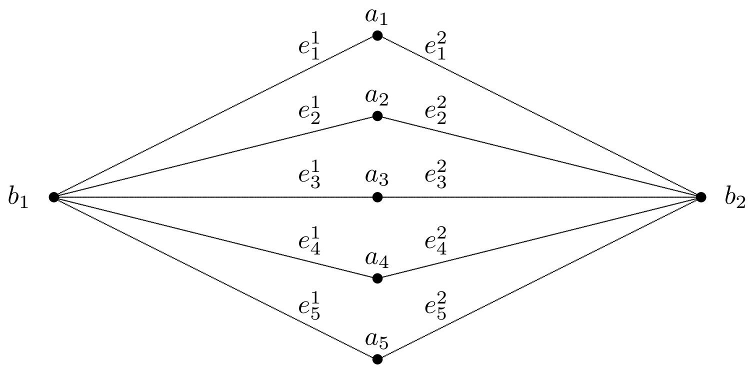

Fig. 1. The graph G

Lemma 1. *For each $d \geq 2$, if $1 \leq i < j \leq 2^{d+1}$ and*

$$(H_{d+1}(i), H_{d+1}(j)) \in E(I[d+1]) \setminus E(\overline{H_{d+1}}),$$

then exactly one of the following conditions holds:

(1) $1 \leq i < j \leq 2^d$, $(i, j) \neq (1, 2^d)$ *and* $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$,

(2) $i = 1$ *and* $j = 2^d$,

(3) $2^d + 1 \leq i < j \leq 2^{d+1}$, $(i, j) \neq (2^d + 1, 2^{d+1})$ *and* $(H_d \circ R_d(i), H_d \circ R_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$,

(4) $i = 2^d + 1$ *and* $j = 2^{d+1}$,

(5) $2 \leq i \leq 2^d - 1$ *and* $i = R_d(j)$. $\square$

Let G be the graph shown in Figure 1.

Given a permutation $\sigma \in S_5$, let φ_σ and φ_σ^+ be permutations of edges of G such that

$$\varphi_\sigma(e_i^j) = e_{\sigma(i)}^j \qquad \text{and} \qquad \varphi_\sigma^+(e_i^j) = e_{\sigma(i)}^{3-j}.$$

Furthermore, let

$$\sigma_1 = (3\ 5),$$
$$\sigma_2 = (1\ 3)(2\ 4\ 5),$$
$$\sigma_3 = (1\ 2)(4\ 5)$$

be permutations of the set $\{1, \ldots, 5\}$.

The following lemma can be easily verified.

Lemma 2.

(6) For each $e, e' \in E(G)$ and $\sigma \in S_5$, the edges e and e' have the same number of vertices in common as the edges $\varphi_\sigma(e)$ and $\varphi_\sigma(e')$, and the same as the edges $\varphi_\sigma^+(e)$ and $\varphi_\sigma^+(e')$. If e and e' have one vertex in

common, then it belongs to $\{a_1, \ldots, a_5\}$*, if and only if the common vertex of* $\varphi_\sigma(e)$ *and* $\varphi_\sigma(e')$ *belongs to* $\{a_1, \ldots, a_5\}$*, and if and only if the common vertex of* $\varphi^+_\sigma(e)$ *and* $\varphi^+_\sigma(e')$ *belongs to* $\{a_1, \ldots, a_5\}$*.*

(7) *For each* $e \in E(G)$ *the edges* $\varphi_{\sigma_1}(e)$ *and* $\varphi^+_{\sigma_2}(e)$ *are vertex disjoint, and the edges* $\varphi_{\sigma_1}(e)$ *and* $\varphi^+_{\sigma_3}(e)$ *are vertex disjoint.*

□

Now we can state our key lemma. We claim in it the existence, for each $d \geq 2$, of a closed walk of length 2^d in G which, after combining it with the Hamiltonian cycle $\overline{H_d}$, will provide a construction of long snakes. The walk will start from a vertex belonging to the set $\{a_1, \ldots, a_5\}$, will not use any edge twice in turn and will posses a certain property with respect to the Hamiltonian cycle $\overline{H_d}$. Namely, if we regard this walk and the Hamiltonian cycle $\overline{H_d}$ as sequences of length 2^d, the walk as a sequence of edges and $\overline{H_d}$ as a sequence of vertices, then any two edges corresponding to two nonconsecutive vertices of $\overline{H_d}$ being neighbours in $I[d]$ will be vertex disjoint.

