A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR
PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 4

A COLLECTION OF HANDOUTS ON
PARTIAL DIFFERENTIAL EQUATIONS (PDE's)

CHAPTER 4

Introduction to the

Heat Conduction Model

1. Introduction to Partial Differential Equations (PDE’s)
2. Derivation of the Heat Conduction Equation Using Conservation of Energy
3. The Boundary Value Problem for the Heat Conduction Model
4. Method of Separation of Variables
5. Derivation and Solution of the ODE Eigenvalue Problem
6. General Solution of the PDE and Boundary Conditions
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Handout #1 INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS (PDES) Prof. Moseley

We begin by giving several examples of partial differential equations. For each equation, be able to identify its common name and the physical situation that it is used to model. To obtain a complete model of the dynamic behavior of a physical situation, boundary and initial conditions must be specified for the partial differential equation. For steady-state (equilibrium) behavior, only boundary conditions are needed.

PDE	COMMON NAME	USED TO MODEL
1. $u_{tt} = c^2 u_{xx}$ $(u_{tt} - c^2 u_{xx} = 0)$ (homogeneous)	1-(space) dimensional wave equation	Dynamic behavior of the vibrations in a taut elastic string with no external force applied to the string.
2. $u_t = c^2 u_{xx}$ $(u_t - c^2 u_{xx} = 0)$ (homogeneous)	1-(space) dimensional heat (diffusion) equation	Dynamic behavior of heat conduction in a rod (bar) with no external heat source applied to the rod.
3. $u_{xx} + u_{yy} = 0$ (homogeneous) time- plate.	2-(space) dimensional Laplace's equation	1. Steady state deformation of a drum head with boundary deformation and no external force applied to the drum head. 2. Steady state heat conduction in a plate with an external non-varying heat source applied to the
4. $u_{xx} + u_{yy} = f(x,y)$ (nonhomogeneous)	2-(space) dimensional Poisson's equation	1. Steady state deformation of an elastic drum head with fixed boundary deformation and an external non-time-varying force (e.g. gravity or a mechanical force) on the drum head.. 2. Steady state heat conduction in a plate with an external non-time-varying heat source applied to the plate.
5. $u_{tt} = c^2 (u_{xx} + u_{yy})$ $(u_{tt} - c^2 (u_{xx} + u_{yy}) = 0)$ (homogeneous)	2-(space) dimensional wave equation	Dynamic behavior of the vibrations of an elastic drum head with no external force applied to the drum head.

6. $u_t = c^2 (u_{xx} + u_{yy})$ $(u_t - c^2 (u_{xx} + u_{yy}) = 0)$ (homogeneous)	2-(space) dimensional heat (diffusion) equation	Dynamic behavior of heat conduction in a plate with no external heat source applied to the plate.
7. $u_{xx} + u_{yy} + u_{zz} = 0$ (homogeneous) (Using the ∇ notation $\nabla \cdot \nabla u = \nabla^2 u = 0$)	3-(space) dimensional Laplace's equation	1. Steady state deformation of an elastic body in space with boundary deformations but no external force applied internally to the body. 2. Steady state heat flow in a body with no external heat source applied internally to the body.
8. $u_{xx} + u_{yy} + u_{zz} = f(x,y)$ (nonhomogeneous) (Using the ∇ notation $\nabla \cdot \nabla u = \nabla^2 u = 0$)	3-(space) dimensional Poisson's equation	1. Steady state deformation of an elastic body in space with an external non-time-varying force (e.g. gravity) applied internally to the body. 2. Steady state heat flow in a body with no external non-time-varying heat source applied internally to the body.
9. $u_{tt} = c^2 (u_{xx} + u_{yy} + u_{zz})$ $(u_{tt} - c^2 (u_{xx} + u_{yy} + u_{zz}) = 0)$ (homogeneous) (Using the ∇ notation $u_{tt} = \nabla \cdot \nabla u = \nabla^2 u$)	3-(space) dimensional wave equation	Dynamic behavior of the vibrations of an elastic body with no external force applied internally to the body.
10. $u_t = c^2 (u_{xx} + u_{yy} + u_{zz})$ $(u_t - c^2 (u_{xx} + u_{yy} + u_{zz}) = 0)$ (homogeneous) (Using the ∇ notation $u_t = \nabla \cdot \nabla u = \nabla^2 u$)	3-(space) dimensional heat (diffusion) equation	Dynamic behavior of heat conduction in a body in space with no external heat source applied internally to the body.

PDE's DERIVATION OF THE HEAT CONDUCTION EQUATION
Handout #2 USING CONSERVATION OF ENERGY Prof. Moseley

To develop a mathematical model for heat conduction in a rod, we first use conservation of energy to determine how heat flows in within the rod. This results in the heat (or diffusion) equation. We begin this process by noting some facts about heat conduction. These are a result of experimentation, observation, and measurement to validate theoretical physics.

