`A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 4

A COLLECTION OF HANDOUTS ON
PARTIAL DIFFERENTIAL EQUATIONS (PDE's)

CHAPTER 3

Fourier Series

1. Periodic Functions
2. Facts about the Sine and Cosine Function
3. Fourier Theorem and Computation of Fourier Series Coefficients
4. Even and Odd Functions
5. Sine and Cosine Series

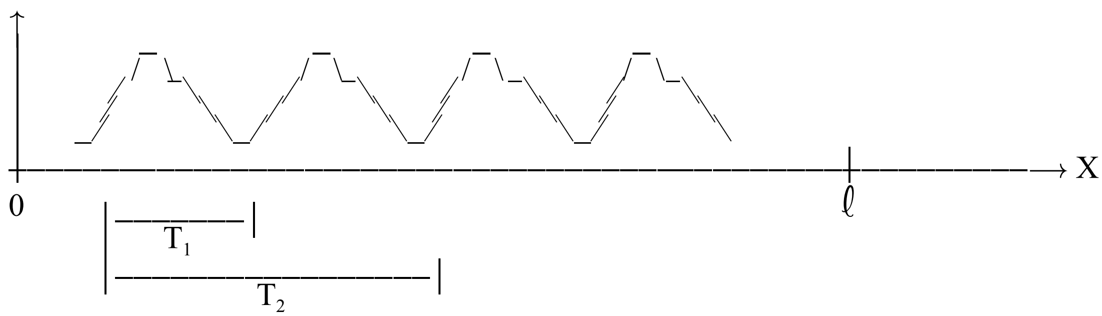
Handout #1 PERIODIC FUNCTIONS Professor Moseley

We consider the general problem of trying to represent a periodic function by an infinite series of sine and cosine functions. One application is the heat conduction model. Another is the response of an electric circuit to a square wave. Nonhomogeneous superposition can be used to obtain the spectrum of frequencies in the forcing function.

DEFINITION. A function f:**R** → **R** is **periodic of period T** if ∀ x ∈ **R** we have f(x + T) = f(x). The smallest positive period is called the **fundamental period**.

We can check this property either graphically or analytically (or both).

EXAMPLE.



THEOREM. If f:**R** → **R** is is a periodic function with period T, then 2T, 3T, ... and – T, – 2T, ... are also periods.

(Idea of) PROOF. (A proof would require **mathematical induction**.)
Let us assume that f is periodic of period T. We prove that 2T is a period.

STATEMENT	REASON
f(x + 2T) = f((x + T) + T)	(Theorems from) Algebra
= f(x + T)	Hypothesis (assumption) that f is periodic of period T
= f(x)	Hypothesis that f is periodic of period T

Since f(x + 2T) = f(x), 2T is a period. (The general case requires **mathematical induction**.)

THEOREM. If f:**R** → **R** is is a periodic function with period T, then there need not be a fundamental period.

PROOF. We give a **counter example** to the statement "Every periodic function has a fundamental period". Let f(x) = 4 (or any constant function). Then f(x+T) = f(x) = 4 ∀ x, T ∈

ℝ. Since all real numbers are periods, there is no smallest positive period and hence no fundamental period.

DEFINITION. If $f:(-\ell,\ell\,]\to\mathbb{R}$, then we define the **periodic extension** , f_{pe} , of f by $f_{pe}(x+2n\ell) = f(x) \quad \forall\, x \in (-\ell,\ell\,]$ and $\forall\, n \in \mathbf{Z}$.

DEFINITION. Let $PC^1_{fs}(\mathbf{R},\mathbf{R};\ell)$ = {f∈F(**R**,**R**):1) f is periodic of period 2ℓ, 2) f and f' are piecewise continuous on [−ℓ,ℓ], and 3) f(x) =(f(x+) + f(x−)/2 at points of discontinuity},
$PC^1_{fs}([-\ell,\ell],\mathbf{R};\ell)$ be $PC^1_{fs}(\mathbf{R},\mathbf{R};\ell)$ ns in with their domains restricted to [−ℓ,ℓ]
$PC^1_{ffs}([-\ell,\ell],\mathbf{R};\ell)$ be th $PC^1_{fs}([-\ell,\ell],\mathbf{R};\ell)$ for which the fourier series is finite.

EXERCISES on Periodic Functions

EXERCISE. Sketch the periodic extensions of:

1. $f(x) = x \qquad x \in (-2, 2]$
2. $f(x) = x^2 \qquad x \in (-1, 1]$

Handout #2 FACTS ABOUT THE SINE AND COSINE FUNCTIONS Professor Moseley

THEOREM #1. (Facts about Trigonometric Functions)

1. $\sin(\frac{n\pi}{\ell}$ x) and $\frac{n\pi}{\ell}$ x) where $n \in \mathbf{Z}$.

are periodic with fundamental period $T = \frac{2\ell}{n}$

All are periodic of (common) period 2ℓ.