Lemma 3. *For every* $d \geq 2$ *there is a function* $\Phi_d : [2^d] \to E(G)$ *such that*

(8) *if* $1 \leq i \leq 2^d - 1$*,* $j = i + 1$*, or* $i = 2^d$*,* $j = 1$*, then* $\Phi_d(i)$ *and* $\Phi_d(j)$ *have exactly one vertex* v_i *in common such that* $v_i \in \{a_1, \ldots, a_5\}$ *for* i *even, and* $v_i \in \{b_1, b_2\}$ *for* i *odd, and*

(9) *if* $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$*, then* $\Phi_d(i)$ *and* $\Phi_d(j)$ *are vertex disjoint.*

Proof. We shall prove the lemma by induction on d. In order to make the induction work, we shall define functions

$$\Phi^{k,l}_d : [2^d] \to E(G)$$

for each $d \geq 2$ and

$$(k, l) \in I = \{(1, 1), (1, 3), (1, 4), (2, 3), (2, 4)\}$$

such that each of the following conditions holds:

(10) $\Phi^{k,l}_d(1) = e^1_k$ and $\Phi^{k,l}_d(2^d) = e^2_l$,

(11) if $1 \leq i \leq 2^d - 1$, then $\Phi^{k,l}_d(i)$ and $\Phi^{k,l}_d(i + 1)$ have exactly one vertex v_i in common such that $v_i \in \{a_1, \ldots, a_5\}$ for i even and $v_i \in \{b_1, b_2\}$ for i odd,

(12) if $2 \leq i \leq 2^d - 1$ and $(k, l) \neq (1, 1) \neq (k', l')$, then $\Phi^{k,l}_d(i) = \Phi^{k',l'}_d(i)$,

(13) if $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$, then $\Phi^{k,l}_d(i)$ and $\Phi^{k,l}_d(j)$ are vertex disjoint.

In other words, the function $\Phi^{k,l}_d$ will describe a walk in G starting from the vertex a_k and the edge e^1_k, ending in the edge e^2_l and the vertex a_l, and having all the properties we require for the walk described by the function Φ_d, i.e., it will not use any edge twice in turn and its edges corresponding to two nonconsecutive vertices of $\overline{H_d}$ being neighbours in $I[d]$ will be vertex disjoint. Also, given $d \geq 2$, all the walks described by $\Phi_d^{k,l}$, for $(k,l) \in I \setminus \{(1,1)\}$, will differ only at the first and the last vertices.

The construction of such functions will complete the proof of Lemma 3 because if we set $\Phi_d = \Phi_d^{1,1}$, then (9) will follow from (13), and (8) will follow from (10) and (11).

Set

$$(\Phi_2^{k,l}(1), \Phi_2^{k,l}(2), \Phi_2^{k,l}(3), \Phi_2^{k,l}(4)) = (e_k^1, e_5^1, e_5^2, e_l^2).$$

If $(k,l) \neq (1,1)$, then let

$$\Phi_{d+1}^{k,l}(i) = \begin{cases} \varphi_{\sigma_1} \circ \Phi_d^{k,3}(i) & \text{if } 1 \leq i \leq 2^d, \\ \varphi_{\sigma_2}^+ \circ \Phi_d^{\sigma_2^{-1}(l),4} \circ R_d(i) & \text{if } 2^d + 1 \leq i \leq 2^{d+1}, \end{cases}$$

and set

$$\Phi_{d+1}^{1,1}(i) = \begin{cases} \varphi_{\sigma_1} \circ \Phi_d^{1,3}(i) & \text{if } 1 \leq i \leq 2^d, \\ \varphi_{\sigma_3}^+ \circ \Phi_d^{2,4} \circ R_d(i) & \text{if } 2^d + 1 \leq i \leq 2^{d+1}, \end{cases}$$

where R_d is the order reversing bijection defined in Section 1.

