Math 793C — Topology for Analysis

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Theorem (uniform space subspace product).

Each uniform space is uniformly isomorphic to a subspace of a product of pseudometric spaces. A uniform space with separating uniformity is uniformly isomorphic to a subspace of a product of metric spaces.

Proof. Let $(X, \mathscr{D})$ be a uniform space.

- Let $\mathscr{P}$ be the family of all uniformly continuous pseudometrics on X.
- For each $d \in \mathscr{P}$, let (Y_d, d) be the pseudometric space, where $Y_d = X$.
- Let $Y = \prod \{Y_d : d \in \mathscr{P}\}$ be the uniform space with the product uniformity.
- Let $f : X \to Y$ be defined by $f(x) = (y_d)_{d \in \mathscr{P}}$, where $y_d = x$ for each $d \in \mathscr{P}$.
- It is clear that f if injective.
- We show that f is a uniform isomorphism onto $f[X]$ (relative to $\mathscr{D}$ and the subspace uniformity $\mathscr{D}'$ on $f[X]$ inherited from the product uniformity on Y).
  - For each $d \in \mathscr{P}$, let $\mathscr{D}_d$ be the uniformity on $Y_d = X$ that is induced by d.
  - For each $d \in \mathscr{P}$, the composition $\pi_d \circ f$ is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{D}_d$, where $\pi_d : Y \to Y_d$ is the projection.
    - $\pi_d \circ f : X \to Y_d$ is the identity function $X \to X$.
    - Since the pseudometric d is uniformly continuous relative to $\mathscr{D}$, it follows that $\pi_d \circ f$ is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{D}_d$.

- Let $\mathscr{D}'' = \left\{\hat{f}^{-1}[D'] : D' \in \mathscr{D}'\right\}$.
- Since f is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{D}'$, it follows that $\mathscr{D}'' \subseteq \mathscr{D}$.
- Note that $\pi_d \circ f : X \to Y_d$ is uniformly continuous relative to $\mathscr{D}''$ and $\mathscr{D}_d$.

  It suffices to show that $f : X \to Y$ is uniformly continuous relative to $\mathscr{D}''$ and the product uniformity on Y.
  * Let D be a member of the product uniformity on Y.
  * Then $D' := D \cap (f[X] \times f[X])$ is a member of $\mathscr{D}'$.
  * It follows that $\hat{f}^{-1}[D] = \hat{f}^{-1}[D'] \in \mathscr{D}''$.
- Since $\mathscr{D}$ is the weak uniformity on X induced by the family $\{(\mathscr{D}_d, \pi_d \circ f), d \in \mathscr{P}\}$, it follows that $\mathscr{D} \subseteq \mathscr{D}''$.
- Thus $\mathscr{D} = \mathscr{D}''$ and so f is a uniform isomorphism onto $f[X]$.

- Assume that $\mathscr{D}$ is separating.
- For each $d \in \mathscr{P}$, let Z_d be a subspace of X obtained by selecting exactly one element from each equivalence class of the equivalence relation $\sim_d$ on X defined by $x \sim_d y$ iff $d(x, y) = 0$.
- For each $d \in \mathscr{P}$, let $\mathscr{D}'_d$ be the uniformity on Z_d induced by the metric d.
- Let $Z = \prod \{Z_d : d \in \mathscr{P}\}$ be the product uniform space.
- Let $g : X \to Z$ be defined by $g(x) = (z_d : d \in \mathscr{P})$, where z_d is the unique element of Z_d such that $z_d \sim_d x$.
- Since $\mathscr{D}$ is separating and since $\mathscr{D}$ is induced by $\mathscr{P}$, it follows that if $x, y \in X$ are distinct, then there is $d \in \mathscr{P}$ with $d(x, y) > 0$.

