Math 793C Topology for Analysis

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Exercise (pseudometric uniformity topology).

Let X be a set, d be a pseudometric on X and $\mathscr{D}$ be the uniformity on X that is induced by d. Prove that both d and $\mathscr{D}$ induce the same topology on X.

Solution. If $x \in X$ and $\varepsilon > 0$, then let

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

be the open ball (with respect to d) centered at x with radius ε.

- Let τ be the topology on X that is induced by d and τ' be the topology induced by $\mathscr{D}$.
- Let $U \in \tau$. To show that $U \in \tau'$, we take any $x \in U$ and find $D \in \mathscr{D}$ such that $D[x] \subseteq U$.
  - Let $x \in U$. There is $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq U$.
  - Let $D = \{\langle y, z\rangle \in X \times X : d(x, y) < \varepsilon\}$. Then $D \in \mathscr{D}$.
  - Moreover, $D[x] = B_d(x, \varepsilon)$ implying that $D[x] \subseteq U$.
- Let $U \in \tau'$. To show that $U \in \tau$, we take $x \in U$ and find $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq U$.
  - Since $U \in \tau'$, there is $D \in \mathscr{D}$ such that $D[x] \subseteq U$.
  - There is $\varepsilon > 0$ such that $\langle x, y\rangle \in D$ whenever $d(x, y) < \varepsilon$.
  - Then $B_d(x, \varepsilon) \subseteq D[x]$ so $B_d(x, \varepsilon) \subseteq U$.

Example (pseudometric topology uniformity).

Let X be the open interval $(-\pi/2, \pi/2)$. Let d be the standard metric on X (with $d(x,y) = |x-y|$) and ρ be the metric defined by

$$\rho(x,y) = |\tan(x) - \tan(y)|.$$

Let $\mathscr{D}_d$ and $\mathscr{D}_\rho$ be the uniformities induced by d and ρ, respectively.

- Then $\mathscr{D}_d \subseteq \mathscr{D}_\rho$ but these uniformities are distinct.

- Note that both d and ρ induce the same topology on X.
- The topology on X that is induced by d is the same as the topology induced by $\mathscr{D}_d$. Similarly for ρ and $\mathscr{D}_\rho$.
- Thus $\mathscr{D}_d$ and $\mathscr{D}_\rho$ are different uniformities on X that induce the same topology.

Uniformity base.

A *uniformity base* on a set X is a filter base of a some uniformity on X.

- Explicitely, a *uniformity base* on X is a family $\mathscr{B}$ of subsets of $X \times X$ that is a filter base on $X \times X$ and such that the filter on $X \times X$ that is induced by $\mathscr{B}$ is a uniformity on X.
- The theorem on uniformity base gives a necessary and sufficient condition for a family of subsets of $X \times X$ to be a uniformity base on X.

Theorem (uniformity base).

Let X be a set and $\mathscr{B}$ be a nonempty family of subsets of $X \times X$. Then $\mathscr{B}$ is a uniformity base on X if and only if the following conditions hold:

1. Each $D \in \mathscr{B}$ is a reflexive relation on X.
2. If $D \in \mathscr{B}$, then there exists $B \in \mathscr{B}$ with $B \subseteq D^{-1}$.
3. If $D \in \mathscr{B}$, then there exists $B \in \mathscr{B}$ with $B \circ B \subseteq D$.
4. If $B, D \in \mathscr{B}$, then there exists $E \in \mathscr{B}$ with $E \subseteq B \cap D$.

Proof. Assume that $\mathscr{B}$ satisfies the listed conditions.

