Math 793C Topology for Analysis

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Uniform space.

A *uniform space* is a set X with a uniformity on X.

- Each pseudometric space is a uniform space, see uniformity from pseudometric.
- Each uniform space is a topological space, see topology from uniformity.

Uniformity from pseudometric.

Let d be a pseudometric on a set X. The uniformity $\mathscr{D}$ on X that is *induced* by d is obtained as follows. Given $D \subseteq X \times X$, we include D as a member of $\mathscr{D}$ if and only if there exists $\varepsilon > 0$ such that $\langle x, y \rangle \in D$ whenever $d(x, y) < \varepsilon$.

- For the proof that the resulting family $\mathscr{D}$ is a uniformity on X see the proof of the theorem on uniformity from pseudometric.
- See some examples of uniformity from pseudometric.

Topology from uniformity.

Let $\mathscr{D}$ be a uniformity on a set X. The topology τ on X that is *induced* by $\mathscr{D}$ is obtained as follows. Given $U \subseteq X$, we include U as a member of τ if and only if for every $x \in U$ there exists $D \in \mathscr{D}$ such that $D[x] \subseteq U$, where $D[x] = \{y \in X : \langle x, y \rangle \in D\}$.

- We also say that τ is the *topology of* $\mathscr{D}$.
- For the proof that the resulting family τ is a topology on X see the proof of the theorem on topology from uniformity.
- Different uniformities may induce the same topology. See examples (pseudometric topology uniformity).

Topology from uniformity interior.

Let $\mathscr{D}$ be a uniformity on a set X. If $A \subseteq X$, and

$$U = \{x \in A : (\exists D \in \mathscr{D})\, D[x] \subseteq A\},$$

then U is open in the topology on X that is induced by $\mathscr{D}$. Moreover, U is the interior of A.

Proof.

- First note that when we show that U is open, it follows easily that U is the interior of A.

  Suppose that B is open with $U \subseteq B \subseteq A$. We show that $B \subseteq U$ so $U = B$.

  - Let $x \in B$. Since B is open, there is $D \in \mathscr{D}$ with $D[x] \subseteq B$.
  - Then $D[x] \subseteq A$ and it follows that $x \in U$. Thus $U = B$.

- Now we show that U is open. Let $x \in U$. We need to find $E \in \mathscr{D}$ with $E[x] \subseteq U$.

  - Since $x \in U$, there is $D \in \mathscr{D}$ with $D[x] \subseteq A$.
  - Let $E \in \mathscr{D}$ be such that $E \circ E \subseteq D$. We show that $E[x] \subseteq U$.
    - Let $y \in E[x]$. To show that $y \in U$, it suffices to verify that $E[y] \subseteq A$.
    - Let $z \in E[y]$. Then $\langle x, y\rangle \in E$ and $\langle y, z\rangle \in E$, implying that $\langle x, z\rangle \in E \circ E \subseteq D$.
    - It follows that $z \in D[x] \subseteq A$ so $E[y] \subseteq A$.

Topology from uniformity nbhds.

Let $\mathscr{D}$ be a uniformity on a set X and τ be the topology on X induced by $\mathscr{D}$. Then, for each $x \in X$, the family $\mathscr{U}_x = \{D[x] : D \in \mathscr{D}\}$ is the nbhd system at x (the family of all nbhds of x).

Proof.

- Let $x \in X$ and A be a nbhd of x. We show that $A \in \mathscr{U}_x$.

  - Since A is a nbhd of x, there is open U with $x \in U \subseteq A$.
  - Thus there is $D \in \mathscr{D}$ with $D[x] \subseteq U$.
  - Define $E = D \cup \{\langle x, y\rangle \in X \times X : y \in A\}$.

- Then $A = E[x]$ and $E \in \mathscr{D}$ so $A \in \mathscr{U}_x$.
    - $*$ The definition of E implies that $A \subseteq E[x]$.
    - $*$ Since $D[x] \subseteq A$, it follows that $E[x] \subseteq A$ so $E[x] = A$.
    - $*$ Since $D \subseteq E$ and since $\mathscr{D}$ is a filter, it follows that $E \in \mathscr{D}$.
    - $*$ Since A is of the form $E[x]$ for $E \in \mathscr{D}$, it follows that $A \in \mathscr{U}_x$.

- Let $A \in \mathscr{U}_x$. We show that A is a nbhd of x.
  - Since $A \in \mathscr{U}_x$, there is $E \in \mathscr{D}$ with $A = E[x]$.
  - Let $U = \{y \in A : (\exists D \in \mathscr{D})\, D[y] \subseteq A\}$.
  - Theorem (topology from uniformity interior) implies that U is open.
  - Since U is open and $x \in U \subseteq A$, it follows that A is a nbhd of x.

Theorem (topology from uniformity Hausdorff).

Let $\mathscr{D}$ be a uniformity on a set X and τ be the topology on X induced by $\mathscr{D}$ (topology from uniformity). Then τ is Hausdorff if and only if $\mathscr{D}$ is separating (uniformity).

Proof.

- Assume that τ is Hausdorff. Let $x, y \in X$ be distinct. To show that $\mathscr{D}$ is separating, we need to find $D \in \mathscr{D}$ with $\langle x, y \rangle \notin D$.
  - Since τ is Hausdorff, there are disjoint open sets $U, V \subseteq X$ with $x \in U$ and $y \in V$.
  - Since U is open and $x \in U$, there is $D \in \mathscr{D}$ with $D[x] \subseteq U$.
  - Since $y \notin U$, it follows that $y \notin D[x]$ so $\langle x, y \rangle \notin D$.

- Assume that $\mathscr{D}$ is separating. Let $x, y \in X$ be distinct. We show that there exists $D \in \mathscr{D}$ with $D[x] \cap D[y] = \varnothing$.
  - Since $\mathscr{D}$ is separating, there is $E \in \mathscr{D}$ such that $\langle x, y \rangle \notin E$.
  - There is $F \in \mathscr{D}$ be such that $F \circ F \subseteq E$. Let $D = F \cap F^{-1}$.
  - The definition of uniformity implies that $D \in \mathscr{D}$.
  - We show that $D[x] \cap D[y] = \varnothing$.
    - $*$ Suppose, for a contradiction, that $z \in D[x] \cap D[y]$.
    - $*$ Then $\langle x, z \rangle \in D \subseteq F$ and $\langle y, z \rangle \in D \subseteq F^{-1}$ so $\langle z, y \rangle \in F$.
    - $*$ Consequently, $\langle x, y \rangle \in F \circ F \subseteq E$, which is a contradiction.

- Theorem (topology from uniformity nbhds) implies that $D[x]$ and $D[y]$ are nbhds of x and y, respectively, so it follows that τ is Hausdorff.