Math 793C Topology for Analysis

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Weak uniformity.

Let A and X be sets, let $(X_\alpha, \mathscr{E}_\alpha)$ be a uniform space and $f_\alpha : X \to X_\alpha$ be any function for each $\alpha \in A$. For each $\alpha \in A$, let $\mathscr{B}_\alpha = \left\{\hat{f}_\alpha^{-1}[E] : E \in \mathscr{E}_\alpha\right\}$ and let $\mathscr{S} = \bigcup\{\mathscr{B}_\alpha : \alpha \in A\}$. Theorem (inverse image uniformity) implies that $\mathscr{B}_\alpha$ is a uniformity base for each $\alpha \in A$ and Theorem (union uniformity subbase) implies that $\mathscr{S}$ is a uniformity subbase on X. The uniformity $\mathscr{D}$ on X that is induced by $\mathscr{S}$ is called the *weak uniformity* on X induced by the family $\{(\mathscr{E}_\alpha, f_\alpha) : \alpha \in A\}$.

(1) f_α is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{E}_\alpha$ for each $\alpha \in A$.

Proof. Let $\alpha \in A$ and $E \in \mathscr{E}_\alpha$.

- Then $\hat{f}_\alpha^{-1}[E] \in \mathscr{B}_\alpha \subseteq \mathscr{D}$.
- Thus f_α is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{E}_\alpha$.

(2) $\mathscr{D}$ is the smallest uniformity on X such that for each $\alpha \in A$ the function f_α is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{E}_\alpha$.

Proof. Let $\mathscr{D}'$ be a uniformity on X such that f_α is uniformly continuous relative to $\mathscr{D}'$ and $\mathscr{E}_\alpha$ for each $\alpha \in A$. We show that $\mathscr{D} \subseteq \mathscr{D}'$.

- If $\alpha \in A$, then the uniform continuity of f_α implies that $\mathscr{B}_\alpha \subseteq \mathscr{D}'$. Thus $\mathscr{S} \subseteq \mathscr{D}'$.
- Since $\mathscr{D}$ is the filter on $X \times X$ with subbase $\mathscr{S}$, it follows that $\mathscr{D} \subseteq \mathscr{D}'$.

  In general, if $\mathscr{T}$ is a subbase for a filter $\mathscr{F}$ on a set Y and $\mathscr{G}$ is any filter on Y that contains $\mathscr{T}$, then $\mathscr{F} \subseteq \mathscr{G}$.

- Let $F \in \mathscr{F}$. Then $F = Y$ or $\bigcap \mathscr{T}' \subseteq F$ for some finite and nonempty $\mathscr{T}' \subseteq \mathscr{T}$.
- If $F = Y$, then $F \in \mathscr{G}$.
- Assume that $\bigcap \mathscr{T}' \subseteq F$ for some finite and nonempty $\mathscr{T}' \subseteq \mathscr{T}$.
- Since $\mathscr{T}' \subseteq \mathscr{G}$ and since filters are closed under finite intersections and taking supersets, it follows that $F \in \mathscr{G}$.

Theorem (weak uniformity topology).

Let A and X be sets, let $(X_\alpha, \mathscr{E}_\alpha)$ be a uniform space and $f_\alpha : X \to X_\alpha$ be any function for each $\alpha \in A$. For each $\alpha \in A$, consider X_α to be a topological space with the topology induced by $\mathscr{E}_\alpha$. Let $\mathscr{D}$ be the weak uniformity on X induced by the family $\{(\mathscr{E}_\alpha, f_\alpha) : \alpha \in A\}$ and τ be the weak topology on X induced by the same family of functions (and the topologies τ_α on X_α induced by $\mathscr{E}_\alpha$). Then τ is the topology induced by $\mathscr{D}$.

Proof. Let τ' be the topology induced by $\mathscr{D}$.

