Math 793C Topology for Analysis

Jerzy Wojciechowski Spring 2020

Class 31 April 8

Lemma (continuous pseudometric).

Let X be a set and $(D_n : n \in \mathbb{N})$ be a sequence of symmetric reflexive relations on X such that $D_1 = X \times X$ and

$$D_{n+1} \circ D_{n+1} \circ D_{n+1} \subseteq D_n$$

for each $n \in \mathbb{N}$. Then there exists a pseudometric d on X such that

$$D_n \subseteq \{\langle x, y\rangle \in X \times X : d(x,y) < 2^{-n}\} \subseteq D_{n-1}$$

for each $n \geq 2$.

Proof. Note that $D_1 \supseteq D_2 \supseteq \dots$.

- Define $f : X \times X \to [0,\infty)$ by $f(x,y) = 2^{-n}$, where n is the smallest element of $\mathbb{N}$ with $\langle x, y\rangle \notin D_n$ if such n exists and $f(x,y) = 0$ otherwise.
- For each $x, y \in X$, let $S(x,y)$ be the set of all finite sequences in X with at least two terms such that the first term is equal to x and the last is equal to y.
- Define $d : X \times X \to [0,\infty)$ by

  $$d(x,y) = \inf\left\{\sum_{i=1}^{n} f(x_i, x_{i+1}) : (x_1,\dots,x_{n+1}) \in S(x,y)\right\}.$$
- We verify that d is a pseudometric on X.

  We check that the following conditions hold:

  - $d(x,x) = 0$ for each $x \in X$.

    Let $x \in X$. Since $f(x,x) = 0$, it follows that $d(x,x) = 0$.
  - $d(x,y) = d(y,x)$ for each $x, y \in X$.

    Let $x, y \in X$. Let

$P := \{\sum_{i=1}^{n} f(x_i, x_{i+1}) : (x_1, \ldots, x_{n+1}) \in S(x,y)\}$

and

$Q := \{\sum_{i=1}^{n} f(x_i, x_{i+1}) : (x_1, \ldots, x_{n+1}) \in S(y,x)\}$.

    * Since D_i is symmetric for each $i \in \mathbb{N}$, it follows that $f(w,z) = f(z,w)$ for every $w, z \in X$. Thus $P = Q$.
    * Since $d(x,y) = \inf P$ and $d(y,x) = \inf Q$, the conclusion holds.

  - $d(x,z) \le d(x,y) + d(y,z)$ for any $x, y, z \in X$.

    * Let $x, y, z \in X$. Let
      $P := \{\sum_{i=1}^{n} f(x_i, x_{i+1}) : (x_1, \ldots, x_{n+1}) \in S(x,y)\}$,
      $Q := \{\sum_{i=1}^{n} f(x_i, x_{i+1}) : (x_1, \ldots, x_{n+1}) \in S(y,z)\}$,
      $R := \{\sum_{i=1}^{n} f(x_i, x_{i+1}) : (x_1, \ldots, x_{n+1}) \in S(x,z)\}$.
    * Then $p + q \in R$ for any $p \in P$ and $q \in Q$.
    * Thus $\inf R \le p + q$ for any $p \in P$ and $q \in Q$.
    * Thus $\inf R \le \inf P + \inf Q$ and the conclusion holds.

- Let $n \ge 2$ and $E_n = \{\langle x, y\rangle \in X \times X : d(x,y) < 2^{-n}\}$.

- We show that $D_n \subseteq E_n$.

  - Let $\langle x, y\rangle \in D_n$.
  - If $\langle x, y\rangle \in D_m$ for each $m \in \mathbb{N}$, then $f(x,y) = 0$, which implies that $d(x,y) = 0$.
  - Otherwise, there is the smallest $m \in \mathbb{N}$ such that $\langle x, y\rangle \notin D_m$.
  - Then $m > n$ so $f(x,y) = 2^{-m} < 2^{-n}$.
  - If follows that $d(x,y) \le f(x,y) < 2^{-n}$ so $\langle x, y\rangle \in E_n$.

- We show that $E_n \subseteq D_{n-1}$.

  - Let $\langle x, y\rangle \in E_n$. Then $d(x,y) < 2^{-n}$.
  - There is a sequence $(x_1, x_2, \ldots, x_{m+1})$ with $x_1 = x$ and $x_{m+1} = y$ such that

    $\sum_{i=1}^{m} f(x_i, x_{i+1}) < 2^{-n}$.
  - We use induction on $m \in \mathbb{N}$ to show that $\langle x, y\rangle \in D_{n-1}$.

