

A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND
NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED
MATHEMATICIANS

DE CLASS NOTES 4

A COLLECTION OF HANDOUTS ON
PARTIAL DIFFERENTIAL EQUATIONS (PDE's)

CHAPTER 5

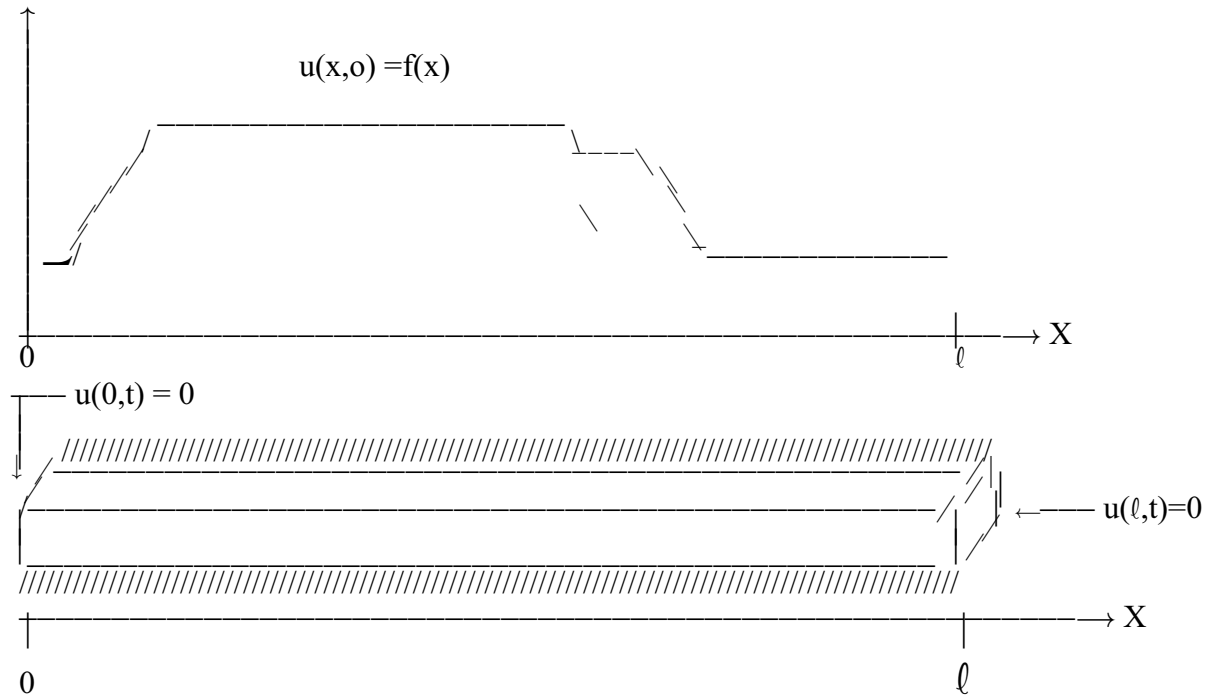
Solution of the Heat Conduction Model

1. Review of the Heat Conduction Model with Homogeneous Boundary Conditions
2. Solution of the Heat Conduction Model with Homogeneous Boundary Conditions Using Fourier Series
3. Review of Steps in Solving a PDE Model with Homogeneous Boundary Conditions
4. Solution of the Heat Conduction Model with Nonhomogeneous Boundary Conditions
5. Rod (or Bar) with Insulated Ends
6. Review of Steps in Solving a Homogeneous PDE's with Nonhomogeneous Boundary Conditions

The model for heat conduction in a rod of length ℓ which is insulated on the sides is given below. Recall that we require the temperature at both ends to have zero temperature (i.e. Dirichlet boundary conditions). Later, we will see how to handle arbitrary temperatures (nonhomogeneous conditions) and insulated ends (Neumann conditions).

	PDE	$u_t = \alpha^2 u_{xx}$	$0 < x < \ell, t > 0$
BVP	BC	$u(0, t) = 0, u(\ell, t) = 0$ (homogeneous boundary conditions)	$t > 0$
	IC	$u(x, 0) = f(x)$ (initial temperature distribution)	$0 < x < \ell$

We represent this BVP schematically as follows:



Recall that

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi a/\ell)^2 t} \sin\left(\frac{n\pi}{\ell} x\right)$$

is a family of solutions to the PDE $u_t = \alpha^2 u_{xx}$ and the homogeneous BC's $u(0,t) = 0$, $u(\ell,t) = 0$. That is, a "linear combination" of an infinite number of linearly independent solutions is in the null space of the linear operator defined by the PDE and the BC's and in some sense gives the "general" solution to the problem defined by the PDE and the BCs. Clearly the dimension of this null space is infinite.. We assume no convergence problems and blindly try to use (13) to satisfy the IC in the BVP