- We show that $\tau' \subseteq \tau$.
  - Assume that $U \in \tau'$. If $U = X$ or $U = \varnothing$, then $U \in \tau$.
  - Assume that $\varnothing \neq U \neq X$. Let $x \in U$.
  - There is $D \in \mathscr{D}$ with $D[x] \subseteq U$.
  - We find $U_x \in \tau$ with $x \in U_x \subseteq D[x]$.
    - Since $D \neq X \times X$, there is finite nonempty $A' \subseteq A$ and $E_\alpha \in \mathscr{E}_\alpha$ for each $\alpha \in A'$ such that
      $\bigcap \left\{ \hat{f_\alpha}^{-1}[E_\alpha] : \alpha \in A' \right\} \subseteq D.$
    - For each $\alpha \in A'$, let $V_\alpha \in \tau_\alpha$ be such that
      $f_\alpha(x) \in V_\alpha \subseteq E_\alpha[f_\alpha(x)].$
    - Then $f_\alpha^{-1}[V_\alpha] \in \tau$ for each $\alpha \in A'$ so
      $U_x := \bigcap \left\{ f_\alpha^{-1}[V_\alpha] : \alpha \in A' \right\} \in \tau.$
    - Since $f_\alpha(x) \in V_\alpha$ for each $\alpha \in A'$, it follows that $x \in U_x$.
    - We show that $U_x \subseteq D[x]$.
      - Let $y \in U_x$.
      - Then $f_\alpha(y) \in V_\alpha \subseteq E_\alpha[f_\alpha(x)]$ for each $\alpha \in A$' .
      - Thus $\langle f_\alpha(x), f_\alpha(y) \rangle \in E_\alpha$ for each $\alpha \in A'$ so $\langle x, y \rangle \in D$.
      - Thus $y \in D[x]$.
  - Then $U = \bigcup \{U_x : x \in U\}$ so $U \in \tau$.
- We show that $\tau \subseteq \tau'$.

- Assume that $U \in \tau$. If $U = X$ or $X = \varnothing$, then $U \in \tau'$.
- Assume that $\varnothing \neq U \neq X$. Let $x \in U$.
- We find $D \in \mathscr{D}$ with $D[x] \subseteq U$.
    - Since $U \neq X$, there is finite nonempty $A' \subseteq A$ and $V_\alpha \in \tau_\alpha$ for each $\alpha \in A'$ such that
    $$x \in \bigcap \left\{ f_\alpha^{-1}[V_\alpha] : \alpha \in A' \right\} \subseteq U.$$
    - For each $\alpha \in A'$, we have $f_\alpha(x) \in V_\alpha$ so there is $E_\alpha \in \mathscr{E}_\alpha$ such that $E_\alpha[f_\alpha(x)] \subseteq V_\alpha$.
    - Let $D := \bigcap \left\{ \hat{f_\alpha}^{-1}[E_\alpha] : \alpha \in A' \right\}$. Then $D \in \mathscr{D}$.
    - We show that $D[x] \subseteq U$.
        - Let $y \in D[x]$. Then $\langle x, y \rangle \in D$.
        - Thus $\langle f_\alpha(x), f_\alpha(y) \rangle \in E_\alpha$ for each $\alpha \in A'$.
        - It follows that
        $$f_\alpha(y) \in E_\alpha[f_\alpha(x)] \subseteq V_\alpha$$
        so $y \in f_\alpha^{-1}[V_\alpha]$ for each $\alpha \in A'$.
        - Thus $y \in U$.
- It follows that $U \in \tau'$.

Theorem (uniform continuity weak uniformity).

Let A and X be sets, let $(X_\alpha, \mathscr{E}_\alpha)$ be a uniform space and $f_\alpha : X \to X_\alpha$ be any function for each $\alpha \in A$. Let $\mathscr{D}$ be the weak uniformity on X induced by the family $\{(\mathscr{E}_\alpha, f_\alpha) : \alpha \in A\}$. Let $\{Y, \mathscr{E}\}$ be a uniform space and $g : Y \to X$ be a function. Then g is uniformly continuous (relative to $\mathscr{E}$ and $\mathscr{D}$) if and only if the composition $f_\alpha \circ g$ is uniformly continuous (relative to $\mathscr{E}$ and $\mathscr{E}_\alpha$).

Proof. Assume that g is uniformly continuous.

- For each $\alpha \in A$, the function f_α is uniformly continuous, so $f_\alpha \circ g$ is uniformly continuous.

Now assume that $f_\alpha \circ g$ is uniformly continuous for each $\alpha \in A$.

- Let $D \in \mathscr{D}$. We show that $\hat{g}^{-1}[D] \in \mathscr{E}$.
    - If $D = X \times X$, then $\hat{g}^{-1}[D] = Y \times Y \in \mathscr{E}$.
    - Otherwise, there exists finite nonempty $A' \subseteq A$ and $E_\alpha \in \mathscr{E}_\alpha$ for each $\alpha \in A'$ such that
    $$B := \bigcap \left\{ \hat{f_\alpha}^{-1}[E_\alpha] : \alpha \in A' \right\} \subseteq D.$$
    - Since $f_\alpha \circ g$ is uniformly continuous for each $\alpha \in A'$, it follows that

$$\hat{g}^{-1}\left[\hat{f_\alpha}^{-1}[E_\alpha]\right] = \widehat{f_\alpha \circ g}^{-1}[E_\alpha] \in \mathscr{E}$$

for each $\alpha \in A'$.