  - Let $x, y \in X$ be distinct.
  - Since $\mathscr{D}$ is separating, there is $D \in \mathscr{D}$ with $\langle x, y\rangle \notin D$.
  - Since $\mathscr{D}$ is induced by $\mathscr{P}$, there exists finite nonempty $\mathscr{P}' \subseteq \mathscr{P}$ such that $\bigcap \{D_d : d \in \mathscr{P}'\} \subseteq D$, where D_d is a member of the uniformity on X that is induced by d for each $d \in \mathscr{P}'$.
  - For each $d \in \mathscr{P}'$ let $\varepsilon_d > 0$ be such that $d(w, z) < \varepsilon_d$ implies that $\langle w, z\rangle \in D_d$ for any $w, z \in X$.
  - Since $\langle x, y\rangle \notin D$, it follows that there is $d \in \mathscr{P}'$ such that $d(x, y) \geq \varepsilon_d$.
  - Thus we have $d \in \mathscr{P}$ with $d(x, y) > 0$.
- Thus g is an injection.

- We show that g is a uniform isomorphism onto $g[X]$ (relative to $\mathscr{D}$ and the subspace uniformity $\mathscr{E}$ on $g[X]$ inherited from the product uniformity on Z).
  - For each $d \in \mathscr{P}$, let $\mathscr{D}_d$ be the uniformity on X that is induced by the pseudometric d.
  - Note that $\mathscr{D}'_d = \mathscr{D}_d \cap (Z_d \times Z_d)$.
  - We show that for each $d \in \mathscr{P}$, the composition $\pi_d \circ g$ is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{D}'_d$, where $\pi_d : Z \to Z_d$ is the projection.
    - Let $d \in \mathscr{P}$ be fixed.
    - For each $x \in X$, denote $x_d := (\pi_d \circ g)(x)$.
    - Let $D' \in \mathscr{D}'_d$ be arbitrary and define
      $D := \{\langle x, y\rangle \in X \times X : \langle x_d, y_d\rangle \in D'\}$.
    - It remains to show that $D \in \mathscr{D}$.
      - Since the pseudometric d is uniformly continuous relative to $\mathscr{D}$, it follows that $\mathscr{D}_d \subseteq \mathscr{D}$. Thus it suffices to show that $D \in \mathscr{D}_d$.
      - Since $D' \in \mathscr{D}'_d$, there is $\varepsilon > 0$ be such that for any $w, z \in Z_d$ with $d(w, z) < \varepsilon$ we have $\langle w, z\rangle \in D'$.
      - Let $D'' := \{\langle x, y\rangle \in X \times X : d(x, y) < \varepsilon\}$. Note that $D'' \in \mathscr{D}_d$.
      - If $\langle x, y\rangle \in D''$, then $d(x_d, y_d) = d(x, y) < \varepsilon$ so $\langle x_d, y_d\rangle \in D'$, which implies that $\langle x, y\rangle \in D$.
      - Since $D'' \subseteq D$ and $D'' \in \mathscr{D}_d$, it follows that $D \in \mathscr{D}_d$.
  - Let $\mathscr{F} = \left\{\hat{g}^{-1}[E] : E \in \mathscr{E}\right\}$.
  - Since g is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{E}$, the definition of $\mathscr{F}$ implies that $\mathscr{F} \subseteq \mathscr{D}$.
  - Note that $\pi_d \circ g : X \to Z_d$ is uniformly continuous relative to $\mathscr{F}$ and $\mathscr{D}'_d$ for each $d \in \mathscr{P}$.
    - For each $d \in \mathscr{P}$, the projection π_d is uniformly continuous relative to the product uniformity on Z and $\mathscr{D}'_d$.
    - Thus, it suffices to show that $g : X \to Z$ is uniformly continuous relative to $\mathscr{F}$ and the product uniformity on Z.
    - Let D be a member of the product uniformity on Z.
    - Then $E := D \cap (g[X] \times g[X])$ is a member of $\mathscr{E}$.
    - It follows that $\hat{g}^{-1}[D] = \hat{g}^{-1}[E] \in \mathscr{F}$.
  - Since $\mathscr{D}$ is the weak uniformity on X induced by the family $\{(\mathscr{D}'_d, \pi_d \circ g), d \in \mathscr{P}\}$, it follows that $\mathscr{D} \subseteq \mathscr{F}$.
  - Thus $\mathscr{D} = \mathscr{F}$ and so g is a uniform isomorphism onto $g[X]$.

Exercise (completely regular subspace product).

Let (X, τ) be a topological space. Prove that X is completely regular if and only if it is homeomorphic to a subspace of a product of pseudometric spaces.