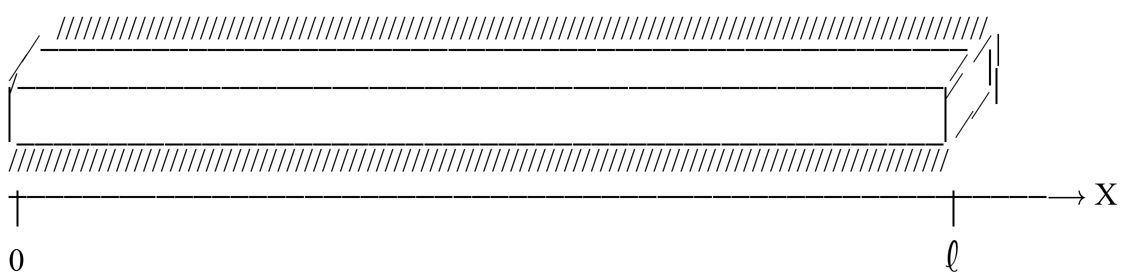
1. Conservation of energy. Heat energy is neither created or destroyed in the rod or bar.
2. Fourier's Law (Newton's "Microscopic or Continuous" Law of Cooling) The rate at which heat energy H is transferred through an area A is proportional to A and to the directional derivative of the temperature field normal to the area
3. Heat flows in the direction of decreasing temperature T (i.e. hot to cold). This is a result of the convention of using a scale that assumes that hotter bodies (i.e. those with more internal energy) have a higher temperature than cooler bodies (those with less internal heat energy).

Thus, in general

$$\frac{dH}{dt} = kA(-\nabla T\cdot\hat{u}_n) = -kA\left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)\cdot(n_x\hat{i} + n_y\hat{j} + n_z\hat{k})$$

$$= -kA\left(\frac{\partial T}{\partial x}n_x + \frac{\partial T}{\partial y}n_y + \frac{\partial T}{\partial z}n_z\right)$$

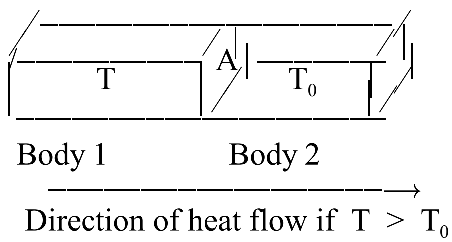
where $\nabla\cdot T$ is the gradient of the scalar field of temperatures $T = T(x, y, z, t) = T(\vec{x}, t)$, $\hat{u}_n = n_x\hat{i} + n_y\hat{j} + n_z\hat{k}$ is the unit normal to the area A which points in the positive direction and has unit length, and k is the proportionality constant. For a long rod (or bar) of length ℓ that is insulated on its sides, we assume that $T = T(x,t)$ (i.e. that the temperature does not vary over the cross section and hence heat flows only in the direction of $-\partial T/\partial x$.



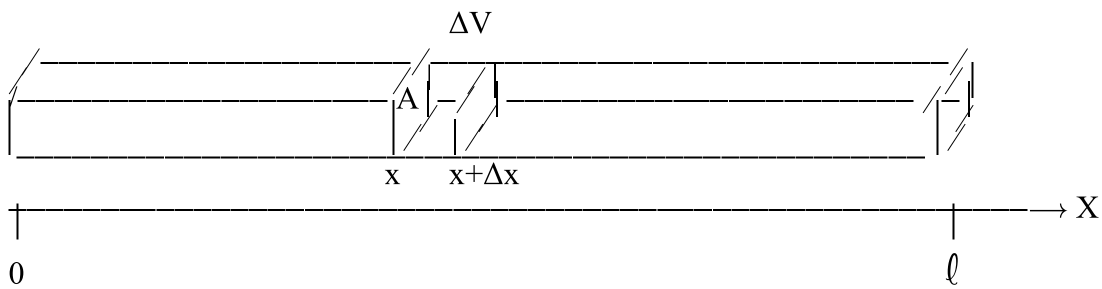
Before proceeding, we review Newton's Law of Cooling as developed for first order ODEs: When two bodies are in contact with one kept at a constant temperature T_0, then the rate at which the temperature of the other body changes is proportional to the difference in temperature between the two bodies. That is,

$$\frac{dT}{dt} = -k\,(T - T_0). \tag{1}$$

The minus sign insists that heat flow from hot to cold. The difference in temperatures $(T - T_0)$ corresponds to the directional derivative in the microscopic (continuous) model.



We might refer to Eq. (1) as the macroscopic (or discrete) form of Newton's Law of Cooling since it gives only macroscopic effects. Realistically, we cannot sustain a jump discontinuity between two bodies in contact.



We apply the Law of Conservation of energy to the volume ΔV by calculating the rate at which the heat is flowing out of the volume ΔV in two ways. We first compute the heat flowing across the boundary. To do this we apply Fourier's Law (Newton's "Microscopic or Continuous" Law of Cooling) to the two cross sectional boundaries, the one at x and the one at $x+\Delta x$. Let

$T(x,t)$ = the temperature distribution over the rod at time t and
$H_{\Delta V}$ = Quantity of heat energy in the volume ΔV.

Then Fourier's Law implies

Rate at which Heat energy crosses any cross section A (to the right) $= -K A \dfrac{\partial T}{\partial x}$

where $\partial T/\partial x$ is evaluated at that cross section and

K = thermal conductivity $\left(\dfrac{\text{calories}}{\text{cm deg sec}}\right)$ = proportionality constant.

For any volume in space, the rate at which heat energy crosses the boundary of the volume is given by the surface integral (i.e. we sum up the heat flow through all of the boundaries):

Rate at which heat energy crosses the boundary $= -K \iint_S \nabla \cdot \hat{u}_n dA$.

