Math 793C Topology for Analysis

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Induced product function.

If $f : X \to Y$, then we will denote by $\hat{f}$ the function $X \times X \to Y \times Y$ defined by

$$\hat{f}(\langle x, y\rangle) = \langle f(x), f(y)\rangle.$$

If $\mathscr{D}$ and $\mathscr{E}$ are uniformities on X and Y, respectively, then f is uniformly continuous if and only if $\hat{f}^{-1}[E] \in \mathscr{D}$ for every $E \in \mathscr{E}$.

Discrete uniformity.

Let X be a set. The *discrete uniformity* on X is the family $\mathscr{D}$ of all $D \subseteq X \times X$ such that D is a reflexive relation on X.

- It is clear that $\mathscr{D}$ is a uniformity.
- The topology on X induced by $\mathscr{D}$ is the discrete topology.

  Proof.

  - Let $D = \{\langle x, x\rangle : x \in X\}$ be the diagonal of X. Then $D \in \mathscr{D}$.
  - Let $x \in X$. Then $D[x] = \{x\}$ so $\{x\}$ is open in the topology τ on X that is induced by $\mathscr{D}$
  - Thus τ is the discrete topology.

Exercise (inverse image uniformity).

Let $X = \mathbb{N}$, $Y = \{0, 1\}$ and let $\mathscr{E}$ be the discrete uniformity on Y. Define $f : X \to Y$ by $f(n) = 0$ if n is even and $f(n) = 1$ otherwise. Let $\mathscr{D} := \left\{\hat{f}^{-1}[E] : E \in \mathscr{E}\right\}$, where $\hat{f} : X \times X \to Y \times Y$ is the induced product function. Prove that $\mathscr{D}$ is not a uniformity on X.

Solution. Note that $\mathscr{E} = \{E_1, E_2, E_3, E_4\}$, where

- $E_1 = \{\langle 0,0\rangle, \langle 1,1\rangle\}$,
- $E_2 = Y \times Y \setminus \{\langle 0,1\rangle\}$,
- $E_3 = Y \times Y \setminus \{\langle 1,0\rangle\}$ and
- $E_4 = Y \times Y$.

Let $A := \{2n : n \in \mathbb{N}\}$ and $B := \{2n-1 : n \in \mathbb{N}\}$.

- Then $\mathscr{D} = \left\{\hat{f}^{-1}[E_1], \hat{f}^{-1}[E_2], \hat{f}^{-1}[E_3], \hat{f}^{-1}[E_4]\right\}$, where
  - $\hat{f}^{-1}[E_1] = A \times A \cup B \times B$,
  - $\hat{f}^{-1}[E_2] = \mathbb{N} \times \mathbb{N} \setminus A \times B$,
  - $\hat{f}^{-1}[E_3] = \mathbb{N} \times \mathbb{N} \setminus B \times A$,
  - $\hat{f}^{-1}[E_4] = \mathbb{N} \times \mathbb{N}$.
- Note that $\mathscr{D}$ is not a filter on $X \times X$ since
  - $A \times A \cup B \times B \in \mathscr{D}$, but
  - $A \times A \cup B \times B \cup \{\langle 1,2\rangle\} \notin \mathscr{D}$.
- Since a uniformity on X is a filter on $X \times X$, it follows that $\mathscr{D}$ is not a uniformity on X.

Theorem (inverse image uniformity).

Let X, Y be sets, $\mathscr{E}$ be a uniformity base on Y, and $f : X \to Y$ be any function. Let $\mathscr{B} = \left\{\hat{f}^{-1}[E] : E \in \mathscr{E}\right\}$, where $\hat{f} : X \times X \to Y \times Y$ is the induced product function. Then $\mathscr{B}$ is a uniformity base on X.

Proof. By Theorem (uniformity base), it suffices to verify the following conditions.

1. Each $D \in \mathscr{B}$ is a reflexive relation on X.
   - Let $D \in \mathscr{B}$. Then $D = \hat{f}^{-1}[E]$ for some $E \in \mathscr{E}$.
   - Let $x \in X$. Then $\langle f(x), f(x)\rangle \in E$ since E is a reflexive relation on Y.
   - Hence $\langle x, x\rangle \in D$ and so D is a reflexive relation on X.
2. If $D \in \mathscr{B}$, then there exists $B \in \mathscr{B}$ with $B \subseteq D^{-1}$.
   - Let $D \in \mathscr{B}$. Then $D = \hat{f}^{-1}[E]$ for some $E \in \mathscr{E}$.