    * First, we prove that:
      (!) if $w, z \in X$ and $f(w,z) < 2^{-n}$, then $\langle w, z\rangle \in D_n$.

      · Since $f(w,z) < 2^{-n}$, it follows that either $\langle w, z\rangle \in D_k$ for all $k \in \mathbb{N}$ or $f(w,z) = 2^{-k}$ for some $k > n$.
      · In the former case, it is clear that $\langle w, z\rangle \in D_n$.

- In the later case, k is the smallest element of $\mathbb{N}$ with $\langle x, y\rangle \notin D_k$.
- Since $k > n$, it follows that $\langle x, y\rangle \in D_n$.

* If $m = 1$, then $f(x, y) < 2^{-n}$ so (!) implies that $\langle x, y\rangle \in D_{n-1}$.
* Assume that $m \geq 2$ and, as inductive hypothesis, that:
  - For any $k \in \mathbb{N}$ and $\ell \in \{1, 2, \ldots, m-1\}$, if $(y_1, y_2, \ldots, y_{\ell+1})$ is a sequence of elements of X such that
    $\sum_{i=1}^{\ell} f(y_i, y_{i+1}) < 2^{-k}$,
    then $\langle y_1, y_{\ell+1}\rangle \in D_{k-1}$.
* We show that $\langle x, y\rangle \in D_{n-1}$.
  - We show that there exists $j \in \{1, 2, \ldots, m\}$ such that
    $\sum_{i=1}^{j} f(x_i, x_{i+1}) < 2^{-n-1}$,
    whenever $j \geq 2$, and
    $\sum_{i=j+1}^{m} f(x_i, x_{i+1}) < 2^{-n-1}$,
    when $j \leq m-1$.
  - If $\sum_{i=1}^{m} f(x_i, x_{i+1}) < 2^{-n-1}$, then taking $j := m$ works.
  - Otherwise, let $j \in \{1, 2, \ldots, m-1\}$ be as small as possible with $\sum_{i=1}^{j+1} f(x_i, x_{i+1}) \geq 2^{-n-1}$.
  - If $j \geq 2$, then $\sum_{i=1}^{j} f(x_i, x_{i+1}) < 2^{-n-1}$ as required.
  - Since $\sum_{i=1}^{m} f(x_i, x_{i+1}) < 2^{-n}$, it follows that
    $\sum_{i=j+1}^{m} f(x_i, x_{i+1}) < 2^{-n-1}$ as required.
  - The inductive hypothesis implies that $\langle x_1, x_j\rangle \in D_n$ and $\langle x_{j+1}, x_{m+1}\rangle \in D_n$.
  - We have $f(x_j, x_{j+1}) < 2^{-n}$, so (!) implies that $\langle x_j, x_{j+1}\rangle \in D_n$.
  - Since $D_n \circ D_n \circ D_n \subseteq D_{n-1}$, it follows that $\langle x, y\rangle \in D_{n-1}$.

Exercise (continuous pseudometric).

Let $\mathscr{D}$ be a uniformity on a set X and $(D_n : n \in \mathbb{N})$ be a sequence of symmetric members of $\mathscr{D}$ such that $D_1 = X \times X$ and

$$D_{n+1} \circ D_{n+1} \circ D_{n+1} \subseteq D_n$$

for each $n \in \mathbb{N}$. Let d be a pseudometric on X such that

$$D_n \subseteq \{\langle x, y\rangle \in X \times X : d(x, y) < 2^{-n}\} \subseteq D_{n-1}$$

for each $n \geq 2$.

- Prove that d is uniformly continuous relative to $\mathscr{D}$.
- Let $\mathscr{B} = \{D_n : n \in \mathbb{N}\}$. Prove that $\mathscr{B}$ is a uniformity base on X.
- Let $\mathscr{E}$ be the uniformity on X that is induced by $\mathscr{B}$. Prove that $\mathscr{E}$ is the uniformity induced by the pseudometric d.

Solution.

- d is uniformly continuous relative to $\mathscr{D}$.