Since the sides are insulated, there is no heat flow out of the sides. Since the heat flow <u>into</u> ΔV is $\frac{\partial H_{\Delta V}}{\partial t}$, the heat flow <u>out of</u> ΔV is given by

$$-\frac{\partial H_{\Delta V}}{\partial t} = (\text{ Heat flow out at } x+\Delta x) - (\text{ Heat flow in at } x)$$

$$= (-K\,A\,T_x(x+\Delta x,t)) - (-K\,A\,T_x(x,t)) \qquad (T_x(x,t) = \partial T/\partial x)$$

$$= -K\,A\,(T_x(x+\Delta x,t) - T_x(x,t)) \tag{2}$$

On the other hand, we can also calculate the heat flow out of the volume ∇V as follows:

$H_{\Delta V}$ = Quantity of heat energy in ΔV = Sum of the energy in each dV in ΔV

$$= \iiint_{\Delta V} s\rho T(x,t)dV \qquad \int_x^{x+\Delta x} T(x,t)dx\,)\,A \qquad \approx s\,\rho\,A\,T(x,t)\,\Delta x$$

where s = specific heat $\left(\frac{\text{calories}}{\text{gm deg}}\right)$

ρ = mass density $\left(\frac{\text{gm}}{\text{cm}^3}\right)$

Units check (dimensional analysis)

$$s\,\rho\,A\,T\,\Delta x \sim \left(\frac{\text{calories}}{\text{gm deg}}\quad \frac{\text{gm}}{\text{cm}^3}\quad ()\right)(\text{cm}^2)(\text{deg})(\text{cm}) = (\text{calories})$$

Since the heat flow <u>into</u> ΔV is $\frac{\partial H_{\Delta V}}{\partial t}$, the heat flow <u>out of</u> ΔV is

$$-\frac{\partial H_{\Delta V}}{\partial t} \approx -\,s\,\rho\,A\,T_t\,\Delta x \tag{3}$$

where $T_t = \partial T/\partial t$. Equating our two expressions for $-\partial H/\partial t$, Equations (2) and (3), we obtain

$$-K\,A\,(T_x(x,t) - T_x(x,t)) \approx -\,s\,\rho\,A\,T_t\,\Delta x. \tag{4}$$

Solving for T_t we obtain

$$T_t = \frac{K}{s\rho} \frac{T_x(x+\Delta x, t) - T_x(x, t)}{\Delta x}$$

Letting $\Delta x \longrightarrow 0$ we obtain

$$T_t = \alpha^2 T_{xx} \qquad \text{where } \alpha^2 = K/(s\rho) > 0. \qquad (6).$$

Note that α^2 depends only on the properties of the material. This derivation can be done more generally in three space dimensions by using Gauss's divergence theorem. Since we will use T for a different function in the process of separation of variables, we rewrite (6) using the standard notation with temperature being u as

$$u_t = \alpha^2 u_{xx}. \qquad (7)$$

PDE's — Handout #3

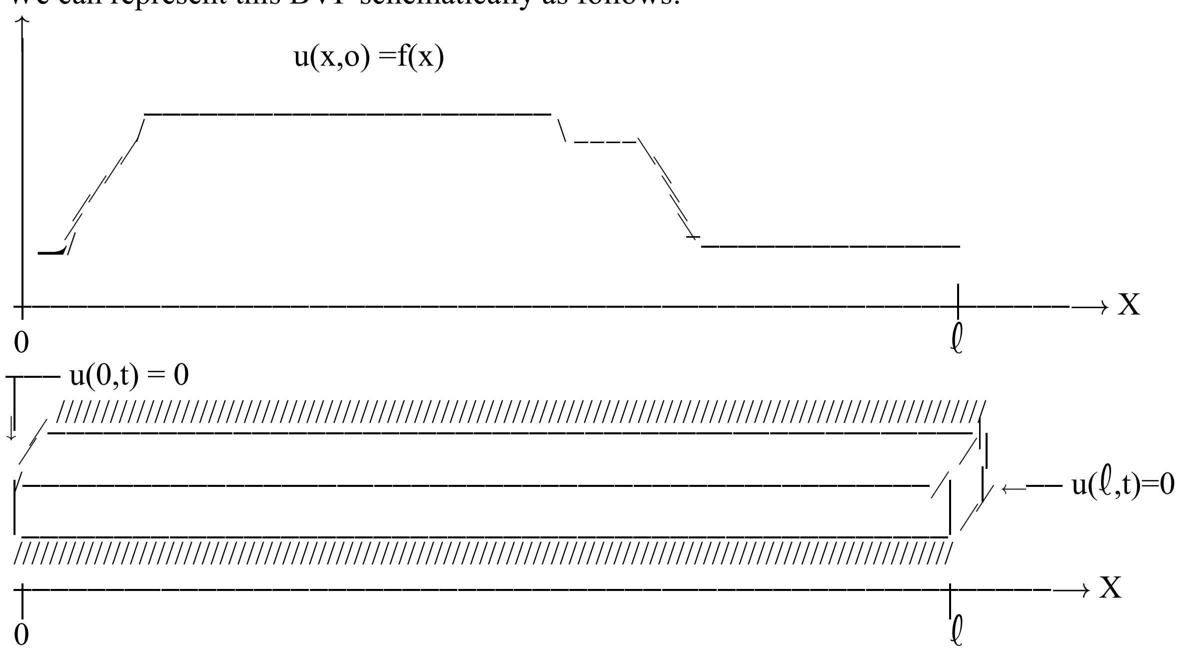
THE BOUNDARY VALUE PROBLEM FOR THE HEAT CONDUCTION MODEL

Prof. Moseley

The heat equation can be written in the standard form L[u] = 0, where $L[u] = u_t - \alpha^2 u_{xx}$. The null space of the linear operator L is infinite dimensional. To obtain a model for heat conduction in a rod of length ℓ which is insulated on the sides (i.e. whose solution uniquely determines the value of the temperature at each point in the rod for all nonnegative time), we must add boundary conditions at the ends of the rod and an initial temperature distribution at t = 0 over the length of the rod to the PDE $u_t = \alpha^2 u_{xx}$. Recall that the PDE forces conservation of energy internally in the rod. The assumption that the sides are insulated implies that the problem is one dimensional and forces conservation of energy on the sides. To assure conservation of energy, we must also specify conditions at both ends of the rod. Several possibilities arise. However, for mathematical convenience we begin with homogeneous Direchet boundary conditions. That is, we require the temperature at both ends to be zero. Later, we will see how to handle arbitrary temperatures (nonhomogeneous conditions) and insulated ends (Neuman conditions).