In the induction construction performed above, the walk w corresponding to $\Phi_{d+1}^{k,l}$ ($(k,l) \neq (1,1)$) is obtained from the walks w_1 and w_2 described by $\Phi_d^{k,3}$ and $\Phi_d^{\sigma_2^{-1}(l),4}$. To obtain w, we permute the edges of w_1 with φ_{σ_1}, and the edges of w_2 with $\varphi_{\sigma_2}^+$, getting w_1' and w_2', then we reverse the order of edges of w_2' getting w_2'', and finally we identify the last vertex of w_1' with the first vertex of w_2''.

It can be checked directly that for $d = 2$ the conditions (10)–(13) are satisfied. Given $d \geq 2$, let us assume that the conditions (10)–(13) are satisfied for d. We shall prove that they are satisfied for $d+1$.

Proof of Condition (10). If $(k,l) \in I \setminus \{(1,1)\}$, then

$$\Phi_{d+1}^{k,l}(1) = \varphi_{\sigma_1} \circ \Phi_d^{k,3}(1) = \varphi_{\sigma_1}(e_k^1) = e_k^1,$$

and

$$\Phi_{d+1}^{k,l}(2^{d+1}) = \varphi_{\sigma_2}^+ \circ \Phi_d^{\sigma_2^{-1}(l),4}(1) = \varphi_{\sigma_2}^+(e_{\sigma_2^{-1}(l)}^1) = e_l^2.$$

For $(k,l) = (1,1)$ the proof is analogous.

Proof of Condition (11). We have to show that if $1 \leq i \leq 2^{d+1} - 1$, then $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(i+1)$ have exactly one vertex in common, which belongs to $\{a_1, \ldots, a_5\}$ for i even and to $\{b_1, b_2\}$ for i odd. If $i \neq 2^d$, it follows from condition (11) of the induction hypothesis and condition (6) of Lemma 2. If $i = 2^d$ and $(k,l) \neq (1,1)$, then the edges

$$\Phi_{d+1}^{k,l}(i) = \varphi_{\sigma_1} \circ \Phi_d^{k,3}(2^d) = \varphi_{\sigma_1}(e_3^2) = e_5^2,$$

and

$$\Phi_{d+1}^{k,l}(i+1) = \varphi_{\sigma_2}^+ \circ \Phi_d^{\sigma_2^{-1}(l),4}(2^d) = \varphi_{\sigma_2}^+(e_4^2) = e_5^1,$$

have the vertex a_5 in common, so (11) holds. If $(k,l) = (1,1)$, then the proof is analogous.

Proof of Condition (12). We have to show that if $2 \le i \le 2^{d+1} - 1$ and $(k,l) \neq (1,1) \neq (k',l')$ then $\Phi_{d+1}^{k,l}(i) = \Phi_{d+1}^{k',l'}(i)$. If $2^d \neq i \neq 2^d + 1$, then this follows from the condition (12) of the induction hypothesis, otherwise

$$\Phi_{d+1}^{k,l}(2^d) = e_5^2 = \Phi_{d+1}^{k',l'}(2^d),$$

and

$$\Phi_{d+1}^{k,l}(2^d + 1) = e_5^1 = \Phi_{d+1}^{k',l'}(2^d + 1).$$

Proof of Condition (13). Let us fix $i < j$ such that

$$(H_{d+1}(i), H_{d+1}(j)) \in E(I[d+1]) \setminus E(\overline{H_{d+1}}).$$

We have to show that for each $(k,l) \in I$, $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex disjoint.