- Since $\mathscr{B} \neq \varnothing$ and since condition (4) holds, it follows that $\mathscr{B}$ is a filter base on $X \times X$.
- Let $\mathscr{D} = \{D \subseteq X : (\exists B \in \mathscr{B})\, B \subseteq D\}$. Then $\mathscr{D}$ is the filter on $X \times X$ that is induced by $\mathscr{B}$.
- We show that $\mathscr{D}$ is a uniformity on X.
  - We verify that each $D \in \mathscr{D}$ is a reflexive relation on X.
    - Let $D \in \mathscr{D}$ and $x \in X$. We need to show that $\langle x, x\rangle \in D$.
    - There is $B \in \mathscr{B}$ with $B \subseteq D$.
    - Each member of $\mathscr{B}$ is a reflexive relation on X so $\langle x, x\rangle \in B$.
    - Since $B \subseteq D$, it follows that $\langle x, x\rangle \in D$.
  - We verify that $D^{-1} \in \mathscr{D}$ for every $D \in \mathscr{D}$.
    - Let $D \in \mathscr{D}$. There is $B \in \mathscr{B}$ with $B \subseteq D$.
    - Condition (2) implies that there is $A \in \mathscr{B}$ with $A \subseteq B^{-1}$.
    - Since $B^{-1} \subseteq D^{-1}$, it follows that $A \subseteq D^{-1}$ so $D^{-1} \in \mathscr{D}$.
  - We verify that for every $D \in \mathscr{D}$ there exists $E \in \mathscr{D}$ with $E \circ E \subseteq D$.
    - Let $D \in \mathscr{D}$. There is $B \in \mathscr{B}$ with $B \subseteq D$.
    - Condition (3) implies that there is $E \in \mathscr{B}$ with $E \circ E \subseteq B$.
    - Then $E \in \mathscr{D}$ and $E \circ E \subseteq D$.

Assume that $\mathscr{B}$ is a uniformity base on X.

- Let $\mathscr{D} = \{D \subseteq X : (\exists B \in \mathscr{B})\, B \subseteq D\}$. Then $\mathscr{D}$ is a uniformity on X.
- We verify that $\mathscr{B}$ satisfies the listed conditions.
  - Let $D \in \mathscr{B}$. Since $D \subseteq D$, it follows that $D \in \mathscr{D}$.
  - Since $\mathscr{D}$ is a uniformity, D is a reflexive relation on X so (1) holds.
  - Since $\mathscr{D}$ is a uniformity, $D^{-1} \in \mathscr{D}$ so there is $B \in \mathscr{B}$ with $B \subseteq D^{-1}$. Thus (2) holds.
  - Since $\mathscr{D}$ is a uniformity, there is $A \in \mathscr{D}$ with $A \circ A \subseteq D$. There is $E \in \mathscr{B}$ with $E \subseteq A$. Then $E \circ E \subseteq A \circ A$ so $E \circ E \subseteq D$. Thus (3) holds.
  - Let $B, D \in \mathscr{B}$. Then $B, D \in \mathscr{D}$ so $B \cap D \in \mathscr{D}$, implying that there is $E \in \mathscr{B}$ with $E \subseteq B \cap D$. Thus (4) holds.

Symmetric surrounding.

Let $\mathscr{D}$ be a uniformity on X and $D \in \mathscr{D}$ be a *surrounding* in $\mathscr{D}$. We say that D is symmetric iff $D^{-1} = D$.

The family of all symmetric surroundings in $\mathscr{D}$ is a uniformity base that induces the uniformity $\mathscr{D}$.

Exercise (symmetric surroundings base).

Let $\mathscr{D}$ be uniformity on a set X. Prove that the family of all symmetric surroundings in $\mathscr{D}$ is a uniformity base that induces the uniformity $\mathscr{D}$.

Solution. Let $\mathscr{B} = \{D \in \mathscr{D} : D \text{ is symmetric}\}$. By Theorem (uniformity base), it suffices to verify the following conditions:

1. Each $D \in \mathscr{B}$ is a reflexive relation on X.

   If $D \in \mathscr{B}$, then $D \in \mathscr{D}$. Each member of $\mathscr{D}$ is a reflexive relation on X.

2. If $D \in \mathscr{B}$, then there exists $B \in \mathscr{B}$ with $B \subseteq D^{-1}$.

   If $D \in \mathscr{B}$, then D is symmetric so $D^{-1} = D$. Thus $B := D$ satisfies the requirements.