	PDE	$u_t = \alpha^2 u_{xx}$	$0 < x < \ell$, $t > 0$
BVP	BC	$u(0, t) = 0$, $u(\ell,t) = 0$	$t > 0$
		(homogeneous Dirichet boundary conditions)	
	IC	$u(x,0) = u_0(x)$	$0 < x < \ell$
		(initial temperature distribution)	

We can represent this BVP schematically as follows:



To formulate the heat conduction in a rod problem as a linear mapping problem we let D $= (-\ell,\ell)\times(0,\infty)$, $\bar{D} = [-\ell,\ell]\times[0,\infty)$, $A(\bar{D},\mathbf{R};\ell) = \{u(x,\bar{D}\ F(\ ,\mathbf{R})$: $u\in A(D\bar{D})\cap C(\ ,\mathbf{R})\}$ and $L_{HC}(\alpha^2,\ell):A_{HC}(\bar{D},\mathbf{R};\ell) \to A(D,\mathbf{R})$ be defined by $L_{HC}(\alpha^2,\ell)[u] = u_t - \alpha^2 u_{xx}$. Now let $A_{HC}(\bar{D},\mathbf{R};\ell) = \{u(x,t)\in A_{HC}(\bar{D},\mathbf{R}))$: $u(0,t) = 0$ and $u(\ell,t) = 0$ for $t > 0\}$, $L_{HC,0}(\alpha^2,\ell):A_{HC}(\bar{D},\mathbf{R}) \to A(D,\mathbf{R})$ be defined by $L_{HC,0}(\alpha^2,\ell)[u] = u_t - \alpha^2 u_{xx}$, Thus we incorporate the boundary conditions into the domain of the operator. Often, instead of specifying a notation for these cases, we let $L[u] = u_t - \alpha^2 u_{xx}$ and describe the domain we intend. However, it is important to recall that the domain is part of the definition of an operator. Hence you should have a clear understanding of the domain of interest in any particular discussion. Now let $N_{L_{HC,0}(\alpha^2,\ell)}$ be the null space of $L_{HC,0}(\alpha^2,\ell)$ and $A_{HC,0,o,fs}(\bar{D},\mathbf{R};\alpha^2,\ell) = \{u(x,t\ N_{L_{HC,0}(\alpha^2,\ell)}\ PC^1_{fs,0}([0,\ell],\mathbf{R};\ell)\ \bar{D}\}$ and $A_{HC,0,o,fs}(\ ,\mathbf{R};\alpha^2,\ell$ $\{u(x,t)\in N_{L_{HC,0}(\alpha^2,\ell)}$: $PC^1_{fs,0}([0,\ell],\mathbf{R};\ell)\ \bar{D}\}$. Hence $A_{HC,0,o,}\bar{D}\ N_{L_{HC,0}(\alpha^2,\ell)} \subseteq A_{HC,o,fs}(\ ,\mathbf{R}) \subseteq A_{HC,0}(\bar{D},\mathbf{R}) \subseteq A_{L}\bar{D},\mathbf{R};\ell)$. We would like a "basis $N_{L_{HC,0}(\alpha^2,\ell)}$. We will obtain a linearly independent set that is a **Hamel basis** for $A_{HC,0,0,ffs}(\bar{D},\mathbf{R};\alpha^2,\ell)$ and a **Schauder basis** for $A_{HC,o,fs}(\bar{D},\mathbf{R})$.

Recall that to solve the homogeneous linear system with constant coefficients given by

$$\bar{x}' - A\bar{x} = 0 \qquad (1)$$

we assumed (i.e. guessed) that there was a solution of the form $\bar{x} \quad \bar{\xi} \quad e^{rt}$ where r is a constant and $\bar{\xi}$ is a constant (i.e. independent of t, but the components can be different) vector to be determined later. This yielded an eigenvalue problem whose solution yielded a basis for the null space of the operator L[] = $\bar{x}' \quad - \bar{x}$. To solve the linear PDE ("with constant linear coefficients") given by

$$u_t = \alpha^2 u_{xx} \qquad (2)$$

and the BCs $u(0, t) = 0$, $u(\ell,t) = 0$ $t > 0$ (homogeneous Dirichet boundary conditions) we could assume (i.e. guess and hope that there is) a solution of the form $u(x,t) = X(x)\, e^{rt}$. Our hopes would be fulfilled. However, we need not assume the form of the time function. This need not be guessed, but can be determined by the process. Since this makes our method more general, we assume (i.e. need only guess and hope that there is) a solution of the form

$$u(x,t) = X(x)\, T(t). \qquad (3)$$

and substitute into the Equation (2) using

$$u_t = \frac{\partial(XT)}{\partial t} \qquad = X(x)\, T'(t) \quad \text{where } T'(t) = \frac{dT}{dt}$$

$$u_{xx} = \frac{\partial^2(XT)}{\partial x^2} \qquad = X''(x)\, T(t) \quad \text{where } X''(x) = \frac{d^2T}{dx^2}$$