	PDE	$u_t = \alpha^2 u_{xx}$	$0 < x < \ell$, $t > 0$
BVP	BC	$u(0,t) = 0$, $u(\ell,t) = 0$	$t > 0$
	IC	$u(x,0) = f(x)$	$0 < x < \ell$

Letting $t = 0$ in (1) we obtain

$$u(x,0) = \sum_{n=1}^{\infty} c_n e^{-(n\pi a/\ell)^2 (0)} \sin\left(\frac{n\pi}{\ell} x\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{\ell} x\right) =$$

Hence we require

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{\ell} x\right) \quad \forall x \in [0,\ell] \quad (2)$$

"Clearly" we can satisfy (2) by taking the c_n 's to be the Fourier Sine series coefficients for $f(x)$. That is, we let

$$c_m = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{m\pi}{\ell} x\right) dx \quad \text{for } m = 1, 2, 3, \dots$$

REVIEW OF STEPS IN SOLVING A HOMOGENEOUS PDE
WITH HOMOGENOUS BOUNDARY CONDITIONS

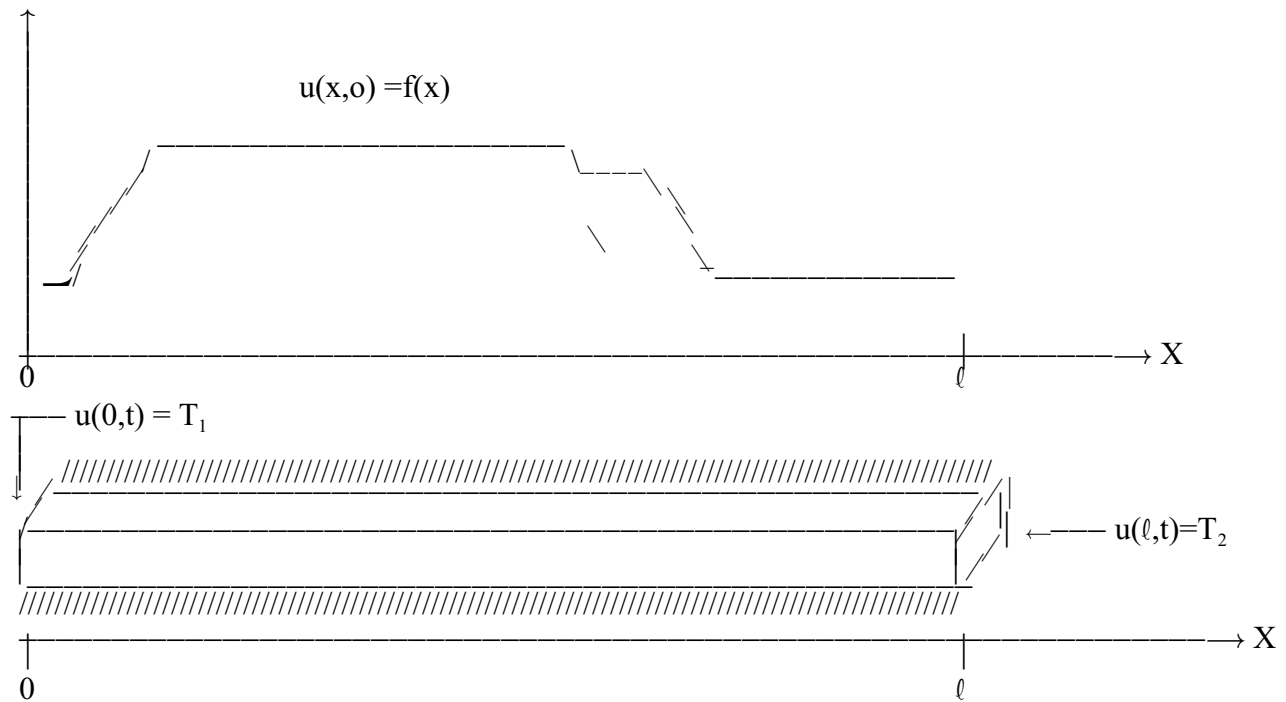
Handout #3

1. Use Separation of Variables (for PDE's) to obtain two ODE's with an arbitrary constant from the PDE (typically, a spacial ODE and a temporal ODE).
2. Use the Boundary Conditions (BC) for the PDE to obtain BC's for the spacial ODE and hence an Eigen Value Problem (EVP) for the spacial ODE.
3. Solve the EVP for the ODE. Typically, it is self adjoint so that the eigen values are real.
Make a Table.
 - a. Find all of the negative eigen values or show that none exist.
 - b. Determine if zero is an eigen value.
 - c. Find all of the positive eigen values or show that none exist.
4. Solve the time ODE and express the solutions in terms of the eigen values for the space ODE.
5. Using the solution of the eigen value problem and the solution of the temporal ODE write an expression for the family of solutions to the PDE and its BC's. We refer to this as the "general solution" of the PDE and its BC's.
6. From the "general solution" of the PDE and its BC's, apply the initial condition and use Fourier series to find the solution of a Boundary Value Problem (BVP) consisting of a PDE, its BC's and its Initial Condition (IC).
7. List the results of the above six steps to solving a BVP for a PDE.