- There is $F \in \mathscr{E}$ with $F \subseteq E^{-1}$. Let $B = \hat{f}^{-1}[F]$.
- Then $B \in \mathscr{B}$ and $B \subseteq D^{-1}$.

  We show that $B \subseteq D^{-1}$.

  - Let $\langle x, y \rangle \in B$.
  - Then $\langle f(x), f(y) \rangle \in F \subseteq E^{-1}$ so $\langle f(y), f(x) \rangle \in E$.
  - Thus $\langle y, x \rangle \in \hat{f}^{-1}[E] = D$ and $\langle x, y \rangle \in D^{-1}$.

3. If $D \in \mathscr{B}$, then there exists $B \in \mathscr{B}$ with $B \circ B \subseteq D$.

   - Let $D \in \mathscr{B}$. Then $D = \hat{f}^{-1}[E]$ for some $E \in \mathscr{E}$.
   - There is $F \in \mathscr{E}$ with $F \circ F \subseteq E$. Let $B = \hat{f}^{-1}[F]$.
   - Then $B \in \mathscr{B}$ and $B \circ B \subseteq D$.

     We show that $B \circ B \subseteq D$.

     - Let $\langle x, z \rangle \in B \circ B$. There is $y \in X$ with $\langle x, y \rangle \in B$ and $\langle y, z \rangle \in B$.
     - Then $\langle f(x), f(y) \rangle \in F$ and $\langle f(y), f(z) \rangle \in F$ so $\langle f(x), f(z) \rangle \in E$.
     - Since $\hat{f}(\langle x, z \rangle) \in E$, it follows that $\langle x, z \rangle \in \hat{f}^{-1}[E] = D$.

4. If $B, D \in \mathscr{B}$, then there exists $E \in \mathscr{B}$ with $E \subseteq B \cap D$.

   - Let $B, D \in \mathscr{B}$. Then $B = \hat{f}^{-1}[F]$ and $D = \hat{f}^{-1}[G]$ for some $F, G \in \mathscr{E}$.
   - Let $H \in \mathscr{E}$ be such that $H \subseteq F \cap G$. Let $E = \hat{f}^{-1}[H]$.
   - If $\langle x, y \rangle \in E$, then $\langle f(x), f(y) \rangle \in H \subseteq F \cap G$.
   - Thus $\langle x, y \rangle \in B \cap D$ and so $E \subseteq B \cap D$.

Inverse image uniformity.

Let X, Y be sets, $\mathscr{E}$ be a uniformity base on Y and $f : X \to Y$ be a function. The *inverse image uniformity* on X induced by $\mathscr{E}$ and f is the uniformity induced by the uniformity base $\left\{ \hat{f}^{-1}[E] : E \in \mathscr{E} \right\}$.

- See Theorem (inverse image uniformity) for the proof that $\left\{ \hat{f}^{-1}[E] : E \in \mathscr{E} \right\}$ is a uniformity base on X.
- In particular, if $\mathscr{E}$ is a uniformity on Y and $f : X \to Y$ is a function, then we get a uniformity on X induced by the uniformity base $\left\{ \hat{f}^{-1}[E] : E \in \mathscr{E} \right\}$.

Theorem (uniform continuity inverse uniformity).

Let X, Y be sets, $\mathscr{E}$ be a uniformity on Y, and $f : X \to Y$ be any function. Let $\mathscr{B} = \left\{ \hat{f}^{-1}[E] : E \in \mathscr{E} \right\}$ and let $\mathscr{D}$ be the uniformity on X that is induced by the uniformity base $\mathscr{B}$.

- f is uniformly continuous relative to $\mathscr{D}$ and $\mathscr{E}$.
- If $\mathscr{D}'$ is any uniformity on X such that $f : X \to Y$ is uniformly continuous relative to $\mathscr{D}'$ and $\mathscr{E}$, then $\mathscr{D} \subseteq \mathscr{D}'$.

Proof.

- Let $E \in \mathscr{E}$. Then $\hat{f}^{-1}[E] \in \mathscr{B}$. Since $\mathscr{B} \subseteq \mathscr{D}$, it follows that $\hat{f}^{-1}[E] \in \mathscr{D}$. Thus f is uniformly continuous.
- Assume that $\mathscr{D}'$ is any uniformity on X such that $f : X \to Y$ is uniformly continuous relative to $\mathscr{D}'$ and $\mathscr{E}$. We show that $\mathscr{D} \subseteq \mathscr{D}'$.
  - Let $D \in \mathscr{D}$. There is $B \in \mathscr{B}$ with $B \subseteq D$.
  - There is $E \in \mathscr{E}$ be such that $B = \hat{f}^{-1}[E]$.
  - Since f is uniformly continuous relative to $\mathscr{D}'$ and $\mathscr{E}$, it follows that $B \in \mathscr{D}'$.
  - Since $\mathscr{D}'$ is a uniformity, $B \in \mathscr{D}'$ and $B \subseteq D$, it follows that $D \in \mathscr{D}'$.