  - Let $\varepsilon > 0$ and $A_\varepsilon = \{\langle x, y\rangle \in X \times X : d(x, y) < \varepsilon\}$.
  - To show that d is uniformly continuous relative to $\mathscr{D}$, it suffices to show that $A_\varepsilon \in \mathscr{D}$.
  - There is $n \in \mathbb{N}$ with $2^{-n} \leq \varepsilon$. Then $D_n \subseteq A_\varepsilon$.
  - Since $D_n \in \mathscr{D}$, it follows that $A_\varepsilon \in \mathscr{D}$.

- $\mathscr{B}$ is a uniformity base on X.

  We need to verify the following conditions:

  - Each member of $\mathscr{B}$ is a reflexive relation on X.

    Holds by assumption.
  - For each $B \in \mathscr{B}$, there exist $D \in \mathscr{B}$ with $D \subseteq B^{-1}$.

    Holds since the members of $\mathscr{B}$ are symmetric.
  - For each $B \in \mathscr{B}$, there exists $D \in \mathscr{B}$ with $D \circ D \subseteq B$.

    If $B = D_n$, then $D := D_{n+1}$ satisfies the requirements.
  - For each $B, D \in \mathscr{B}$, there exists $E \in \mathscr{B}$ with $E \subseteq B \cap D$.

    If $B = D_n$ and $D = D_m$, then let $E := D_k$, where $k = \max\{n, m\}$.

- $\mathscr{E}$ is the uniformity induced by the pseudometric d.

  Let $\mathscr{E}'$ be the uniformity induced by d.

  - Let $E \in \mathscr{E}'$. We show that $E \in \mathscr{E}$.

    * There is $\varepsilon > 0$ such that $A_\varepsilon \subseteq E$, where
      $$A_\varepsilon = \{\langle x, y\rangle \in X \times X : d(x, y) < \varepsilon\}.$$
    * If $n \in \mathbb{N}$ satisfies $2^{-n} \leq \varepsilon$, then $D_n \subseteq A_\varepsilon$ so $D_n \subseteq E$ and hence $E \in \mathscr{E}$.

  - Let $E \in \mathscr{E}$. We show that $E \in \mathscr{E}'$.

    * There is $n \in \mathbb{N}$ with $D_n \subseteq E$.
    * If $x, y \in X$ and $d(x, y) < 2^{-n-1}$, then $\langle x, y\rangle \in D_n$ so $\langle x, y\rangle \in E$.
    * It follows that $E \in \mathscr{E}'$.

Pseudometrizable uniform space.

A uniform space $(X, \mathscr{D})$ is *pseudometrizable* iff there exists a pseudometric d on X such that $\mathscr{D}$ is the uniformity induced by d, (see uniformity from pseudometric).

- $(X, \mathscr{D})$ is *metrizable* iff there exists a metric d on X such that $\mathscr{D}$ is the uniformity induced by d.
- Note that $(X, \mathscr{D})$ is metrizable if and only if it is pseudometrizable and $\mathscr{D}$ is separating.

    - If $(X, \mathscr{D})$ is metrizable, then it is pseudometrizable and $\mathscr{D}$ is separating.
    - Assume that $(X, \mathscr{D})$ is pseudometrizable and $\mathscr{D}$ is separating.
        * Let d be a pseudometric on X that induces $\mathscr{D}$.
        * Since $\mathscr{D}$ is separating, d must be a metric.
        * Thus $(X, \mathscr{D})$ is metrizable.

Theorem (pseudometrizable uniform space).

A uniform space is pseudometrizable if and only if it has a countable base.

Proof. Assume that $(X, \mathscr{D})$ is a pseudo-metrizable uniform space.

- Let d be a pseudometric on X that induces $\mathscr{D}$.
- Define $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$, where

$$B_n = \{\langle x, y\rangle \in X \times X : d(x, y) < 1/n\}$$

  for each $n \in \mathbb{N}$.
- We show that $\mathscr{B}$ is a base for $\mathscr{D}$.

    - Since $B_n \in \mathscr{D}$ for each $n \in \mathbb{N}$, we have $\mathscr{B} \subseteq \mathscr{D}$.
    - It remains to show that for each $D \in \mathscr{D}$ there is $n \in \mathbb{N}$ with $B_n \subseteq D$.
    - Let $D \in \mathscr{D}$. There is $\varepsilon > 0$ such that $\langle x, y\rangle \in D$ for each $x, y \in X$ with $d(x, y) < \varepsilon$.
    - Let $n \in \mathbb{N}$ be such that $1/n \leq \varepsilon$. Then $B_n \subseteq D$.

