Math 793C — Topology for Analysis

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Theorem (surrounding interior).

Let $\mathscr{D}$ be a uniformity on a set X. Consider X to be a topological space with the topology induced by $\mathscr{D}$. If $D \in \mathscr{D}$ and D° is the interior of D with respect to the product topology on $X \times X$, then $D^\circ \in \mathscr{D}$.

Proof. Let $D \in \mathscr{D}$. Note that

$$D^\circ = \{\langle x, y\rangle \in D : (\exists E \in \mathscr{D})\, E[x] \times E[y] \subseteq D\}.$$

Let $F \in \mathscr{D}$ be symmetric and such that $F \circ F \circ F \subseteq D$. Since

$$F \circ F \circ F = \bigcup \{F[x] \times F[y] : \langle x, y\rangle \in F\},$$

if follows that $F \subseteq D^\circ$. Thus $D^\circ \in \mathscr{D}$.

We show that

$$F \circ F \circ F = \bigcup \{F[x] \times F[y] : \langle x, y\rangle \in F\}.$$

- Proof of $\subseteq$.
  - Let $\langle z, w\rangle \in F \circ F \circ F$.
  - Then there are $x, y \in X$ such that $\langle z, x\rangle, \langle x, y\rangle, \langle y, w\rangle \in F$.
  - Since F is symmetric, it follows that $\langle x, z\rangle \in F$.
  - Thus $z \in F[x]$ and $w \in F[y]$ so $\langle z, w\rangle \in F[x] \times F[y]$.
  - Hence $\langle z, w\rangle \in \bigcup \{F[x] \times F[y] : \langle x, y\rangle \in F\}$.
- Proof of $\supseteq$.
  - Let $\langle z, w\rangle \in \bigcup \{F[x] \times F[y] : \langle x, y\rangle \in F\}$.
  - Thus there are $x, y \in X$ with $\langle x, z\rangle, \langle x, y\rangle, \langle y, w\rangle \in F$.
  - Since F is symmetric, it follows that $\langle z, x\rangle \in F$.
  - Hence $\langle z, w\rangle \in F \circ F \circ F$.

Exercise (symmetric open uniformity base).

Let $\mathscr{D}$ be a uniformity on a set X and let $\mathscr{B}$ be the family of all members of $\mathscr{D}$ that are symmetric and open (in the product topology on $X \times X$, where the topology on X is induced by $\mathscr{D}$). Prove that $\mathscr{B}$ is a base for $\mathscr{D}$.

Solution. Since $\mathscr{B} \subseteq \mathscr{D}$, it suffices to show that for every $D \in \mathscr{D}$, there exists $B \in \mathscr{B}$ with $B \subseteq D$. Let $D \in \mathscr{D}$. Take $B := D^\circ \cap (D^\circ)^{-1}$.

- By Theorem (surrounding interior) we have $D^\circ \in \mathscr{D}$, which implies that $B \in \mathscr{D}$.
- It is clear that B is symmetric.
- To show that B is open, it suffices to show that $(D^\circ)^{-1}$ is open.
  - Let $\langle x, y\rangle \in (D^\circ)^{-1}$. Then $\langle y, x\rangle \in D^\circ$.
  - Since D° is open, there is $E \in \mathscr{D}$ with $E[y] \times E[x] \subseteq D^\circ$.
  - Then $E[x] \times E[y] \subseteq (D^\circ)^{-1}$ so $(D^\circ)^{-1}$ is open.
- Since B is open and symmetric, it follows that $B \in \mathscr{B}$.
- Since $B \subseteq D^\circ \subseteq D$, the proof is complete.

Theorem (closure uniform topology).

Let $\mathscr{D}$ be a uniformity on a set X.

- If $A \subseteq X$, then the closure of A (with respect to the topology on X that is induced by $\mathscr{D}$) is given by

  $$\overline{A} = \bigcap \{D[A] : D \in \mathscr{D}\},$$

  where $D[A] = \bigcup \{D[x] : x \in A\}$.
- If $M \subseteq X \times X$, then the closure of M (with respect to the product topology on $X \times X$) is given by

  $$\overline{M} = \bigcap \{D \circ M \circ D : D \in \mathscr{D}\}.$$

Proof. Let $A \subseteq X$ and $x \in X$.

- We show that $x \in \overline{A}$ if and only if $x \in D[A]$ for each $D \in \mathscr{D}$.
  - We have $x \in \overline{A}$ if and only if $D[x] \cap A \neq \varnothing$ for each $D \in \mathscr{D}$.
  - Given $D \in \mathscr{D}$, we have $D[x] \cap A \neq \varnothing$ if and only if $\langle x, y\rangle \in D$ for some $y \in A$ which holds if and only if $x \in D^{-1}[A]$.