3. If $D \in \mathscr{B}$, then there exists $B \in \mathscr{B}$ with $B \circ B \subseteq D$.

   - Let $D \in \mathscr{B}$. Then $D \in \mathscr{D}$ so there exists $E \in \mathscr{D}$ with $E \circ E \subseteq D$.
   - Let $B = E \cap E^{-1}$. Then B is symmetric and $B \in \mathscr{D}$ so $B \in \mathscr{B}$.
   - Moreover, $B \circ B \subseteq E \circ E \subseteq D$.

4. If $B, D \in \mathscr{B}$, then there exists $E \in \mathscr{B}$ with $E \subseteq B \cap D$.

   - Let $B, D \in \mathscr{B}$. Then $B, D \in \mathscr{D}$ so $E := B \cap D \in \mathscr{D}$.
   - Since E is symmetric, it follows that $E \in \mathscr{B}$.

Uniformity subbase.

A *uniformity subbase* on a set X is a subset of $X \times X$ that is a subbase of a uniformity on X.

- Recall that each family $\mathscr{S}$ of subsets of $X \times X$ induces a filter $\mathscr{F}$ on $X \times X$. The filter $\mathscr{F}$ is induced by the filter base

$$\mathscr{B} = \{X \times X\} \cup \{\bigcap \mathscr{S}' : \mathscr{S}' \subseteq \mathscr{S} \text{ is finite and nonempty}\}.$$

- Explicitly, the filter $\mathscr{F}$ that is induced by $\mathscr{S}$ is given by:

$\mathscr{F} = \{D \in X \times X : \bigcap \mathscr{S}' \subseteq D : \text{for some finite and nonempty } \mathscr{S}' \subseteq \mathscr{S}\} \cup \{X \times X\}$.

- A family $\mathscr{S}$ of subsets of $X \times X$ is a *uniformity subbase* if and only if the filter on $X \times X$ that is induced by $\mathscr{S}$ is a uniformity.
- See Theorem (uniformity subbase) for a list of conditions on $\mathscr{S}$ that are sufficient for $\mathscr{S}$ to be a uniformity subbase on X.

Theorem (uniformity subbase).

Let X be a set and $\mathscr{S}$ be any family of subsets of $X \times X$. If $\mathscr{S}$ satisfies the following conditions, then $\mathscr{S}$ is a uniformity subbase on X.

1. Each $S \in \mathscr{S}$ is a reflexive relation on X.
2. If $S \in \mathscr{S}$, then there exists $T \in \mathscr{S}$ with $T \subseteq S^{-1}$.
3. If $S \in \mathscr{S}$, then there exists $T \in \mathscr{S}$ with $T \circ T \subseteq S$.

Proof. Assume that $\mathscr{S}$ satisfies the listed conditions. Let

$$\mathscr{B} := \{X \times X\} \cup \{\bigcap \mathscr{S}' : \mathscr{S}' \subseteq \mathscr{S} \text{ is finite and nonempty}\}.$$

We show that $\mathscr{B}$ a uniformity base on X by verifying that $\mathscr{B}$ satisfies all the conditions given in Theorem (uniformity base).

- Each $D \in \mathscr{B}$ is a reflexive relation on X.
  - Let $D \in \mathscr{B}$. If $D = X \times X$, then D is a reflexive relation on X.
  - Otherwise, $D = \bigcap \mathscr{S}'$ for some finite and nonempty $\mathscr{S}' \subseteq \mathscr{S}$.
  - If $x \in X$, then $\langle x, x\rangle \in S$ for every $S \in \mathscr{S}'$ so $\langle x, x\rangle \in D$.
  - Thus D is a reflexive relation on X.
- If $D \in \mathscr{B}$, then there exists $B \in \mathscr{B}$ with $B \subseteq D^{-1}$.
  - Let $D \in \mathscr{B}$. If $D = X \times X$, then $B := D$ satisfies the requirements.
  - Assume that $D = \bigcap \mathscr{S}'$ for some finite and nonempty $\mathscr{S}' \subseteq \mathscr{S}$.
  - Condition (2) implies that for each $S \in \mathscr{S}'$ there is $T_S \in \mathscr{S}$ with $T_S \subseteq S^{-1}$.
  - Let $B = \bigcap \{T_S : S \in \mathscr{S}'\}$. Then $B \in \mathscr{B}$ and $B \subseteq \bigcap \{S^{-1} : S \in \mathscr{S}'\}$.
  - Since $D^{-1} = \bigcap \{S^{-1} : S \in \mathscr{S}'\}$, it follows that $B \subseteq D^{-1}$.