Solution. Let A be a set and (X_α, d_α) be a pseudometric space for each $\alpha \in A$.

- Then X_α is completely regular for each $\alpha \in A$ so $X' := \prod \{X_\alpha : \alpha \in A\}$ is completely regular.
- If X is homeomorphic to a subspace of X', then X is completely regular.

Now assume that (X, τ) is a completely regular topological space.

- Let A be the set of all continuous $f : X \to [0, 1]$.
- Let $Z := X \times [0, 1]$ and let d be the pseudometric on Z given by $d(\langle x, a\rangle, \langle y, b\rangle) = |a - b|$.
- Let $\varphi : X \to Z^A$ be defined as follows:
  - If $x \in X$, then let $\varphi(x) : A \to Z$ be such that $\varphi(x)(f) = \langle x, f(x)\rangle$ for every $f \in A$.

Hint: Show that $\varphi : X \to \varphi[X]$ is a homeomorphism.

Exercise.

Theorem (uniformity completely regular).

Let (X, τ) be a topological space. There exists a uniformity $\mathscr{D}$ on X that induces the topology τ if and only if τ is completely regular. It follows that there exists a separating uniformity on X that induces τ if and only if τ is Tychonoff.

Proof. Assume that there exists a uniformity $\mathscr{D}$ on X that induces the topology τ.

- Then $(X, \mathscr{D})$ is uniformly isomorphic to a subspace X' of a product $Y := \prod \{X_\alpha : \alpha \in A\}$, where X_α is a pseudometric space for each $\alpha \in A$.
- Let τ_α be the topology on X_α induced by the corresponding pseudometric.
- Since τ_α is completely regular for each $\alpha \in A$, it follows that the subspace topology on X' induced by the product topology on Y is also completely regular.
- Thus τ is completely regular.

Assume that τ is completely regular.

- The exercise above implies that X is homeomorphic to a subspace X' of a product $Y := \prod \{X_\alpha : \alpha \in A\}$, where X_α is a pseudometric space for each $\alpha \in A$.
- Let $\mathscr{D}'$ be the relative uniformity on X' that is induced by the product uniformity on Y.
- Let $\mathscr{D} = \left\{\hat{h}^{-1}[D] : D \in \mathscr{D}'\right\}$, where $h : X \to X'$ is a homeomorphism.
- Then $\mathscr{D}$ is a uniformity on X that induces τ.
  - Note that $\hat{h} : X \times X \to X' \times X'$ is a bijection such that for any $D \subseteq X \times X$ we have $\hat{h}[D] \in \mathscr{D}'$ if and only if $D \in \mathscr{D}$.
  - Let τ' be the subspace topology on X' inherited from the product topology on Y. Then τ' is induced by the uniformity $\mathscr{D}'$ on X'.
  - Since for any $U \subseteq X$, we have $U \in \tau$ if and only if $h[U] \in \tau'$, it follows that $\mathscr{D}$ induces τ on X.
    * Let $x \in X$. We show that $\{D[x] : D \in \mathscr{D}\}$ is a nbhd base at x, relative to τ.
    * If $D \in \mathscr{D}$, then $\hat{h}[D] \in \mathscr{D}'$ so $\hat{h}[D][h(x)]$ is a nbhd of $h(x)$ with respect to τ'.
    * It follows that $h^{-1}\left[\hat{h}[D][h(x)]\right]$ is a nbhd of x with respect to τ.
    * We show that $h^{-1}\left[\hat{h}[D][h(x)]\right] = D[x]$.

      The following are equivalent:
      · $y \in h^{-1}\left[\hat{h}[D][h(x)]\right]$,
      · $h(y) \in \hat{h}[D][h(x)]$,
      · $\langle h(x), h(y)\rangle \in \hat{h}[D]$,
      · $\langle x, y\rangle \in D$,
      · $y \in D[x]$.
    * It follows that $D[x]$ is a nbhd of x with respect to τ.
    * Let U be any nbhd of x with respect to τ.
    * We show that there is $D \in \mathscr{D}$ with $D[x] \subseteq U$.
      · $h[U]$ is a nbhd of $h(x)$ with respect to τ', so there is $D' \in \mathscr{D}'$ with $D'[h(x)] \subseteq h[U]$.
      · Let $D \in \mathscr{D}$ be such that $D' = \hat{h}[D]$.
      · As proved above, we have $h^{-1}[D'[h(x)]] = D[x]$.
      · If $y \in D[x]$, then $h(y) \in D'[h(x)] \in h[U]$ so $y \in U$.
      · Thus $D[x] \subseteq U$.
    * Thus $\{D[x] : D \in \mathscr{D}\}$ is a nbhd base at x, relative to τ.