In the proof we shall assume that $(k,l) \neq (1,1)$. For $(k,l) = (1,1)$ the proof is analogous. By Lemma 1, one of the conditions (1)–(5) holds.

If (1) holds, then by condition (13) of the induction hypothesis, $\Phi_d^{k,3}(i)$ and $\Phi_d^{k,3}(j)$ are vertex disjoint. By condition (6) of Lemma 2, $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex disjoint.

If (3) holds, then by condition (13) of the induction hypothesis, $\Phi_d^{\sigma_2^{-1}(l),4}(R_d(i))$ and $\Phi_d^{\sigma_2^{-1}(l),4}(R_d(j))$ are vertex disjoint. By condition (6) of Lemma 2, $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex disjoint.

If (2) holds, then $\Phi_{d+1}^{k,l}(i) = e_k^1$, $\Phi_{d+1}^{k,l}(j) = e_5^2$, and if (4) holds, then $\Phi_{d+1}^{k,l}(i) = e_5^1$, $\Phi_{d+1}^{k,l}(j) = e_l^2$ are vertex disjoint.

If (5) holds, then by condition (12) of the induction hypothesis,

$$\Phi_d^{k,3}(i) = \Phi_d^{\sigma_2^{-1}(l),4}(i) = e,$$

for some $e \in E(G)$. Hence

$$\Phi_{d+1}^{k,l}(i) = \varphi_{\sigma_1}(e),$$

and

$$\Phi_{d+1}^{k,l}(j) = \varphi_{\sigma_2}^{+}(e).$$

By condition (7) of Lemma 2, the edges $\Phi_{d+1}^{k,l}(i)$ and $\Phi_{d+1}^{k,l}(j)$ are vertex disjoint., completing the proof of Lemma 3. $\square$

3. The Lower Bound

In this section we shall prove the main result of this note.

Theorem 4. *For each $d_0 \geq 2$, the length $S(d_0)$ of the longest induced cycle in $I[d_0]$ satisfies the following lower bound*

$$S(d_0) \geq \frac{9}{64} 2^{d_0}.$$

Proof. It is known [3,5,6] that $S(2) = 4$, $S(3) = 6$, $S(4) = 8$, $S(5) = 14$, $S(6) = 26$, so we can assume that $d_0 \geq 7$.

Let us fix $d_0 = d + 5$, $d \geq 2$. Assume that we have a sequence of induced paths

$$(v_1^1, v_1^2, \dots, v_1^{r_1}), (v_2^1, v_2^2, \dots, v_2^{r_2}), \dots, (v_{2^d}^1, \dots, v_{2^d}^{r_{2^d}})$$

in $I[5]$ such that

(14) if $1 \leq i \leq 2^d - 1$ and $j = i + 1$ (or $i = 2^d$ and $j = 1$), then the paths $(v_i^1, \dots, v_i^{r_i})$ and $(v_j^1, \dots, v_j^{r_j})$ have only the vertex $v_i^{r_i} = v_j^1$ in common,

(15) if $(H_d(i), H_d(j)) \in E(I[d]) \setminus E(\overline{H_d})$ (H_d and $\overline{H_d}$ are defined in section 1), then $(v_i^1, \dots, v_i^{r_i})$ and $(v_j^1, \dots, v_j^{r_j})$ are vertex disjoint,

(16) if $1 \leq i \leq 2^d$, then $r_i = 4$ for $i \equiv 1$ or $2 \pmod 4$, and $r_i = 5$ for $i \equiv 3$ or $0 \pmod 4$.

Let us regard $I[d+5]$ as $I[d] \times I[5]$. To construct an induced cycle C_{d+5} in $I[d+5]$ we consider $I[d+5]$ as the d-dimensional cube $I[d]$ with each vertex being a copy of $I[5]$. Let us take the jth path $(v_j^1, \dots, v_j^{r_j})$ in the copy of $I[5]$ corresponding to the vertex $H_d(j)$ in $I[d]$; see Figure 2 for the case $d = 3$.