Recall the model for heat conduction in a rod of length ℓ which is insulated on the sides and zero (homogeneous) temperature conditions. The nonhomogeneous problem allows the temperature at the ends to have values other than zero. Later, we consider insulated ends (Niemann conditions) where the (partial) derivative, rather than the temperature is specified.

	PDE	$u_t = \alpha^2 u_{xx}$	$0 < x < \ell, t > 0$
BVP	BC	$u(0, t) = T_1, u(\ell, t) = T_2$ (nonhomogeneous boundary conditions)	$t > 0$
	IC	$u(x, 0) = f(x)$ (initial temperature distribution)	$0 < x < \ell$

We represent this BVP schematically as follows:



Physically, we expect $u(x,t)$ to approach a steady-state or equilibrium solution. Recall that for the homogeneous problem, the steady-state solution was zero. Recall also that we expect that the general solution of a nonhomogeneous problem to be of the form

$$u(x,t) = u_p(x,t) + u_h(x,t)$$

where u_p is a particular solution to the nonhomogeneous and u_h is the general solution of the associated homogeneous (complementary) equation. Thus it is reasonable for u_p to be the steady-state solution and u_h to be the previously found general solution to the homogeneous problem. Thus to find u_p , we take u_p independent of time, i.e. let $u_p(x,t) = X_{ss}(x)$. Thus since

$$\frac{\partial u_p(x,t)}{\partial t} - \frac{\partial X_{ss}(x)}{\partial t} = 0,$$

$$\frac{\partial u_p(x,t)}{\partial x} - \frac{\partial X_{ss}(x)}{\partial x} = \frac{\partial^2 u_p(x,t)}{\partial x^2} - \frac{\partial^2 X_{ss}(x)}{\partial x^2} = X''(x)$$

X_{ss} must satisfy the ODE boundary value problem

ODE $X_{ss}'' = 0$

BCs $X_{ss}(0) = T_1, \quad X_{ss}(\ell) = T_2.$

SOLUTION OF ODE BVP

Solving the ODE we obtain $X(x) = c_1 x + c_2$. Applying the BCs, we obtain

$$c_1(0) + c_2 = T_1 \Rightarrow c_2 = T_1$$

$$c_1(\ell) + c_2 = T_2 \quad c_1 = \frac{T_2 - c_1}{\ell} = \frac{T_2 - T_1}{\ell}$$

Hence $X_{ss}(x) = \left[\frac{T_2 - T_1}{\ell} \right] x + T_1$. (Check that the BCs are satisfied.)

So we have

$$u(x,t) = u_p(x,t) + u_h(x,t) = X_{ss}(x) + u_h(x,t)$$

$$= \left[\frac{T_2 - T_1}{\ell} \right] x + \sum_{n=1}^{\infty} c_n e^{-(n\pi/\ell)^2 t} \sin\left(\frac{n\pi}{\ell} x\right)$$

Applying the initial condition $u(x,0) = f(x)$ we obtain the requirement

$$f(x) = \left[\frac{T_2 - T_1}{\ell} \right] x + \sum_{n=1}^{\infty} c_n e^{-(n\pi/\ell)^2 t} \sin\left(\frac{n\pi}{\ell} x\right)$$

or

$$f(x) - \left[\frac{T_2 - T_1}{\ell} \right] x = \sum_{n=1}^{\infty} c_n e^{-(n\pi/\ell)^2 t} \sin\left(\frac{n\pi}{\ell} x\right)$$

"Clearly" the c_n 's are the Fourier Sine series coefficients for

$$\tilde{f}(x) = f(x) \frac{T_2 - T_1}{\ell} x + T_1.$$

That is we have

$$c_m = \frac{2}{\ell} \int_0^{\ell} \tilde{f}(x) \sin\left(\frac{m\pi}{\ell} x\right) dx \quad \text{for } m = 1, 2, 3, \dots$$

$$= \frac{2}{\ell} \int_0^{\ell} \left[f(x) - \left[\frac{T_2 - T_1}{\ell} \right] x + T_1 \right] \sin\left(\frac{m\pi}{\ell} x\right) dx \quad \text{for } m = 1, 2, 3, \dots$$