  - Thus $\hat{g}^{-1}[B] \in \mathscr{E}$.
  - Since $\hat{g}^{-1}[B] \subseteq \hat{g}^{-1}[D]$, it follows that $\hat{g}^{-1}[D] \in \mathscr{E}$.

- Thus g is uniformly continuous.

Product uniformity.

Let A be a set and $(X_\alpha, \mathscr{E}_\alpha)$ be a uniform space for each $\alpha \in A$. Let $X = \prod \{X_\alpha : \alpha \in A\}$. The *product uniformity* $\mathscr{D}$ on X is the weak uniformity induced by the family $\{(\mathscr{E}_\alpha, \pi_\alpha) : \alpha \in A\}$, where $\pi_\alpha : X \to X_\alpha$ is the projection.

- For each $\alpha \in A$, let $\mathscr{B}_\alpha = \left\{\hat{\pi}_\alpha^{-1}[E] : E \in \mathscr{E}_\alpha\right\}$ and let $\mathscr{S} = \bigcup \{\mathscr{B}_\alpha : \alpha \in A\}$. Then $\mathscr{S}$ is a subbase for $\mathscr{D}$.

- The product uniformity on X is the smallest uniformity such that π_α is uniformly continuous for each $\alpha \in A$.

Corollary (product uniformity topology).

Let A be a set and $(X_\alpha, \mathscr{E}_\alpha)$ be a uniform space for each $\alpha \in A$. Let $X = \prod \{X_\alpha : \alpha \in A\}$ with the product uniformity $\mathscr{D}$. For each $\alpha \in A$, consider X_α to be a topological space with the topology induced by $\mathscr{E}_\alpha$. Then the product topology on X is the topology induced by $\mathscr{D}$.

Proof. It is a special case of Theorem (weak uniformity topology).

Corollary (uniform continuity product uniformity).

Let A be a set and $(X_\alpha, \mathscr{E}_\alpha)$ be a uniform space for each $\alpha \in A$. Let $X = \prod \{X_\alpha : \alpha \in A\}$ with the product uniformity $\mathscr{D}$. Let $\{Y, \mathscr{E}\}$ be a uniform space and $g : Y \to X$ be a function. Then g is uniformly continuous (relative to $\mathscr{E}$ and $\mathscr{D}$) if and only if the composition $\pi_\alpha \circ g$ is uniformly continuous (relative to $\mathscr{E}$ and $\mathscr{E}_\alpha$).

Proof. It is a special case of Theorem (uniform continuity weak uniformity).

Theorem (uniformly continuous pseudometric).

Let $\mathscr{D}$ be a uniformity on a set X and d be a pseudometric on X. Consider $X \times X$ equipped with the product uniformity and let $\mathbb{R}$ have the standard uniformity (uniformity induced by the standard metric on $\mathbb{R}$). The function $d : X \times X \to \mathbb{R}$ is uniformly continuous if and only if the identity function $g : X \to X$ is uniformly continuous relative to $\mathscr{D}$ and the uniformity $\mathscr{E}$ on X that is induced by d (which holds if and only if $\mathscr{E} \subseteq \mathscr{D}$).

Proof. Assume that d is uniformly continuous.

- We show that $\mathscr{E} \subseteq \mathscr{D}$.

  - Let $E \in \mathscr{E}$.
  - There exists $\varepsilon > 0$ such that $\langle x, y\rangle \in E$ for any $x, y \in X$ with $d(x,y) < \varepsilon$.
  - Let $S = \{\langle a, b\rangle \in \mathbb{R} \times \mathbb{R} : |a - b| < \varepsilon\}$. Then S is a member of the standard uniformity on $\mathbb{R}$.
  - Since d is uniformly continuous, it follows that $\hat{d}^{-1}[S]$ is a member of the product uniformity on $X \times X$.
  - Thus there exist $B, D \in \mathscr{D}$ with
    $$\hat{\pi_1}^{-1}[B] \cap \hat{\pi_2}^{-1}[D] \subseteq \hat{d}^{-1}[S],$$
    where $\pi_1, \pi_2 : X \times X \to X$ are the projections.
  - We prove that $B \subseteq E$.