Omitting the function notation we obtain

$$X\,T' = \alpha^2 X''\, T. \qquad (4)$$

If we had assumed that $u(x,t) = X(x)\, e^{rt}$, then since e^{rt} would have appeared on both sides, we would have obtained $r X = \alpha^2 X''$, a second order linear homogeneous ODE with constant coefficients. Instead, we use separation of the variables. Dividing both sides by $\alpha^2 X T$, we obtain

$$\frac{XT'}{\alpha^2 XT} \qquad \frac{\alpha^2 X''T}{\alpha^2 XT}$$

or

$$\frac{T'}{\alpha^2 T} \qquad \frac{X''}{X} \qquad = -\lambda = \text{constant} \tag{5}$$

Note that we have some choices. Since α^2 is a constant, we could put it on either side of the equation. It is not a priori evident that this choice will make the notation slightly easier later.

If we had assumed $u(x,t) = X(x)\, e^{rt}$, then the e^{rt} factor would "cancel" and we would have obtained $r/\alpha^2 = X''/X$ = constant. The standard argument for the more general technique is as follows: Since the left side of Equation (5) ($T'/(\alpha^2 T)$) is a function of t only and the other side

(X''/X) is a function of x only, the only way that they can be equal (i.e. the only way to obtain solutions of the form u = XT) is for both sides to be constant. Whether you believe the argument justifies setting both sides equal to the same constant is in some sense unimportant, if we can come up with a solution of any form by any method, we can check that it is a solution by substituting it back into the equation. We make no uniqueness claim, either that all solutions are of the form obtained, or that we have obtained all solutions of this form. The choice of $-\lambda$ (instead of $+\lambda$) will make the notation easier later, but again this is not a priori apparent. The various choices are not substantively different, and any is "correct".

Equation (5) yields the two ODEs

$$X'' + \lambda X = 0 \qquad \text{(space equation)} \tag{6}$$

$$T' + \lambda\alpha^2 T = 0 \qquad \text{(time equation)} \tag{7}$$

The lack of minus signs in these two equations explains our choice of $-\lambda$. The next step is to derive an ODE eigenvalue problem using the homogeneous boundary conditions and the space equation. Not having to deal with α^2 in this process explains why we put it with the time equation. The EVP results from addressing the linear operator consisting of $L[u] = u_t - \alpha^2 u_{xx}$ and the BCs $u(0, t) = 0$, $u(\ell,t) = 0$, $t > 0$ (homogeneous Dirichet boundary conditions). The dimension of the null space of the linear operator (even with the restrictions imposed by the boundary conditions) is infinite.

Handout #5 | DERIVATION AND SOLUTION OF THE ODE EIGENVALUE PROBLEM | Prof. Moseley

Recall that by assuming a solution of the form u(x,t) = X(x) T(t) and applying the process of separation of variables to the heat conduction equation (PDE) we obtained the two ODEs

$$X'' + \lambda X = 0 \qquad \text{(space equation)} \qquad (1)$$

$$T' + \lambda\alpha^2 T = 0 \qquad \text{(time equation)} \qquad (2)$$

Also recall that the BCs for the BVP for the model of heat conduction in a rod with homogeneous Dirichet BCs (both ends of the rod are at zero degrees) are given by
BC $\quad u(0, t) = 0\ ,\ \ u(\ell,t) = 0 \qquad t > 0$ (homogeneous Dirichet boundary conditions)

Applying these two conditions to u(x,t) = X(x) T(t), we obtain

$$u(0,t) = X(0)\, T(t) = 0,\ u(\ell,t) = X(\ell)\, T(t) = 0 \qquad \text{for all } t > 0 \qquad (3)$$

For Equation (3) to hold for all $t > 0$, we must have either $T(t) = 0$ for all $t>0$, which means $u(x,t) = X(x)\, T(t) = 0$ for all $t > 0$ (u=0 is the trivial solution which satisfies the BC and the initial condition $u(x,0) = 0$) or $X(0) = 0$ and $X(\ell) = 0$ for all $t > 0$ (a more likely prospect to obtain a collection of solutions. Putting the space ODE with these BCs, we obtain the ODE BVP

$$X'' + \lambda X = 0 \qquad (4)$$

$$X(0) = 0, \qquad X(\ell) = 0 \qquad \forall\, t > 0 \qquad (5)$$

We solve this ODE EVP using the standard procedure. As previously indicated, this will yield a basis of the null space of the linear operator defined by $L[u] = u_t - \alpha^2 u_{xx}$ and the BCs $u(0, t) = 0$, $u(\ell,t) = 0$, for all $t > 0$ (homogeneous Dirichet boundary conditions). Recall that the dimension of this null space is infinite even with the restrictions imposed by the BCs.