- Thus $x \in \overline{A}$ if and only if $x \in D^{-1}[A]$ for each $D \in \mathscr{D}$.
- Since each $D \in \mathscr{D}$ contains a symmetric member of $\mathscr{D}$, it follows that $x \in \overline{A}$ if and only if $x \in D[A]$ for each $D \in \mathscr{D}$.

  We show that the following are equivalent:

  1. $x \in D^{-1}[A]$ for ever $D \in \mathscr{D}$;
  2. $x \in D[A]$ for every $D \in \mathscr{D}$.

  We prove that (1) implies (2).

  * Assume that (1) holds. Let $D \in \mathscr{D}$.
  * There is a symmetric $E \in \mathscr{D}$ with $E \subseteq D$.
  * (1) implies that $x \in E^{-1}[A]$.
  * Since E is symmetric, it follows that $x \in E[A]$.
  * Thus $x \in D[A]$ and so (2) holds.

  The proof the (2) implies (1) is similar.

Let $M \subseteq X \times X$ and $\langle x, y\rangle \in X \times X$.

- We show that $\langle x, y\rangle \in \overline{M}$ if and only if $\langle x, y\rangle \in D \circ M \circ D$ for every $D \in \mathscr{D}$.

  - We have $\langle x, y\rangle \in \overline{M}$ if and only if $(D[x] \times D[y]) \cap M \neq \varnothing$ for each symmetric $D \in \mathscr{D}$.
  - Given a symmetric $D \in \mathscr{D}$, we have $(D[x] \times D[y]) \cap M \neq \varnothing$ if and only if there exists $\langle w, z\rangle \in M$ with $\langle x, w\rangle, \langle z, y\rangle \in D$ which holds if and only if $\langle x, y\rangle \in D \circ M \circ D$.
  - Thus $\langle x, y\rangle \in \overline{M}$ if and only if $\langle x, y\rangle \in D \circ M \circ D$ for every symmetric $D \in \mathscr{D}$.
  - Since each $D \in \mathscr{D}$ contains a symmetric member of $\mathscr{D}$, it follows that $x \in \overline{M}$ if and only if $\langle x, y\rangle \in D \circ M \circ D$ for every $D \in \mathscr{D}$.

Theorem (symmetric closed uniformity base).

Let $\mathscr{D}$ be a uniformity on a set X and let $\mathscr{B}$ be the family consisting of all members of $\mathscr{D}$ that are symmetric and closed (in the product topology on $X \times X$, where X has the topology induced by $\mathscr{D}$). Then $\mathscr{B}$ is a base for $\mathscr{D}$.

Proof. Let $D \in \mathscr{D}$.

- We show that there exists a closed $G \in \mathscr{D}$ with $G \subseteq D$.

  - There is $E \in \mathscr{D}$ with $E \circ E \circ E \subseteq D$.
  - The closure of E (in the product topology on $X \times X$) is

    $$\overline{E} = \bigcap \{F \circ E \circ F : F \in \mathscr{D}\}.$$

    - It follows that $\overline{E} \subseteq E \circ E \circ E \subseteq D$.
    - Since $E \in \mathscr{D}$ and since $\mathscr{D}$ is a filter on $X \times X$, it follows that $\overline{E} \in \mathscr{D}$.
    - Thus $G := \overline{E}$ is a closed member of $\mathscr{D}$ with $G \subseteq D$.

- Since G is closed in $X \times X$, it follows that G^{-1} is also closed.

    - Let $\langle x, y \rangle \in (X \times X) \smallsetminus G^{-1}$.
    - Then $\langle y, x \rangle \in (X \times X) \smallsetminus G$.
    - Since $(X \times X) \smallsetminus G$ is open, there exists $E \in \mathscr{D}$ with

      $E[y] \times E[x] \subseteq (X \times X) \smallsetminus G$.
    - Then $E[x] \times E[y] \subseteq (X \times X) \smallsetminus G^{-1}$ so $(X \times X) \smallsetminus G^{-1}$ is open.

- If $K := G \cap G^{-1}$, then $K \in \mathscr{D}$ and $K \subseteq D$. Moreover, K is closed and symmetric so $K \in \mathscr{B}$.

Since each $D \in \mathscr{D}$ contains a member of $\mathscr{B}$, it follows that $\mathscr{B}$ is a base for $\mathscr{D}$.

Exercise (uniform continuity pseudometric).

Let (X, d) and (Y, ρ) be pseudometric spaces and let $\mathscr{D}$ and $\mathscr{E}$ be the uniformities induced by d and ρ on X and Y, respectively. Prove that $f : X \to Y$ is uniformly continuous (relative to $\mathscr{D}$ and $\mathscr{E}$) (see uniform continuity) if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.

Solution. Assume that f is uniformly continuous. Let $\varepsilon > 0$.