    We have $D^{-1} = \bigcap \{S^{-1} : S \in \mathscr{S}'\}$ since
    - $\langle x, y\rangle \in D^{-1}$ iff

    - $\langle y, x\rangle \in D$ iff
    - $\langle y, x\rangle \in S$ for every $S \in \mathscr{S}'$ iff
    - $\langle x, y\rangle \in S^{-1}$ for every $S \in \mathscr{S}'$.

- If $D \in \mathscr{B}$, then there exists $B \in \mathscr{B}$ with $B \circ B \subseteq D$.

  - Let $D \in \mathscr{B}$. If $D = X \times X$, then $B := D$ satisfies the requirements.
  - Assume that $D = \bigcap \mathscr{S}'$ for some finite and nonempty $\mathscr{S}' \subseteq \mathscr{S}$.
  - Condition (3) implies that for each $S \in \mathscr{S}'$ there is $T_S \in \mathscr{S}$ with $T_S \circ T_S \subseteq S$.
  - Let $B := \bigcap \{T_S : S \in \mathscr{S}'\}$. Then $B \in \mathscr{B}$ and $B \circ B \subseteq D$.

    We show that $B \circ B \subseteq D$.

    - Let $\langle x, z\rangle \in B \circ B$. Then there is $y \in X$ with $\langle x, y\rangle \in B$ and $\langle y, z\rangle \in B$.
    - If $S \in \mathscr{S}'$, then $\langle x, y\rangle \in T_S$ and $\langle y, z\rangle \in T_S$ so $\langle x, z\rangle \in S$.
    - Since $\langle x, z\rangle \in S$ for every $S \in \mathscr{S}'$, it follows that $\langle x, z\rangle \in D$.

- If $B, D \in \mathscr{B}$, then there exists $E \in \mathscr{B}$ with $E \subseteq B \cap D$.

  - Let $B, D \in \mathscr{B}$. If $B = X \times X$, then $B \cap D = B \in \mathscr{B}$ so $E := B$ satisfies the requirements. Similarly, when $D = X \times X$.
  - Assume that $B = \bigcap \mathscr{S}'$ and $D = \bigcap \mathscr{S}''$ for some finite and nonempty $\mathscr{S}', \mathscr{S}'' \subseteq \mathscr{S}$. Let $E := B \cap D$.
  - Since $E = \bigcap \mathscr{S}'''$ where $\mathscr{S}''' = \mathscr{S}' \cup \mathscr{S}''$ is a finite and nonempty subset of $\mathscr{S}$, it follows that $E \in \mathscr{B}$.

Exercise (union of uniformities).

Let X be the interval $(-\pi/2, \pi/2)$ and $f, g : X \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & x \le 0; \\ \tan(x) & x \ge 0. \end{cases}$$

and

$$g(x) = \begin{cases} \tan(x) & x \le 0; \\ x & x \ge 0. \end{cases}$$

Let d_1, d_2 be the pseudometrics on X defined by $d_1(x, y) = |f(x) - f(y)|$ and $d_2(x, y) = |g(x) - g(y)|$. Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be the uniformities on X induced by d_1 and d_2, respectively. Prove that $\mathscr{D}_1 \cup \mathscr{D}_2$ is not a uniformity base on X.