Cauchy net.

Let $(X, \mathscr{D})$ be a uniform space and $(x_\alpha : \alpha \in I)$ be a net in X. We say that the net $(x_\alpha : \alpha \in I)$ is *Cauchy* (*$\mathscr{D}$-Cauchy*, *Cauchy relative to* $\mathscr{D}$) iff for each $D \in \mathscr{D}$ there exists $\gamma \in I$ such that $\langle x_\alpha, x_\beta \rangle \in D$ for every $\alpha, \beta \geq \gamma$.

(1) Assume that $\mathscr{D}$ is induced by a pseudometric d on X.

- We say that a net $(x_\alpha : \alpha \in I)$ in X is d-Cauchy iff for every $\varepsilon > 0$, there is $\gamma \in I$ such that $d(x_\alpha, x_\beta) < \varepsilon$ for every $\alpha, \beta \geq \gamma$.
- Then $(x_\alpha : \alpha \in I)$ is d-Cauchy if and only if it is $\mathscr{D}$-Cauchy.
  - Assume that $(x_\alpha : \alpha \in I)$ is d-Cauchy.
  - Let $D \in \mathscr{D}$. We show that there exists $\gamma \in I$ such that $\langle x_\alpha, x_\beta \rangle \in D$ for every $\alpha, \beta \geq \gamma$.
    * There is $\varepsilon > 0$ such that $\langle x, y \rangle \in D$ for any $x, y \in X$ with $d(x, y) < \varepsilon$.
    * Let $\gamma \in I$ such that $d(x_\alpha, x_\beta) < \varepsilon$ for every $\alpha, \beta \geq \gamma$.
    * Then $\langle x_\alpha, x_\beta \rangle \in D$ for every $\alpha, \beta \geq \gamma$.
  - Assume that $(x_\alpha : \alpha \in I)$ is $\mathscr{D}$-Cauchy.
  - Let $\varepsilon > 0$. We show that there exists $\gamma \in I$ such that $d(x_\alpha, x_\beta) < \varepsilon$ for every $\alpha, \beta \geq \gamma$.
    * Let $D := \{\langle x, y \rangle \in X \times X : d(x, y) < \varepsilon\}$. Then $D \in \mathscr{D}$.
    * There exists $\gamma \in I$ such that $\langle x_\alpha, x_\beta \rangle \in D$ for every $\alpha, \beta \geq \gamma$.
    * Thus $d(x_\alpha, x_\beta) < \varepsilon$ for every $\alpha, \beta \geq \gamma$.

(2) Given a directed set I, assume that $I \times I$ has the product direction, that is let $\langle \alpha, \beta \rangle \leq \langle \gamma, \delta \rangle$ iff $\alpha \leq \gamma$ and $\beta \leq \delta$ for any $\alpha, \beta, \gamma, \delta \in I$.

- A net $(x_\alpha : \alpha \in I)$ in X is $\mathscr{D}$-Cauchy if and only if for every $D \in \mathscr{D}$ the net $(\langle x_\alpha, x_\beta \rangle : \langle \alpha, \beta \rangle \in I \times I)$ is eventually in D. (There is $\langle \gamma, \delta \rangle \in I \times I$ such that $\langle x_\alpha, x_\beta \rangle \in D$ for all $\langle \alpha, \beta \rangle \geq \langle \gamma, \delta \rangle$.)
  - Assume that the net $(x_\alpha : \alpha \in I)$ is $\mathscr{D}$-Cauchy.
    * Let $D \in \mathscr{D}$. There is $\gamma \in I$ such that $\langle x_\alpha, x_\beta \rangle \in D$ for all $\alpha, \beta \geq \gamma$.
    * Then $\langle \gamma, \gamma \rangle \in I \times I$ and $\langle x_\alpha, x_\beta \rangle \in D$ for all $\langle \alpha, \beta \rangle \geq \langle \gamma, \gamma \rangle$.
  - Assume that there is $\langle \gamma, \delta \rangle \in I \times I$ such that $\langle x_\alpha, x_\beta \rangle \in D$ for all $\langle \alpha, \beta \rangle \geq \langle \gamma, \delta \rangle$.
    * There is $\xi \in I$ such that $\gamma, \delta \leq \xi$.
    * Then $\langle x_\alpha, x_\beta \rangle \in D$ for all $\alpha, \beta \geq \xi$.
    * Thus the net $(x_\alpha : \alpha \in I)$ is $\mathscr{D}$-Cauchy.