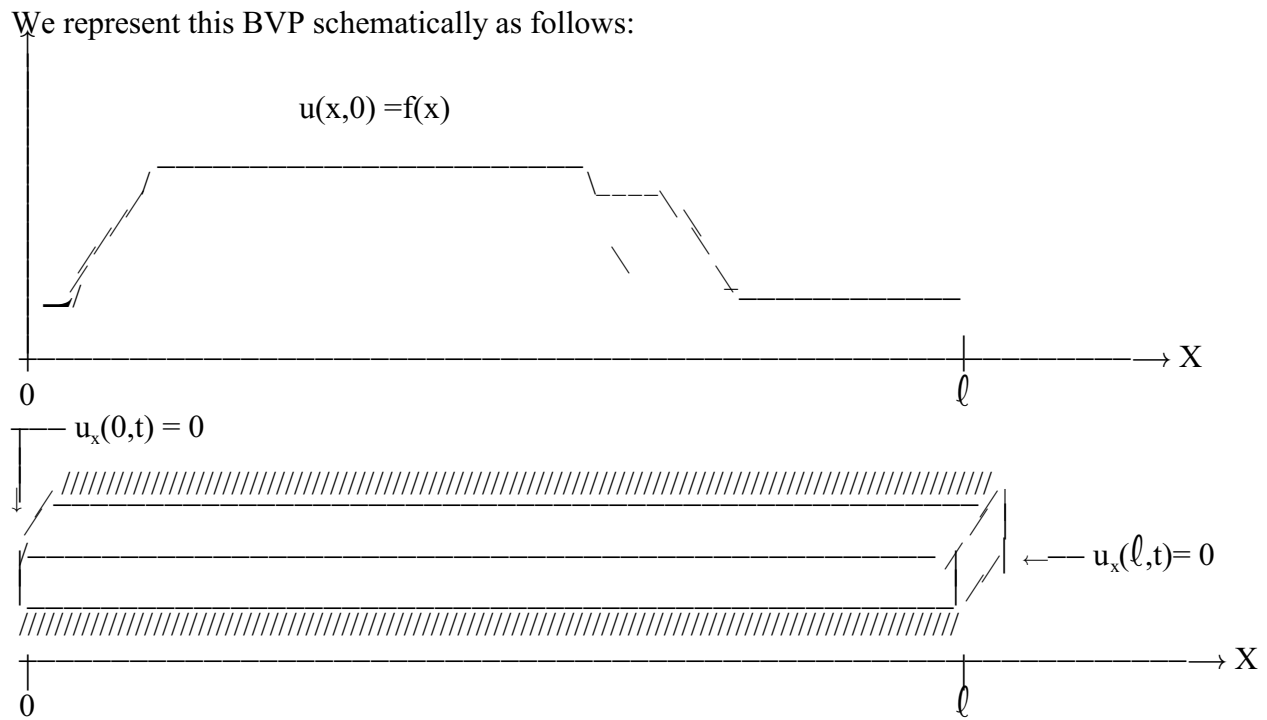
Recall the model for heat conduction in a rod of length ℓ which is insulated on the sides and where the temperature is specified (Dirichlet conditions, homogeneous or nonhomogeneous). For the rod (or bar) with insulated ends, we must consider conditions on the (partial) derivative (i.e. Neumann conditions). Recall that the energy flow through a cross section (e.g. the ends of the rod) is given by Fourier's Law as Rate at which Heat energy crosses any cross section A (to the right) = $-K A u_x$ where K is the thermal conductivity (proportionality constant) and A is the area of the cross section. If the ends are insulated then u_x is zero.

PDE $u_t = \alpha^2 u_{xx} \quad 0 < x < \ell, t > 0$

BVP BC $u_x(0, t) = 0, u_x(\ell, t) = 0 \quad t > 0$
 (homogeneous Neumann boundary conditions)

IC $u(x, 0) = f(x) \quad 0 < x < \ell$
 (initial temperature distribution)

We represent this BVP schematically as follows:



Since the PDE and BC are homogeneous, we expect to be able to find the general solution of these in the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$$

Since the system is closed with no energy entering or leaving, physically, we expect $u(x,t)$ to approach a steady-state or equilibrium solution of a constant temperature. However, this need not be $u(x,t) = 0$. In anticipation of this, we started the series at $n = 0$ and expect $u_0(x,t)$ to be one. That is we expect

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n u_n(x,t) \qquad \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$$

We need to review and redo the steps in solving a BVP for the heat conduction problem with homogeneous Bc's.

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