<u>SOLUTION TO THE ODE EVP.</u> Although it is not readily apparent, it is well known that the operator consisting of $L[X] = X'' + \lambda X$ and the BCs $X(0) = 0$, $X(\ell) = 0$ for all $t > 0$ is self-adjoint. (This is analogous to an operator defined by matrix multiplication being Hermitian.) Thus we know that the eigen values are real. (We also refer to the problem as self-adjoint.) Substituting $X = e^{rt}$ into the ODE we obtain the auxiliary equation:

$$r^2 + \lambda = 0. \qquad \text{Auxiliary Equation}$$

Hence $r_{1,2} = \pm\sqrt{-\lambda}$. We consider three cases:

<u>Case 1.</u> $\lambda < 0$. We have $\lambda < 0$ implies $-\lambda > 0$ so that the roots $r_1 = \sqrt{-\lambda}$ and

$r_2 = -\ \sqrt{-\lambda}$ are real and distinct. Hence the general solution of the ODE is given by

$$X(x) = c_1\ e^{\sqrt{-\lambda}\,x} \qquad e^{-\sqrt{-\lambda}\,x}$$

Applying the BCs we obtain

$$X(0) = 0 = c_1\ e^{\sqrt{-\lambda}\,(0)} \qquad e^{-\sqrt{-\lambda}\,(0)}$$
$$X(\ell) = 0 = c_1\ e^{\sqrt{-\lambda}\,(\ell)} \qquad e^{-\sqrt{-\lambda}\,(\ell)}$$

or

$0 = c_1 + c_2$ which implies $c_2 = -\ c_1$

$0 = c_1\ e^{\sqrt{-\lambda}\,(\ell)} \qquad e^{-\sqrt{-\lambda}\,(\ell)} \qquad e^{\sqrt{-\lambda}\,(\ell)}\ \ 0 = e^{-\sqrt{-\lambda}\,(\ell)} \qquad -\ c_1$

or

$$c_1\ (\ e^{\sqrt{-\lambda}\,(\ell)} \qquad e^{-\sqrt{-\lambda}\,(\ell)} \qquad = 0.$$

Hence either $c_1 = 0$ or $e^{\sqrt{-\lambda}\,(\ell)} \qquad e^{-\sqrt{-\lambda}\,(\ell)} \qquad = 0$. There is something wrong with both of these cases. $c_1 = 0$ implies $c_2 = -c_1 = 0$ as well and hence $X(x) = 0\ \forall\ x \in [0, \ell]$. This is the trivial solution and never part of any basis. We can handle $e^{\sqrt{-\lambda}\,(\ell)} \qquad e^{-\sqrt{-\lambda}\,(\ell)} \qquad = 0$ two ways: (1) Directly using the properties of the exponential function e^x, or (2) dividing the equation by 2 and recalling that $(\ e^{\sqrt{-\lambda}\,(\ell)} \qquad e^{-\sqrt{-\lambda}\,(\ell)} \qquad \sqrt{-\lambda}\,/2 = \sinh(\qquad \ell\ )$. We use method (2) a leave (1) as an exercise. Using method (2) we obtain $\sinh(\ \sqrt{-\lambda} \qquad \ell\ ) = 0$. But the only zero of $\sinh(x)$ is zero (recall the graph of $\sinh(x)$). Hence we obtain $\sqrt{-\lambda} \qquad \ell = 0$. Hence (since $\ell \neq 0$), $\sqrt{-\lambda} \qquad = 0$, so that $\lambda = 0$. But we assumed in this case that $\lambda \neq 0$. Hence it can not be the case that $e^{\sqrt{-\lambda}\,(\ell)} \qquad e^{-\sqrt{-\lambda}\,(\ell)} \qquad = 0$ since this implies that $\lambda = 0$ and this is not allowe in this case. The result is that there are no eigenvalues for $\lambda < 0$ (i.e. no negative eigenvalues)

<u>Case 2.</u> $\lambda = 0$ so that the roots $r_{1,2} = 0, 0.$ are real and repeated. Hence the general solution of the ODE is given by

$$X(x) = c_1 + c_2\,x$$

Applying the BCs we obtain

$X(0) = 0 = c_1 + c_2\,(0)$ which implies $c_1 = 0$

$X(\ell) = 0 = c_1 + c_2\,(\ell)$ so that $c_2 = c_1/\ell = 0$

so that

$X(x) = 0.$ for all $x \in [0,\ell\ ]$

Hence $\lambda = 0$ is not an eigenvalue.

Case 3. $\lambda > 0$ so that the roots $r_{1,2} = \pm \sqrt{-\lambda}$ $\sqrt{-1}\ \sqrt{\lambda}$ $\sqrt{\lambda}$ $= \pm i$ are not real (rec that λ is assumed to be real since the problem is self-adjoint) and distinct. Hence the general solution of the ODE is given by

$$X(x) = c_1 \cos(\sqrt{\lambda}\ x) + c_2 \sin(\sqrt{\lambda}\ x)$$

Applying the BCs we obtain

$$X(0) = 0 = c_1 \cos(\sqrt{\lambda}\ 0) + c_2 \sin(\sqrt{\lambda}\ 0)$$

$$X(\ell) = 0 = c_1 \cos(\sqrt{\lambda}\ \ell) + c_2 \sin(\sqrt{\lambda}\ \ell)$$

or

$$0 = c_1 \cos(0) + c_2 \sin(0)$$

$$0 = c_1 \cos(\sqrt{\lambda}\ \ell) + c_2 \sin(\sqrt{\lambda}\ \ell)$$

or

$$0 = c_1 (1) + c_2\ (0) \qquad \text{which implies} \quad c_1 = 0$$

$$0 = c_1 \cos(\sqrt{\lambda}\ \ell) + c_2 \sin(\sqrt{\lambda}\ \ell) \quad \text{so that} \quad c_2 \sin(\sqrt{\lambda}\ \ell) = 0$$