Exercise (injective inverse image uniformity).

Let X, Y be sets, $\mathscr{E}$ be a uniformity on Y, and $f : X \to Y$ be any injective function. Let $\mathscr{D} = \left\{ \hat{f}^{-1}[E] : E \in \mathscr{E} \right\}$. Prove that $\mathscr{D}$ is a uniformity on X.

Solution. By Theorem (inverse image uniformity), $\mathscr{D}$ is a uniformity base on X.

- To prove that $\mathscr{D}$ is a uniformity on X, it suffices to show that $\mathscr{D}$ is closed under taking supersets.
- Let $D \in \mathscr{D}$ and $D \subseteq D' \subseteq X \times X$. We show that $D' \in \mathscr{D}$.
  - Let $E \in \mathscr{E}$ be such that $D = \hat{f}^{-1}[E]$. Let $E' = \hat{f}[D']$.
  - Since f is injective, it follows that $\hat{f}$ is injective.
    * Let $\langle x, y \rangle, \langle z, w \rangle \in X \times X$ with $\hat{f}(\langle x, y \rangle) = \hat{f}(\langle z, w \rangle)$.
    * Then $f(x) = f(z)$ and $f(y) = f(w)$.
    * Thus $x = z$ and $y = w$ so $\langle x, y \rangle = \langle z, w \rangle$.
  - Since $\hat{f}$ is injective, it follows that $\hat{f}^{-1}[E'] = D'$.
  - Since $D \subseteq D'$, if follows that $E \subseteq E'$ so $E' \in \mathscr{E}$ and hence $D' \in \mathscr{D}$.

Relative uniformity.

Let $(Y, \mathscr{E})$ be a uniform space and $X \subseteq Y$. The *relative uniformity* (*subspace uniformity*) on X is the uniformity $\mathscr{D}$ on X that is induced by the embedding $f : X \to Y$ ($f(x) = x$ for each $x \in X$).

- Explicitly, $\mathscr{D}$ is induced by the uniformity base

  $$\mathscr{B} = \left\{ \hat{f}^{-1}[E] : E \in \mathscr{E} \right\} = \{E \cap (X \times X) : E \in \mathscr{E}\}$$

  which is also called the *trace* of $\mathscr{E}$ on $X \times X$.

- Since the embedding $f : X \to Y$ is injective, $\mathscr{B}$ is a uniformity on X so $\mathscr{D} = \mathscr{B}$.

Exercise (topology relative uniformity).

Let $\mathscr{E}$ be a uniformity on a set Y, let τ be the topology on Y that is induced by $\mathscr{E}$, let $X \subseteq Y$ and let $\mathscr{D}$ be the relative uniformity on X. Prove that the topology on X that is induced by $\mathscr{D}$ is the relative (subspace) topology with respect to τ.

Solution. Let τ' be the topology on X that is induced by $\mathscr{D}$ and τ'' be the subspace topology on X inherited from τ .

- We show that $\tau' \subseteq \tau''$.
  - Let $U \subseteq X$ be open in τ'. Let $x \in U$ be arbitrary.
  - There is $D \in \mathscr{D}$ such that $D[x] \subseteq U$.
  - There is $E \in \mathscr{E}$ with $D = E \cap (X \times X)$.
  - Then $E[x]$ is a nbhd of x in Y with respect to τ so there is $V \in \tau$ with $x \in V \subseteq E[x]$.
  - Since $E[x] \cap X = D[x]$, it follows that $V \cap X \subseteq D[x] \subseteq U$.
  - Since $V \cap X \in \tau''$ and since x was an arbitrary point of U, it follows that $U \in \tau''$.
- Now we show that $\tau'' \subseteq \tau'$.
  - Let $U \subseteq X$ be open in τ''.
  - There is $V \subseteq Y$ such that $U = V \cap X$ and $V \in \tau$.
  - Let $x \in U$ be arbitrary.
  - There is $E \in \mathscr{E}$ with $E[x] \subseteq V$.
  - Let $D = E \cap (X \times X)$. Then $D \in \mathscr{D}$.
  - Since $D[x] = E[x] \cap X$, it follows that $D[x] \subseteq U$.
  - Thus $U \in \tau'$.