Solution. Let $\mathscr{B} := \mathscr{D}_1 \cup \mathscr{D}_2$. Let

$$D_1 = \{\langle x, y\rangle \in X \times X : |f(x) - f(y)| < 1\}$$

and

$$D_2 = \{\langle x, y\rangle \in X \times X : |g(x) - g(y)| < 1\}.$$

Then $D_1 \in \mathscr{D}_1$ and $D_2 \in \mathscr{D}_2$ so both D_1 and D_2 belong to $\mathscr{B}$. Let $D = D_1 \cap D_2$.

- We show that there are no $E \in \mathscr{B}$ with $E \subseteq D$.
  - Suppose, for a contradiction, that there exists $E \in \mathscr{B}$ with $E \subseteq D$. Then either $E \in \mathscr{D}_1$ or $E \in \mathscr{D}_2$.
  - Assume first that $E \in \mathscr{D}_1$. Then there exists $\varepsilon > 0$ such that $\langle x, y\rangle \in E$ whenever $|f(x) - f(y)| < \varepsilon$.
  - There are $x, y \in X$ such that $x, y < 0$, $|x - y| < \varepsilon$ and $|\tan(x) - \tan(y)| \geq 1$.
  - Then $|f(x) - f(y)| = |x - y| < \varepsilon$ so $\langle x, y\rangle \in E$.
  - However, $|g(x) - g(y)| = |\tan(x) - \tan(y)| \geq 1$ so $\langle x, y\rangle \notin D_2$ and consequently $\langle x, y\rangle \notin D$. This is a contradiction.
  - Similarly, we get a contradiction when $E \in \mathscr{D}_2$.
- That implies that $\mathscr{B}$ is not a uniformity base on X.

Theorem (union uniformity subbase).

Let X be a set and $\mathfrak{A}$ be a family such that each member of $\mathfrak{A}$ is a uniformity base on X. Then $\bigcup \mathfrak{A}$ is a uniformity subbase on X. In particular, the union of a family of uniformities on X is a uniformity subbase on X.

Proof. Let $\mathscr{S} = \bigcup \mathfrak{A}$. We show that $\mathscr{S}$ a uniformity subbase on X by verifying that $\mathscr{S}$ satisfies all the conditions given in Theorem (uniformity subbase).

- Each $S \in \mathscr{S}$ is a reflexive relation on X.

  If $S \in \mathscr{S}$, then $S \in \mathscr{B}$ for some $\mathscr{B} \in \mathfrak{A}$. Thus S is a reflexive relation on X.
- If $S \in \mathscr{S}$, then there exists $T \in \mathscr{S}$ with $T \subseteq S^{-1}$.
  - If $S \in \mathscr{S}$, then $S \in \mathscr{B}$ for some $\mathscr{B} \in \mathfrak{A}$.
  - Thus there exists $T \in \mathscr{B}$ with $T \subseteq S^{-1}$.
  - Since $\mathscr{B} \subseteq \mathscr{S}$, it follows that $T \in \mathscr{S}$.
- If $S \in \mathscr{S}$, then there exists $T \in \mathscr{S}$ with $T \circ T \subseteq S$.
  - If $S \in \mathscr{S}$, then $S \in \mathscr{B}$ for some $\mathscr{B} \in \mathfrak{A}$.
  - Thus there exists $T \in \mathscr{B}$ with $T \circ T \subseteq S$.
  - Since $\mathscr{B} \subseteq \mathscr{S}$, it follows that $T \in \mathscr{S}$.

Exercise (uniformity base nbhds).

Let $\mathscr{B}$ be a uniformity base on X, let $\mathscr{D}$ be the uniformity on X that is induced by $\mathscr{B}$ and τ be the topology on X that is induced by $\mathscr{D}$. Prove that for each $x \in X$, the family $\mathscr{B}_x = \{B[x] : B \in \mathscr{B}\}$ is a base of the nbhd filter at x (the filter consisting of all nbhds at x) with respect to τ.

Solution. Let $x \in X$ and $\mathscr{U}_x$ be the nbhd filter at x.