    * Let $\langle x, y\rangle \in B$.
    * Note that
      $$\hat{\pi_1}^{-1}[B] = \big\{\langle\langle s,t\rangle, \langle w,z\rangle\rangle \in X^2 \times X^2 : \langle s, w\rangle \in B\big\}.$$
      and
      $$\hat{\pi_2}^{-1}[D] = \big\{\langle\langle s,t\rangle, \langle w,z\rangle\rangle \in X^2 \times X^2 : \langle t, z\rangle \in D\big\}.$$
    * Moreover
      $$\hat{d}^{-1}[S] = \big\{\langle\langle s,t\rangle, \langle w,z\rangle\rangle \in X^2 \times X^2 : |d(s,t) - d(w,z)| < \varepsilon\big\}.$$
    * Since $\langle x, y\rangle \in B$ and $\langle x, x\rangle \in D$, it follows that $\langle\langle x,y\rangle, \langle x,x\rangle\rangle \in \hat{d}^{-1}[S]$ so $d(x,y) < \varepsilon$.
    * Thus $\langle x, y\rangle \in E$.
  - Thus $E \in \mathscr{D}$.

Assume that that $\mathscr{E} \subseteq \mathscr{D}$.

- We show that d is uniformly continuous.

  - Let S be a member of the standard uniformity on $\mathbb{R}$.
  - There is $\varepsilon > 0$ such that $\langle a, b\rangle \in S$ for any $a, b \in \mathbb{R}$ with $|a - b| < \varepsilon$.
  - We show that $\hat{d}^{-1}[S]$ is a member of the product uniformity on $X \times X$.

    * Let $D = \{\langle x, y\rangle \in X \times X : d(x,y) < \varepsilon/2\}$. Then $D \in \mathscr{E}$.
    * Since $\mathscr{E} \subseteq \mathscr{D}$, it follows that $D \in \mathscr{D}$.
    * We show that $\hat{\pi_1}^{-1}[D] \cap \hat{\pi_2}^{-1}[D] \subseteq \hat{d}^{-1}[S]$.

      · Let $\langle\langle s,t\rangle, \langle w,z\rangle\rangle \in \hat{\pi_1}^{-1}[D] \cap \hat{\pi_2}^{-1}[D]$.

      - Then $\langle s, w\rangle \in D$ and $\langle t, z\rangle \in D$ so $d(s,w) < \varepsilon/2$ and $d(t,z) < \varepsilon/2$.
      - It follows that $|d(s,t) - d(w,z)| < \varepsilon$.
      - Thus $\langle\langle s,t\rangle, \langle w,z\rangle\rangle \in \hat{d}^{-1}[S]$.
    * Thus $\hat{d}^{-1}[S]$ is a member of the product uniformity on $X \times X$.
  - Thus d is uniformly continuous.

Uniformly continuous pseudometric.

Let $\mathscr{D}$ be a uniformity on a set X and d be a pseudometric on X. We say that d is *uniformly continuous* (relative to $\mathscr{D}$) iff it is uniformly continuous as a function $d : X \times X \to \mathbb{R}$, where $X \times X$ has the product uniformity and $\mathbb{R}$ has the standard uniformity (uniformity induced by the standard metric on $\mathbb{R}$).

Actually, d is uniformly continuous if and only if the identity function $g : X \to X$ is uniformly continuous relative to $\mathscr{D}$ and the uniformity $\mathscr{E}$ on X that is induced by d (which holds if and only if $\mathscr{E} \subseteq \mathscr{D}$).

- See the Theorem (uniformly continuous pseudometric) for a proof of the above characterization.

Explicitly, d is uniformly continuous if and only if for every $\varepsilon > 0$ the set

$$D_\varepsilon := \{\langle x, y\rangle \in X \times X : d(x,y) < \varepsilon\}$$

is a member of $\mathscr{D}$.

Proof. Assume that $D_\varepsilon \in \mathscr{D}$ for every $\varepsilon > 0$.

- We show that the identity function $g : X \to X$ is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{E}$.

  - Let $E \in \mathscr{E}$. There exists $\varepsilon > 0$ such that $\langle x, y\rangle \in E$ whenever $d(x,y) < \varepsilon$.
  - Then $\hat{g}^{-1}[E] = E$ and $D_\varepsilon \subseteq E$, implying that $\hat{g}^{-1}[E] \in \mathscr{D}$.

Now assume that the identity function $g : X \to X$ is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{E}$.

- We show that $D_\varepsilon \in \mathscr{D}$ for every $\varepsilon > 0$.

  - Let $\varepsilon > 0$. Then $D_\varepsilon \in \mathscr{E}$.
  - The uniform continuity of g implies that $\hat{g}^{-1}[D_\varepsilon] \in \mathscr{D}$.
  - Since $\hat{g}^{-1}[D_\varepsilon] = D_\varepsilon$, it follows that that $D_\varepsilon \in \mathscr{D}$.