Now assume that $\mathscr{D}$ has a countable base $\mathscr{B} = \{B_1, B_2, \dots\}$.

- Let $D_1, D_2, \ldots$ be a sequence of members of $\mathscr{D}$ such that
  - $D_1 = X \times X$.
  - D_n is symmetric for each $n \in \mathbb{N}$.
  - $D_{n+1} \circ D_{n+1} \circ D_{n+1} \subseteq D_n$ and $D_{n+1} \subseteq B_n$ for every $n \in \mathbb{N}$.
- Let d be a pseudometric on X such that

$$D_n \subseteq \left\{ \langle x, y \rangle \in X \times X : d(x, y) < 2^{-n} \right\} \subseteq D_{n-1}$$

  for every $n \geq 2$. Such a pseudometric exists by Lemma (continuous pseudometric).
- Let $\mathscr{E}$ be the uniformity on X that is induced by d. We show that $\mathscr{E} = \mathscr{D}$.
  - We show that $\mathscr{E} \subseteq \mathscr{D}$.
    * Let $E \in \mathscr{E}$. There is $\varepsilon > 0$ such that $\langle x, y \rangle \in E$ for every $x, y \in X$ with $d(x, y) < \varepsilon$.
    * Let $n \in \mathbb{N}$ be such that $2^{-n} \leq \varepsilon$. Then $D_n \subseteq E$.
    * Since $D_n \in \mathscr{D}$, it follows that $E \in \mathscr{D}$.
  - We show that $\mathscr{D} \subseteq \mathscr{E}$.
    * Let $D \in \mathscr{D}$. There is $n \in \mathbb{N}$ with $B_n \subseteq D$.
    * Then $D_{n+1} \subseteq B_n$ so $D_{n+1} \subseteq D$. Let $\varepsilon = 2^{-n-2} > 0$.
    * For any $x, y \in X$ with $d(x, y) < \varepsilon$, we have $\langle x, y \rangle \in D_{n+1} \subseteq D$.
    * Thus $D \in \mathscr{E}$.

Example (nonmetrizable uniformity metrizable topology).

Let $X = \Omega_0$ (the set of all countable ordinals) and for each $\alpha \in X$ let

$$B_\alpha = \{\langle x, y \rangle \in X \times X : x = y \text{ or both } x, y > \alpha\}.$$

- Then $\mathscr{B} = \{B_\alpha : \alpha \in X\}$ is a uniformity base on X.
- Let $\mathscr{D}$ be the uniformity on X that is induced by $\mathscr{B}$. Then $\mathscr{D}$ is not metrizable, in particular $\mathscr{D}$ is not the discrete uniformity.
- The topology on X induced by $\mathscr{D}$ is the discrete topology, hence the topology is metrizable.

Proof. We show that:

- $\mathscr{B} = \{B_\alpha : \alpha \in X\}$ is a uniformity base on X.

  We show the following conditions:

  - Every member of $\mathscr{D}$ is a reflexive relation on X.

    This is clear from the definition of B_α.

  - If $B \in \mathscr{B}$, then there exists $D \in \mathscr{B}$ wit $D^{-1} \subseteq B$.

    We have $B^{-1} = B$ for each $B \in \mathscr{B}$.

  - If $B \in \mathscr{B}$, then there is $D \in \mathscr{B}$ with $D \circ D \subseteq B$.

    * Let $\alpha \in X$ be such that $B = B_\alpha$.
    * Since $B_\alpha \circ B_\alpha = B_\alpha$, taking $D := B$ works.

  - If $B, D \in \mathscr{B}$, then there is $E \in \mathscr{B}$ with $E \subseteq B \cap D$.

    * Let $\alpha, \beta \in X$ be such that $B = B_\alpha$ and $D = B_\beta$.
    * Let $\gamma = \max\{\alpha, \beta\}$. Then $E := B_\gamma$ works.

- $\mathscr{D}$ is not metrizable.

  We show that $\mathscr{D}$ has no countable base. Then Theorem (pseudometrizable uniform space) implies that $\mathscr{D}$ is not metrizable.

  - Suppose, for a contradiction, that $\mathscr{A}$ is a countable base for $\mathscr{D}$.

  - For each $A \in \mathscr{A}$, there is $\alpha_A \in X$ with $B_{\alpha_A} \subseteq A$.