(3) A net $(x_\alpha : \alpha \in I)$ in X is $\mathscr{D}$-Cauchy if and only if there exists a subbase $\mathscr{S}$ for $\mathscr{D}$ such that for every $S \in \mathscr{S}$ the net $(\langle x_\alpha, x_\beta\rangle : \langle \alpha, \beta\rangle \in I \times I)$ is eventually in S.

Proof. Assume that the net $(x_\alpha : \alpha \in I)$ is $\mathscr{D}$-Cauchy.

- Then $\mathscr{S} := \mathscr{D}$ satisfies the requirements.

Assume that there exists a subbase $\mathscr{S}$ for $\mathscr{D}$ such that for every $S \in \mathscr{S}$ the net $(\langle x_\alpha, x_\beta\rangle : \langle \alpha, \beta\rangle \in I \times I)$ is eventually in S.

- Let $D \in \mathscr{D}$.
- We show that there exists $\gamma \in I$ such that $\langle x_\alpha, x_\beta\rangle \in D$ for every $\alpha, \beta \geq \gamma$.
    - We can assume $D \neq X \times X$.
    - Let $\mathscr{S}' \subseteq \mathscr{S}$ be finite, nonempty and such that $\bigcap \mathscr{S}' \subseteq D$.
    - For each $S \in \mathscr{S}'$, there is $\gamma_S \in I$ such that $\langle x_\alpha, x_\beta\rangle \in S$ for every $\alpha, \beta \geq \gamma_S$.
    - Let $\gamma \in I$ be such that $\gamma_S \leq \gamma$ for every $S \in \mathscr{S}'$. Such γ exists since $\mathscr{S}'$ is finite and I is directed.
    - If $\alpha, \beta \geq \gamma$, then $\langle x_\alpha, x_\beta\rangle \in D$ as required.

(4) Let $\mathscr{P}$ be a family of pseudometrics on X such that $\mathscr{P}$ induces the uniformity $\mathscr{D}$. Then a net $(x_\alpha : \alpha \in I)$ in X is $\mathscr{D}$-Cauchy if and only if for every $d \in \mathscr{P}$ the net $(d(x_\alpha, x_\beta) : \langle \alpha, \beta\rangle \in I \times I)$ converges to 0.

Proof. Assume that $(x_\alpha : \alpha \in I)$ is a $\mathscr{D}$-Cauchy net in X.

- Let $d \in \mathscr{P}$. Let $\varepsilon > 0$.
- We show that there exists $\gamma \in I$ such that $d(x_\alpha, x_\beta) < \varepsilon$ for every $\alpha, \beta \geq \gamma$.
    - Let $D := \langle \langle x, y\rangle \in X \times X : d(x, y) < \varepsilon\rangle$. Then $D \in \mathscr{D}$.
    - There is $\gamma \in I$ such that $\langle x_\alpha, x_\beta\rangle \in D$ for every $\alpha, \beta \geq \gamma$.
    - Thus $d(x_\alpha, x_\beta) < \varepsilon$ for every $\alpha, \beta \geq \gamma$ as required.

Assume that for every $d \in \mathscr{P}$ the net $(d(x_\alpha, x_\beta) : \langle \alpha, \beta\rangle \in I \times I)$ converges to 0.

- Let $D \in \mathscr{D}$.
- We show that there exists $\gamma \in I$ such that $\langle x_\alpha, x_\beta\rangle \in D$ for every $\alpha, \beta \geq \gamma$.
    - We can assume that $D \neq X \times X$.