Then, let us join $v_i^{r_i}$ from the ith copy of $I[5]$ with v_j^1 from the jth copy of $I[5]$ for all $i \in \{1, \dots, 2^d - 1\}$, $j = i + 1$ and $i = 2^d$, $j = 1$. Hence we have

$$C_{d+5} = ((H_d(1), v_1^1), (H_d(1), v_1^2), \dots, (H_d(1), v_1^{r_1}), (H_d(2), v_2^1), \dots, (H_d(2), v_2^{r_2}), \dots,$$
$$(H_d(2^d), v_{2^d}^1), \dots, (H_d(2^d), v_{2^d}^{r_{2^d}}), (H_d(1), v_1^1)).$$

It is clear that C_{d+5} is a cycle in $I[d+5]$. We claim that it is an induced cycle. Assume that

$$((H_d(i), v_i^k), (H_d(j), v_j^l)) \in E(I[d+5]).$$

Then $H_d(i) = H_d(j)$ or $v_i^k = v_j^l$. If $H_d(i) = H_d(j)$, then $i = j$ and v_i^k, v_j^l are neighbours in $I[5]$, so

$$((H_d(i), v_i^k), (H_d(j), v_j^l)) \in E(C_{d+5}),$$

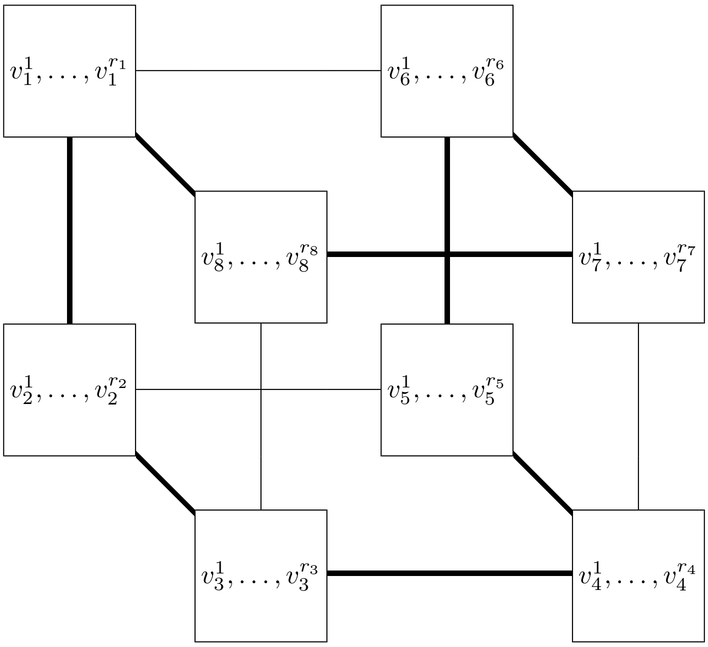

Fig. 2. The cycle C_8

since the path $(v_i^1, \ldots, v_i^{r_i})$ is an induced path. If $v_i^k = v_j^l$, then $H_d(i)$ and $H_d(j)$ are neighbours in $I[d]$, and by (15),

$$(H_d(i), H_d(j)) \in E(\overline{H_d}).$$

By (14), $k = r_i$ and $l = 1$, so

$$((H_d(i), v_i^k), (H_d(j), v_j^l)) \in E(C_{d+5}),$$

and the claim is proved.

By (16), the length of C_{d+5} is equal to

$$9 \cdot 2^{d-1} = \frac{9}{64} 2^{d+5},$$

so to complete the proof of the theorem it is enough to construct a sequence of induced paths satisfying conditions (14)–(16).