  - Since the set $\{\alpha_A : A \in \mathscr{A}\}$ is a countable subset of X, there is $\beta \in X$ such that $\alpha_A < \beta$ for every $A \in \mathscr{A}$.

  - Then $\langle \beta, \beta + 1 \rangle \in B_{\alpha_A} \subseteq A$ for any $A \in \mathscr{A}$, but $\langle \beta, \beta + 1 \rangle \notin B_\beta$.

  - Thus no member of $\mathscr{A}$ is a subset of B_β (which is a member of $\mathscr{D}$).

  - Hence $\mathscr{A}$ is not a base for $\mathscr{D}$, and we have a contradiction.

- The topology induced by $\mathscr{D}$ is the discrete topology.

  - Let $\alpha \in X$. There is $\beta \in X$ with $\alpha < \beta$. Let $D := B_\beta \in \mathscr{D}$.

  - Then $D[\alpha] = \{\alpha\}$ so $\{\alpha\}$ is open in the topology τ on X that is induced by $\mathscr{D}$.

  - Thus τ is the discrete topology.

Uniformity from pseudometric family.

Let $\mathscr{P}$ be a family of pseudometrics on a set X. For each $d \in \mathscr{P}$, let $\mathscr{D}_d$ be the uniformity on X that is induced by d and let $f_d : X \to X$ be the identity function. Let $\mathscr{D}$ be the weak uniformity on X induced by the family $\{(\mathscr{D}_d, f_d) : d \in \mathscr{P}\}$. We say that $\mathscr{D}$ it the uniformity on X that is *induced* by $\mathscr{P}$.

(1) Theorem (union uniformity subbase) implies that $\mathscr{S} = \bigcup\{\mathscr{D}_d : d \in \mathscr{P}\}$ is a uniformity subbase on X. Note that $\mathscr{S}$ induces $\mathscr{D}$.

- For each $d \in \mathscr{P}$ and $D \in \mathscr{D}_d$, we have $\hat{f}_d^{-1}[D] = D$.
- Then $\mathscr{D}_d = \mathscr{D}'_d$, where $\mathscr{D}'_d = \left\{\hat{f}_d^{-1}[D] : D \in \mathscr{D}_d\right\}$ for each $d \in \mathscr{P}$.
- The definition of $\mathscr{D}$ as the weak uniformity means that $\mathscr{D}$ is induced by $\bigcup\{\mathscr{D}'_d : d \in \mathscr{P}\}$ which is equal to $\mathscr{S}$.

(2) $\mathscr{D}$ is the weakest uniformity on X making all the pseudometrics in $\mathscr{P}$ uniformly continuous.

(3) Let $\mathscr{B} = \{B_{\mathscr{Q},\varepsilon} : \mathscr{Q} \subseteq_f \mathscr{P}, \varepsilon > 0\}$, where

$$B_{\mathscr{Q},\varepsilon} = \{\langle x, y\rangle \in X \times X : (\forall d \in \mathscr{Q})\, d(x, y) < \varepsilon\}$$

and $\mathscr{Q} \subseteq_f \mathscr{P}$ means that $\mathscr{Q}$ is a finite subset of $\mathscr{P}$. Then $\mathscr{B}$ is base for $\mathscr{D}$.

Proof.

- We show that $\mathscr{B} \subseteq \mathscr{D}$.
  - Let $B \in \mathscr{B}$. Then there are $\mathscr{Q} \subseteq_f \mathscr{P}$ and $\varepsilon > 0$ such that $B = B_{\mathscr{Q},\varepsilon}$.
  - If $d \in \mathscr{Q}$, then $E_{d,\varepsilon} \in \mathscr{D}_d \subseteq \mathscr{D}$, where

    $$E_{d,\varepsilon} = \{\langle x, y\rangle \in X \times X : d(x, y) < \varepsilon\}.$$
  - It follows that

    $$B = \bigcap\{E_{d,\varepsilon} : d \in \mathscr{Q}\} \in \mathscr{D},$$

    since $\mathscr{D}$ is closed under finite intersections.
- We show that for every $D \in \mathscr{D}$, there is $B \in \mathscr{B}$ with $B \subseteq D$.
  - Let $D \in \mathscr{D}$.