- There is finite $\mathscr{P}' \subseteq \mathscr{P}$ and $\varepsilon_d > 0$ for every $d \in \mathscr{P}'$ such that $\bigcap\{D_d : d \in \mathscr{P}'\} \subseteq D$, where
$$D_d = \{\langle x, y\rangle \in X \times X : d(x, y) < \varepsilon_d\}$$
for every $d \in \mathscr{P}'$.
- For each $d \in \mathscr{P}'$, let $\gamma_d \in I$ be such that $d(x_\alpha, x_\beta) < \varepsilon_d$ for every $\alpha, \beta \geq \gamma_d$.
- Let $\gamma \in I$ be such that $\gamma \geq \gamma_d$ for each $d \in \mathscr{P}'$.
- If $\alpha, \beta \geq \gamma$, then $\langle x_\alpha, x_\beta\rangle \in D_d$ for every $d \in \mathscr{P}'$ so $\langle x_\alpha, x_\beta\rangle \in D$ as required.

Theorem (convergent net is Cauchy).

Let $(x_\alpha : \alpha \in I)$ be a net in a uniform space $(X, \mathscr{D})$. If $(x_\alpha : \alpha \in I)$ converges in the topology on X that is induced by $\mathscr{D}$, then it is $\mathscr{D}$-Cauchy.

Proof. Assume that $(x_\alpha : \alpha \in I)$ converges to $x \in X$.

- Let $D \in \mathscr{D}$.
- Let $E \in \mathscr{D}$ be symmetric and such that $E \circ E \subseteq D$.
- There is $\gamma \in I$ such that $x_\alpha \in E[x]$ for every $\alpha \geq \gamma$.
- If $\alpha, \beta \geq \gamma$, then $\langle x, x_\alpha\rangle \in E$ and $\langle x, x_\beta\rangle \in E$, which implies that $\langle x_\alpha, x_\beta\rangle \in E \circ E \subseteq D$.
- Thus $(x_\alpha : \alpha \in I)$ is $\mathscr{D}$-Cauchy.

Theorem (Cauchy net converges cluster).

Let $(x_\alpha : \alpha \in I)$ be a net in a uniform space $(X, \mathscr{D})$. If $(x_\alpha : \alpha \in I)$ is $\mathscr{D}$-Cauchy and has a cluster point $x \in X$, then it converges to x in the topology induced by $\mathscr{D}$.

Proof. Let $D \in \mathscr{D}$.

- Let $E \in \mathscr{D}$ be such that $E \circ E \subseteq D$.
- There is $\gamma \in I$ such that $\langle x_\alpha, x_\beta\rangle \in E$ for every $\alpha, \beta \geq \gamma$.
- Since x is a cluster point of $(x_\alpha : \alpha \in I)$, there is $\delta \in I$ be such that $\delta \geq \gamma$ and $x_\delta \in E[x]$.
- If $\alpha \geq \delta$, then $\langle x_\delta, x_\alpha\rangle \in E$.
- Since $\langle x, x_\delta\rangle \in E$, it follows that $\langle x, x_\alpha\rangle \in E \circ E \subseteq D$, so $x_\alpha \in D[x]$.
- Thus $(x_\alpha : \alpha \in I)$ converges to x.

Complete uniform space.

Let $(X, \mathscr{D})$ be a uniform space. We say that the space is complete (or that $\mathscr{D}$ is complete) iff every $\mathscr{D}$-Cauchy net converges to some $x \in X$ in the topology induced by $\mathscr{D}$.

Theorem. Assume that $\mathscr{D}$ is induced by a pseudometric d on X. Then $(X, \mathscr{D})$ is $\mathscr{D}$-complete if and only if every d-Cauchy sequence in X converges to some $x \in X$ with respect to the topology on X that is induced by $\mathscr{D}$ (which is the same as the topology induced by d).

Proof. Assume that $(X, \mathscr{D})$ is $\mathscr{D}$-complete.

- Let $(x_n : n \in \mathbb{N})$ be a d-Cauchy sequence in X. It follows that $(x_n : n \in \mathbb{N})$ is $\mathscr{D}$-Cauchy. Thus $(x_n : n \in \mathbb{N})$ converges to some $x \in X$.