Let G' be the subdivision of G obtained by subdividing e_k^1 with two new vertices c_k^1 and c_k^2 in such a way that we get the path (b_1, c_k^1, c_k^2, a_k), and subdividing e_k^2 with three new vertices c_k^4, c_k^5 and c_k^6, giving rise to the path $(a_k, c_k^4, c_k^5, c_k^6, b_2)$, for each $k \leq 5$. Let $c_k^3 = a_k$ and

$$\xi : V[G'] \to V(I[5])$$

be defined as follows. Set

$$\xi(b_1) = (0, 0, 0, 0, 0),$$

$$\xi(c_1^1) = (1,0,0,0,0),$$
$$\xi(c_1^2) = (1,1,0,0,0),$$
$$\xi(c_1^3) = (1,1,0,1,0),$$
$$\xi(c_1^4) = (0,1,0,1,0),$$
$$\xi(c_1^5) = (0,1,1,1,0),$$
$$\xi(c_1^6) = (0,1,1,1,1),$$
$$\xi(b_2) = (1,1,1,1,1),$$

and if $\xi(c_1^i) = (\alpha_1, \dots, \alpha_5)$, then let

$$\xi(c_k^i) = (\alpha_k, \dots, \alpha_5, \alpha_1, \dots, \alpha_{k-1}).$$

It is clear that the function ξ defines an embedding of G' into $I[5]$ such that the image of the subdivision of any edge of G is an induced path in $I[5]$. By Lemma 3, we have the function $\Phi_d : [2^d] \to E(G)$ satisfying conditions (8) and (9). Let us define the path $(v_i^1, \dots, v_i^{r_i})$ to be the image under ξ of the subdivision of the edge $\Phi_d(i)$. Now (16) can be checked directly and, since ξ is an embedding, (8) implies (14), and (9) implies (15), completing the proof of Theorem 4. $\square$

4. Remarks

There is still a gap between the lower bound on the length of snake-in-the-box codes proved in this paper and the best known upper bound. The best upper bound is due to Deimer [5], who showed that

$$S(d) \leq 2^{d-1} - \frac{2^{d-1}}{d(d-5)+7} \qquad \text{for} \qquad d \geq 7.$$

We believe that the upper bound can be further improved to the form of $c2^d$, where c is a constant smaller than 1/2.

Acknowledgements. This research is intended to be a part of a Ph.D. thesis written at the University of Cambridge under the direction of Dr Béla Bollobás. I am grateful to my supervisor for pointing out the problem and his comprehensive help during the writing of this note.

Note added in proof. I was informed by the referees that the lower bound of the form $\lambda 2^d$ was first obtained by Evdokimov [6]. The constant λ he got is very small ($\lambda = 2^{-11}$), but he stated that after certain changes in his construction λ can be increased to $2^{-9}S(8)$.

The idea of his proof is similar to this used in the present proof in the sense that the snake in $I[d] = I[d-d_0] \times I[d_0]$ (where d_0 is a constant) is constructed in such a way that its projection on $I[d-d_0]$ is a Hamiltonian cycle. The rest of the cycle is constructed in quite different way.

In [9] Glagolev and Evdokimov proved a theorem about the chromatic number of a certain infinite graph and stated that it can be used to further increase the constant λ so that $\lambda \in (\frac{3}{16}, \frac{1}{4})$. One of the referees informed me also that in his dissertation, Evdokimov [7] proved that $S(d) \geq 0.26 \cdot 2^d$.

Recently Abbot and Katchalski [2] found completely different way of proving a lower bound of the form $\lambda 2^d$. They use induction in a way resembling the proof of Danzer and Klee [3], but construct the so called accessible snake, which is a snake with some additional paths between its vertices, allowing to keep the induction going without decreasing the ratio of vertices used by the snake (what could not be avoided in the proof of Danzer and Klee).

The upper bound has been lately improved by Solov'jeva [13], who got

$$S(d) \leq 2^{d-1}(1 - \frac{2}{d^2 - d + 2}) \qquad \text{for} \qquad d \geq 7.$$

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