- If $D = X \times X$, then $B_{\varnothing,1} = X \times X \subseteq D$.
- Otherwise, there is finite and nonempty $\mathscr{Q} \subseteq \mathscr{P}$ and $D_d \in \mathscr{D}_d$ for each $d \in \mathscr{Q}$ so that $\bigcap \{D_d : d \in \mathscr{Q}\} \subseteq D$.
- For each $d \in \mathscr{Q}$, let $\varepsilon_d > 0$ be such that $\langle x, y\rangle \in D_d$ whenever $d(x, y) < \varepsilon_d$.
- Let $\varepsilon = \min \{\varepsilon_d : d \in \mathscr{Q}\}$.
- Then $B_{\mathscr{Q},\varepsilon} \subseteq D_d$ for each $d \in \mathscr{Q}$ so $B_{\mathscr{Q},\varepsilon} \subseteq D$.

Theorem (uniformity from pseudometric family).

Let $\mathscr{D}$ be a uniformity on a set X and $\mathscr{P}$ be the family of all pseudometrics on X that are uniformly continuous relative to $\mathscr{D}$. Then $\mathscr{D}$ is induced by $\mathscr{P}$.

Proof. Let $\mathscr{E}$ be the uniformity on X that is induced by $\mathscr{P}$.

- Since $\mathscr{E}$ is the weakest uniformity on X making all the pseudometrics in $\mathscr{P}$ uniformly continuous, it follows that $\mathscr{E} \subseteq \mathscr{D}$.
- We show that $\mathscr{D} \subseteq \mathscr{E}$.
  - Let $D \in \mathscr{D}$.
  - Let $D_1 = X \times X$, let $D_2 \in \mathscr{D}$ be symmetric with $D_2 \subseteq D$ and, for each $n \geq 2$, let $D_{n+1} \in \mathscr{D}$ be symmetric and such that $D_{n+1} \circ D_{n+1} \circ D_{n+1} \subseteq D_n$.
  - Let $\mathscr{B}_D = \{D_1, D_2, \dots\}$.
  - Lemma (continuous pseudometric) and the exercise following it show that there exists a pseudometric $d \in \mathscr{P}$ such that $\mathscr{B}_D$ is a base for the uniformity $\mathscr{E}_d$ on X that is induced by d.
  - Then $D \in \mathscr{B}_D \subseteq \mathscr{E}_d \subseteq \mathscr{E}$.

Exercise (uniformly continuous metric).

Consider $X = \mathbb{R}$ as a uniform space (with the standard uniformity on $\mathbb{R}$), let A be an uncountable set and $Y = X^A$ (that is, Y is the product of uncountably many copies of X). Consider Y as a uniform space with the product uniformity $\mathscr{D}$. Prove that no metric on Y is uniformly continuous relative to $\mathscr{D}$.

Solution. Let d be a pseudometric on Y that is uniformly continuous relative to $\mathscr{D}$. We will show that d is not a metric.

- For each $n \in \mathbb{N}$, let $B_n = \{\langle f, g\rangle \in Y \times Y : d(f, g) < 1/n\}$.
- Since d is uniformly continuous relative to $\mathscr{D}$, it follows that $B_n \in \mathscr{D}$, for each $n \in \mathbb{N}$ (see uniformly continuous pseudometric).
- For each $\alpha \in A$ and $\varepsilon > 0$, let

$$S_{\alpha,\varepsilon} := \{\langle f, g\rangle \in Y \times Y : |f(\alpha) - g(\alpha)| < \varepsilon\}.$$

- Then $\mathscr{S} := \{S_{\alpha,\varepsilon} : \alpha \in A, \varepsilon > 0\}$ is a subbase for $\mathscr{D}$.
- Thus for each $n \in \mathbb{N}$, there is finite $A_n \subseteq A$ such that

$$\{\langle f, g\rangle \in Y \times Y : \forall \alpha \in A_n \; f(\alpha) = g(\alpha)\} \subseteq B_n.$$

- Since A is uncountable, there is $\beta \in A \setminus \bigcup \{A_n : n \in \mathbb{N}\}$.
- Let $f(\alpha) = 0$ for each $\alpha \in A$, let $g(\alpha) = 0$ for every $\alpha \in A \setminus \{\beta\}$ and $g(\beta) = 1$.
- Then $\langle f, g\rangle \in \bigcap \{B_n : n \in \mathbb{N}\}$, which implies that $d(f, g) = 0$.
- Since $f \neq g$, it follows that d is not a metric.