- We find $\delta > 0$ such that $\rho(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.

    - Define $E := \{\langle w, z \rangle \in Y \times Y : \rho(w, z) < \varepsilon\}$. Then $E \in \mathscr{E}$.
    - Define $D := \{\langle x, y \rangle \in X \times X : \langle f(x), f(y) \rangle \in E\}$.
    - Since f is uniformly continuous, it follows that $D \in \mathscr{D}$.
    - Thus there exists $\delta > 0$ such that $\langle x, y \rangle \in D$ whenever $d(x, y) < \delta$.
    - It suffices to show that $\rho(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.

        * Let $x, y \in X$ with $d(x, y) < \delta$.
        * Then $\langle x, y \rangle \in D$ so $\langle f(x), f(y) \rangle \in E$.
        * Hence $\rho(f(x), f(y)) < \varepsilon$ as required.

Now assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.

- We show that f is uniformly continuous.

- Let $E \in \mathscr{E}$. Then there exists $\varepsilon > 0$ such that $\langle x, y \rangle \in E$ whenever $\rho(x, y) < \varepsilon$.
- Let $\delta > 0$ be such that $\rho(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$.
- Let $D' := \{\langle x, y \rangle \in X \times X : d(x, y) < \delta\}$. Then $D' \in \mathscr{D}$.
- Let $D := \{\langle x, y \rangle \in X \times X : \langle f(x), f(y) \rangle \in E\}$.
- We verify that $D \in \mathscr{D}$ by showing that $D' \subseteq D$.
  * Let $\langle x, y \rangle \in D'$.
  * Then $d(x, y) < \delta$ so $\rho(f(x), f(y)) < \varepsilon$.
  * Hence $\langle f(x), f(y) \rangle \in E$, which implies that $\langle x, y \rangle \in D$.
- Since $D \in \mathscr{D}$, if follows that f is uniformly continuous.

Uniform isomorphism.

Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be uniform spaces and $f : X \to Y$ be a function. We say that f is a *uniform isomorphism* iff f is a bijection and both f and f^{-1} are uniformly continuous.

- If there exists a uniform isomorphism $f : X \to Y$, we say that $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ are *uniformly isomorphic* or *uniformly equivalent*.
- The identity function $X \to X$ is a uniform isomorphism.
- If $f : X \to Y$ is a uniform isomorphism, then $f^{-1} : Y \to X$ is also a uniform isomorphism.
- If $(Z, \mathscr{F})$ is another uniform space with $f : X \to Y$ and $h : Y \to Z$ being uniform isomorphisms, then the composition $h \circ f : X \to Z$ is also a uniform isomorphism.

Exercise (uniform continuity implies continuity).

Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be uniform spaces and $f : X \to Y$ be uniformly continuous. Prove that f is continuous when we consider X and Y as topological spaces with the topologies induced by $\mathscr{D}$ and $\mathscr{E}$, respectively. Conclude that if f is a uniform isomorphism, then f is a homeomorphism.

Solution. Let $U \subseteq Y$ be open in the topology on Y that is induced by $\mathscr{E}$.

- We show that $f^{-1}[U]$ is open in the topology on X that is induced by $\mathscr{D}$.
- Let $x \in f^{-1}[U]$. Then $f(x) \in U$ so there exists $E \in \mathscr{E}$ with $E[f(x)] \subseteq U$.
- Since f is uniformly continuous, it follows that

  $$D := \{\langle x, y \rangle \in X \times X : \langle f(x), f(y) \rangle \in E\}$$

is a member of $\mathscr{D}$.

- If $y \in D[x]$, then $\langle x, y\rangle \in D$ and $\langle f(x), f(y)\rangle \in E$.
- Thus $f(y) \in E\,[f(x)] \subseteq U$ and $y \in f^{-1}[U]$.
- We proved that $D[x] \subseteq f^{-1}[U]$, which implies that $f^{-1}[U]$ is open.

Example (continuous not uniformly continuous).

Let X be the open interval $(-\pi/2, \pi/2)$. Let d be the standard metric on X (with $d\,(x, y) = |x - y|$) and ρ be the metric defined by

$$\rho\,(x, y) = |\tan\,(x) - \tan\,(y)|.$$

Let $\mathscr{D}_d$ and $\mathscr{D}_\rho$ be the uniformities induced by d and ρ, respectively.

- Recall that $\mathscr{D}_d \subseteq \mathscr{D}_\rho$ but these uniformities are distinct.
- The uniformities $\mathscr{D}_d$ and $\mathscr{D}_\rho$ induce the same topology on X.
- It follows that the identity function $f : X \to X$ is a homeomorphism, and is uniformly continuous but is not a uniform isomorphism from $(X, \mathscr{D}_\rho)$ to $(X, \mathscr{D}_d)$.