Hence either $c_2 = 0$ or $\sin(\sqrt{\lambda}\ \ell) = 0$. $c_1 = 0$ and $c_2 = 0$ implies $X(x) = 0$ for all $x \in [0,\ell]$. This is the trivial solution and never part of any basis. Thus to obtain any eigenvalues, we must look for solutions of the "auxiliary equation"

$$\sin(\sqrt{\lambda}\ \ell) = 0 \tag{6}$$

The zeros of sin(x) are $n\pi$ where $n \in \mathbf{Z}$. Hence

$$\sqrt{\lambda}\ \ell = n\pi,\ n \in \mathbf{Z} \tag{7}$$

Hence, let

$$\lambda_n = \frac{n^2\pi^2}{\ell^2} \qquad n \in \mathbf{Z} \tag{8}$$

However, there is a problem with $n = 0$. If $n = 0$, then $\lambda = 0$. But we assumed $\lambda > 0$ for this case. Hence we must exclude $n = 0$.

$$\lambda_n = \frac{n^2\pi^2}{\ell^2} \qquad n = \pm 1, 2, 3, \ldots . \tag{9}$$

Note that λ_n is the same whether n is positive or negative. Before excluding negative n, we need to see what the eigenfunctions ("vectors") are associated with n positive and n negative. The dimension of the null space of the linear operator $L[X] = X'' + \lambda X$ is two independent of the value of λ. Hence the dimension of this operator L with the boundary conditions $X(0) = 0$ and $X(\ell) = 0$ must be either zero, one or two. Referring back to the solution of the ODE and the application of the BCs we see that $X(x) = c_1 \cos(\sqrt{\lambda}\ x) + c_2 \sin(\sqrt{\lambda}\ x)$ with $c_1 = 0$. Thus the eigenfunctions ("vectors") which are associated with these eigenvalues are

$$X(x) = c_2 \sin(\sqrt{\lambda}\quad x) = c_2 \sin(n\pi/\ell) \qquad n = \pm 1, 2, 3, \ldots .$$

The eigen spaces are clearly one dimensional. Each eigen space is just the scalar multiples of a given function. Hence a basis for each is just

$$X_n(x) = \sin(\frac{n\pi}{\ell}\quad x). \qquad n = \pm 1, 2, 3, \ldots .$$

However there is a problem. Since $\sin(-x) = -\sin(x)$ we see that the family of scalar multiplies of $\sin(n\pi/\ell)$ for n = 1, 2, 3, ... and that of the scalar multiples of $\sin(-n\pi/\ell)$ for n = 1, 2, 3, ... are exactly the same. Hence n negative generates no additional eigen functions. Summarizing,

TABLE

Eigenvalues		Eigenfunctions
$\lambda_1 = \frac{\pi^2}{\ell^2}$		$X_1 = \sin(\frac{\pi}{\ell}\quad x)$
$\lambda_2 = \frac{4\pi^2}{\ell^2}$	$4\pi^2/\ell^2$	$X_2 = \sin(\frac{2\pi}{\ell}\quad \text{—}\; x)$
$\lambda_3 = \frac{9\pi^2}{\ell^2}$	$9\pi^2/\ell^2$	$X_3 = \sin(\frac{3\pi}{\ell}\quad x)$
.		.
.		.
.		.
$\lambda_n = \frac{n^2\pi^2}{\ell^2}$		$X_n = \sin(\frac{n\pi}{\ell}\quad x)$
.		.
.		.
.		.

When solving an eigen value problem for an ODE always summarize in a table as illustrated above giving several specific values and then the general case.

Handout #5 GENERAL SOLUTION OF THE PDE AND BOUNDARY CONDITIONS Prof. Moseley

We review the results of the process thus far. Applying the method of separation of variables to the PDE

PDE $\quad u_t = \alpha^2 u_{xx}$ (1)

yielded the two ODEs

$X'' + \lambda X = 0$ (space equation) (2)

$T' + \lambda\alpha^2 T = 0$ (time equation) (3)

Application of the BCs $u(0, t) = 0$, $u(\ell,t) = 0$ $t > 0$ (homogeneous Dirichet boundary conditions)
yielded the ODE EVP

$X'' + \lambda X = 0$ (4)

$X(0) = 0, \qquad X(\ell) = 0$ (5)

Solution of this ODE EVP yielded the eigenvalues and eigenfunctions:

TABLE

Eigenvalues		Eigenfunctions
$\lambda_1 = \frac{\pi^2}{\ell^2}$		$X_1 = \sin(\frac{\pi}{\ell} x)$
$\lambda_2 = \frac{4\pi^2}{\ell^2}$	$4\pi^2/\ell^2$	$X_2 = \sin(\frac{2\pi}{\ell} x)$
$\lambda_3 = \frac{9\pi^2}{\ell^2}$	$9\pi^2/\ell^2$	$X_3 = \sin(\frac{3\pi}{\ell} x)$
.		.
.		.
.		.
$\lambda_n = \frac{n^2\pi^2}{\ell^2}$		$X_n = \sin(\frac{n\pi}{\ell} x)$
.		.
.		.
.		.

Solving Equation (3) we obtain the auxiliary equation $r + \lambda\alpha^2 = 0$ so that using

$\lambda = \lambda_n = (n\pi/\ell)^2$ we obtain

$$T(t) = c\, e^{-\lambda\alpha^2 t} \qquad c_n e^{-(n\pi\alpha/\ell)^2 t} =$$

Letting

$$T_n(t) = e^{-(n\pi\alpha/\ell)^2 t}$$

as solutions to the PDE and BCs

PDE $\qquad u_t = \alpha^2 u_{xx}$ (6)

BC $\qquad u(0, t) = 0, \quad u(\ell,t) = 0 \quad t > 0$ (homogeneous Dirichet boundary conditions) (7)

we obtain

$$u_n(x,t) = T_n(t)\, X_n(x) = e^{-(n\pi\alpha/\ell)^2 t} \quad \text{in}(\frac{n\pi}{\ell} x) \tag{8}$$

By superposition (for the homogeneous PDE with homogeneous BC), a (finite) linear combination of (linearly independent) solutions is a solution. Hence we obtain that the family of solutions given by

$$u(x,t) = \sum_{n=1}^{N} c_n e^{-(n\pi\alpha/\ell)^2 t} \sin(\frac{n\pi}{\ell} x)$$

where the c_n's are arbitrary constants. It is interesting to note that as $t \to \infty$, the temperature approaches the trivial, $u(x,t) \to 0$. Thus $u_{ss}(x,t) = 0$ is the steady-state or equilibrium solution.