- Since $\mathscr{B} \subseteq \mathscr{D}$, it follows that $\mathscr{B}_x \subseteq \mathscr{U}_x$.
- It remains to show that for every $U \in \mathscr{U}_x$ there exists $B \in \mathscr{B}$ with $B[x] \subseteq U$.
  - Let $U \in \mathscr{U}_x$. There exists open $V \subseteq X$ with $x \in V \subseteq U$.
  - Thus there is $D \in \mathscr{D}$ with $D[x] \subseteq V$ and there is $B \in \mathscr{B}$ with $B \subseteq D$.
  - Then $B[x] \subseteq U$ as required.

Exercise (uniformity subbase nbhds).

Let $\mathscr{S}$ be a uniformity subbase on X, let $\mathscr{D}$ be the uniformity on X that is induced by $\mathscr{S}$ and τ be the topology on X that is induced by $\mathscr{D}$. Prove that for each $x \in X$, the family $\mathscr{S}_x = \{S[x] : S \in \mathscr{S}\}$ is a subbase of the nbhd filter at x (the filter consisting of all nbhds at x) with respect to τ.

Solution. Let $x \in X$ and $\mathscr{U}_x$ be the nbhd filter at x. Let

$$\mathscr{B}_x = \{X\} \cup \{\bigcap \mathscr{A} : \mathscr{A} \subseteq \mathscr{S}_x \text{ is finite and nonempty}\}.$$

To show that $\mathscr{S}_x$ is a subbase for $\mathscr{U}_x$, it suffices to show that $\mathscr{B}_x$ is a base for $\mathscr{U}_x$. Note that

$$\mathscr{D} = \{X \times X\} \cup \{D \subseteq X \times X : \bigcap \mathscr{S}' \subseteq D \text{ for some finite nonempty } \mathscr{S}' \subseteq \mathscr{S}\}.$$

- First we show that $\mathscr{B}_x \subseteq \mathscr{U}_x$.

  Let $B \in \mathscr{B}_x$.
  - If $B = X$, then $B = D[x]$ with $D := X \times X$. Thus $B \in \mathscr{U}_x$.
  - Assume $B = \bigcap \mathscr{A}$ for some finite and nonempty $\mathscr{A} \subseteq \mathscr{S}_x$.
  - Then $\mathscr{A} = \{S[x] : S \in \mathscr{S}'\}$ for some finite and nonempty $\mathscr{S}' \subseteq \mathscr{S}$.
  - Let $D = \bigcap \mathscr{S}'$. Then $D \in \mathscr{D}$ and $D[x] = \bigcap \mathscr{A}$.

    We have $y \in D[x]$ iff $\langle x, y \rangle \in D$ iff $\langle x, y \rangle \in S$ for all $S \in \mathscr{S}'$ iff $y \in S[x]$ for all $S \in \mathscr{S}'$ iff $y \in \bigcap \mathscr{A}$.
  - Since $B = D[x]$ for some $D \in \mathscr{D}$, it follows that $B \in \mathscr{U}_x$.

- Now we show that for every $U \in \mathscr{U}_x$ there exists $B \in \mathscr{B}_x$ with $B \subseteq U$.

  Let $U \in \mathscr{U}_x$. There exists $D \in \mathscr{D}$ with $D[x] \subseteq U$.

  - If $D = X \times X$, then $U = X \in \mathscr{B}_x$ and $B := U$ satisfies the requirements.
  - Assume that there exists finite and nonempty $\mathscr{S}' \subseteq \mathscr{S}$ with $\bigcap \mathscr{S}' \subseteq D$.
    - Let $\mathscr{A} := \{S[x] : S \in \mathscr{S}'\}$ and $B := \bigcap \mathscr{A}$. Then $B \in \mathscr{B}_x$.
    - We show that $B \subseteq U$.

      Let $y \in B$.
      - Then $y \in S[x]$ for each $S \in \mathscr{S}'$ so $\langle x, y \rangle \in S$ for each $S \in \mathscr{S}'$.
      - Thus $\langle x, y \rangle \in \bigcap \mathscr{S}' \subseteq D$, which implies that $y \in D[x] \subseteq U$.