Assume that every d-Cauchy sequence in X converges.

- Let $(x_\alpha : \alpha \in I)$ be a net in X that is $\mathscr{D}$-Cauchy.
- We find $x \in X$ such that $(x_\alpha : \alpha \in I)$ converges to x.
  - For each $n \in \mathbb{N}$, let
    $$D_n := \{\langle x, y\rangle \in X \times X : d(x, y) < 1/n\}.$$
    Then $D_n \in \mathscr{D}$.
  - Let $\alpha_1 \in I$ be such that $\langle x_\alpha, x_\beta\rangle \in D_1$ for every $\alpha, \beta \geq \alpha_1$.
  - Suppose $n \geq 1$ and $\alpha_n \in I$ is defined. Define $\alpha_{n+1} \in I$ to be such that $\alpha_{n+1} \geq \alpha_n$ and $\langle x_\alpha, x_\beta\rangle \in D_{n+1}$ for every $\alpha, \beta \geq \alpha_{n+1}$.
  - The sequence $(x_{\alpha_n} : n \in \mathbb{N})$ is d-Cauchy.
    * Let $\varepsilon > 0$.
    * Take $n \in \mathbb{N}$ such that $1/n < \varepsilon$.
    * If $m, k \geq n$, then $\alpha_m, \alpha_k \geq \alpha_n$ so $d(x_{\alpha_m}, x_{\alpha_k}) < \varepsilon$.
  - There is $x \in X$ such that $(x_{\alpha_n} : n \in \mathbb{N})$ converges to x.
  - We show that the net $(x_\alpha : \alpha \in I)$ converges to x.
    * Let $\varepsilon > 0$.
    * Let $n \in \mathbb{N}$ be such that $2/n < \varepsilon$ and $d(x_{\alpha_n}, x) < \varepsilon/2$.
    * If $\alpha \geq \alpha_n$, then $d(x_\alpha, x_{\alpha_n}) < 1/n < \varepsilon/2$ so $d(x_\alpha, x) < \varepsilon$.
    * Thus the net $(x_\alpha : \alpha \in I)$ converges to x.

Example (topological group uniformity).

Let (G,τ) be a topological group. For each nbhd U of the identity e of G, let

$$U_L = \{\langle x,y\rangle \in G\times G : x^{-1}y \in U\}$$

and

$$U_R = \{\langle x,y\rangle \in G\times G : xy^{-1} \in U\}.$$

Let $\mathscr{L}' = \{U_L : U \text{ is a nbhd of } e\}$ and $\mathscr{R}' = \{U_R : U \text{ is a nbhd of } e\}$.

- Then each of $\mathscr{L}'$ and $\mathscr{R}'$ is a uniformity base on X.
- Let $\mathscr{L}$ and $\mathscr{R}$ be the uniformities on G that are induced by $\mathscr{L}'$ and $\mathscr{R}'$, respectively. We call $\mathscr{L}$ the *left uniformity* of the topological group, and $\mathscr{R}$ is called the *right uniformity*.
- Then $\mathscr{L}$ induces the topology τ and the same is true for $\mathscr{R}$.

Let G be the set of all real-valued functions on $\mathbb{R}$ that are of the form $g(x) = ax+b$ for some $a,b\in\mathbb{R}$ with $a\neq 0$.

- Then G is a group under composition.

Let $\mathscr{B} = \{B_\varepsilon : \varepsilon > 0\}$ be a family of subsets of G, where B_ε consists of those $g\in G$ that are of the form $g(x) = ax+b$ with $|a-1|<\varepsilon$ and $|b|<\varepsilon$.

- There exists a topology τ on G making it a topological group such that $\mathscr{B}$ is a nbhd base at the identity of G.
- Let $\mathscr{L}$ and $\mathscr{R}$ be the left and right uniformity of G.
- In this case, we have $\mathscr{L}\neq\mathscr{R}$.

  Exercise.
- There does not exist an invariant metric d on G that induces τ. (A metric d on G is *invariant* iff

  $$d(g,h) = d(f\circ g, f\circ h) = d(g\circ f, h\circ f)$$

  for every $f,g,h\in G$.

  Exercise.