Recall that $B_{PC^1_{fs,0}([0,\ell],\mathbf{R};\ell)} = \{\sin(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ $PC^1_{fs,0}([0,\ell],\mathbf{R};\ell)$ nel basis of ar

$PC^1_{fs,0}([0,\ell],\mathbf{R};\ell)$. A $N_{L_{HC,0}(\alpha^2,\ell)}$ that is the null space of $\bar{D}_{HC,0}(\alpha^2,\ell)$ and $A_{HC,0,o,fs}($

$\{u(x,t) \in N_{L_{HC,0}(\alpha^2,\ell)}$: $PC^1_{fs,0}([0,\ell],\mathbf{R};\ell)$ $\bar{D}$ } and $A_{HC,0,o,}$ $N_{L_{HC,0}(\alpha^2,\ell)}$ $\mathbf{R};\alpha^2,\ell) = \{u(x,t) \in$

$PC^1_{fs,0}([0,\ell],\mathbf{R};\ell)$ } so $\bar{D}$ at $A_{HC,0,o,ffs}($ $,\mathbf{R}\bar{D}\ \alpha^2,\ell)$ $N_{L_{HC,0}(\alpha^2,\ell)}($ $,\mathbf{R})\bar{D}$ $\subseteq.$ $A_{HC,0}($ $,\mathbf{R})$.

that $B_{PC^1_{fs,0}([0,\ell],\mathbf{R};\ell)} = \{e^{-(\frac{\alpha k\pi}{\ell})^2} \sin(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ $\bar{D}$ a Hamel basis for $A_{HC,0,o,ffs}($ $,\mathbf{R};\alpha^2,\ell)$ ar

basis for $A_{HC,o,fs}(\bar{D}$ $,\mathbf{R})$. Hence we can solve the initial value problem (IVP) if the initial condition (IC) $u_0(x)$ is in $PC^1_{fs,0}([0,\ell],\mathbf{R};\ell)$ $PC^1_{ffs,0}([0,\ell],\mathbf{R};\ell)$ the IVP is particula

EXAMPLE. Solve

	PDE	$u_t = u_{xx}$	$0 < x < \pi, \quad t > 0$
BVP	BC	$u(0, t) = 0, \quad u(\pi,t) = 0$	$t > 0$
	IC	$u(x,0) = 5\sin(x) + 3\sin(3x) + 2\sin(6x)$	$0 < x < \pi$

Solution. First note that $\alpha^2 = 1$ and $\ell = \pi$. Hence the family of solutions given by (9) is

$$u(x,t) = \sum_{n=1}^{N} c_n e^{-n^2 t} \sin(nx)$$

Thus at t = 0 we obtain

$$u(x,0) = \sum_{n=1}^{N} c_n \sin(nx) \qquad = c_1 \sin(x) + c_2 \sin(2x) + c_3 \sin(3x) + \cdots + c_N \sin(Nx) \tag{11}$$

Matching coefficients, we can satisfy the initial condition

IC $\quad u(x,0) = 5\sin(x) + 3\sin(3x) + 2\sin(6x) \qquad 0 < x < \pi$

if we let $c_1 = 5$, $c_2 = 0$, $c_3 = 3$, $c_3 = c_4 = c_5 = 0$, $c_6 = 2$, and $c_n = 0$ for n = 7, 8, 9, ... Hence the solution is

$$u(x,t) = 5e^{-t}\sin(x) + 3e^{-9t}\sin(3x) + 2e^{-36t}\sin(6x) \tag{12}$$

Note that since the exponent has n^2 as a factor, the higher order terms decay much faster than the lower order terms.

The IC for the above problem was nice. To handle more complicated IC, we consider letting

N go to infinity. If there are no convergence problems, we obtain

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi\alpha/\ell)^2 t} \sin\left(\frac{n\pi}{\ell}x\right)$$

as a family of solutions. That is, a "linear combination" of an infinite number of linearly independent solutions is in the null space of the linear operator defined by the PDE and the BCs and in some sense gives the "general" solution to the problem defined by the PDE and the BCs. Clearly the dimension of this null space is infinite.. We assume no convergence problems and blindly try to use (13) to satisfy the IC in the BVP

	PDE	$u_t = \alpha^2 u_{xx}$	$0 < x < \ell$, $t > 0$
BVP	BC	$u(0, t) = 0$, $u(\ell,t) = 0$	$t > 0$
	IC	$u(x,0) = f(x)$	$0 < x < \ell$

Letting t= 0 in (13) we obtain

$$u(x,0) = \sum_{n=1}^{\infty} c_n e^{-(n\pi/\ell)^2(0)} \sin(\frac{n\pi}{\ell}x)$$

$$= \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{\ell}x)$$

Hence we require

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi}{\ell}x) \quad \forall\, x \in [0,\ell] \tag{14}$$

Expressing an arbitrary function as the infinite sum of sine (and cosine) functions requires Fourier series.