2. $$\int_{-\ell}^{\ell} \cos(\frac{m\pi}{\ell}x)\cos(\frac{n\pi}{\ell}x)dx = \begin{cases} 0 & \text{for all } m,n \in \mathrm{Z} \quad \text{if } m \neq n \\ \ell & \text{for all } m,n \in \mathrm{Z} \quad \text{if } m = n \end{cases}$$

3. $$\int_{-\ell}^{\ell} \sin(\frac{m\pi}{\ell}x)\cos(\frac{n\pi}{\ell}x)dx = 0 \quad \text{for all } m,n \in \mathrm{Z}$$

4. $$\int_{-\ell}^{\ell} \sin(\frac{m\pi}{\ell}x)\sin(\frac{n\pi}{\ell}x)dx = \begin{cases} 0 & \text{for all } m,n \in \mathrm{Z} \quad \text{if } m \neq n \\ \ell & \text{for all } m,n \in \mathrm{Z} \quad \text{if } m = n \end{cases}$$

These properties say that these functions form an orthogonal (perpendicular) set of nonzero functions. A theorem from linear algebra says that such a set is linearly independent. Our Fourier Theorem will tell us the subspace "spanned" by this set. We will see that $B_{PC^1_{ms}([-\ell,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in \mathbf{N}\} \cup \{\sin(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ $PC^1_{ms}([-\ell,\ell],\mathbf{R};\ell)$ is a Hamel basis Schauder basis of $PC^1_{ms}([-\ell,\ell],\mathbf{R};\ell)$.

EXERCISES on Facts about the Sine and Cosine Function

EXERCISE 1. Write a proof of part 1. of Theorem #1 using the definition of periodicity.

EXERCISE 2. Write a proof of part 2. of Theorem #1 using properties of the cosine function.

EXERCISE 3. Write a proof of part 3. of Theorem #1 using properties of trigonometric functions.

EXERCISE 4. Write a proof of part 4. of Theorem #1 using properties of the sine functions.

THEOREM. (Fourier). Let f:**R** → **R**. Suppose
1. f is periodic of period 2ℓ.
2. f and f' are piecewise continuous on $[-\ell,\ell]$.
Then at the points of continuity of f, we have that

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(\frac{m\pi}{\ell}x) + b_m \sin(\frac{m\pi}{\ell})$$

where

$$a_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(\frac{m\pi}{\ell}x)dx \qquad \text{for } m = 0, 1, 2, 3, \ldots$$

$$b_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(\frac{m\pi}{\ell}x)dx \qquad \text{for } m = 1, 2, 3, \ldots .$$

That is, the series converges pointwise, (i.e. for each value of x) to the function f(x).
At points of discontinuity, the series converges to (f(x+) + f(x–)/2 where

f(x+) = $\lim_{t\to x+}$ f(t) and f(x- $\lim_{t\to x-}$ f(t).

We wish to give a name to the function space where we are sure that the fourier series coefficients exist and where we know that the fourier series converges pointwise as explained in the theorem. Hence the subset of F(**R**,**R**) that has the two properties given in the theorem and the property that they are defined at points of discontinuity by (f(x+) + f(x–)/2 we call $PC^1_{f}(\mathbf{R},\mathbf{R};\ell)$. $PC^1_{f}(\mathbf{R},\mathbf{R};\ell)$ is a subspace of F(**R**,**R**) so that it is a vector spa let $PC^1_{f}([-\ell,\ell],\mathbf{R};\ell)$ be $PC^1_{f}(\mathbf{R},\mathbf{R};\ell)$ns in with their domains restricted to $[-\ell$ functions in $PC^1_{f}(\mathbf{R},\mathbf{R};\ell)$ are well-defined once their values on $[-\ell,\ell)$ are known, we see that there is an isomorphism between $PC^1_{f}(\mathbf{R},\mathbf{R};\ell)$ $PC^1_{f}([-\ell,\ell],\mathbf{R};\ell)$ $PC^1_{ffs}([-\ell,\ell],\mathbf{R};\ell)$. We let subspace of $PC^1_{f}([-\ell,\ell],\mathbf{R};\ell)$ for which the fourier series is finite. Since the set

$B_{PC^1_{f}([-\ell,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in N\} \cup \{\sin(\frac{k\pi}{\ell}) : k \in N\}$ is linearly indepe

linearly independent) and every function in $PC^1_{ffs}([-\ell,\ell],\mathbf{R};\ell)$ can be written as a (finite) linear combination of functions in $B_{PC^1_{f}([-\ell,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in N\} \cup \{\sin(\frac{k\pi}{\ell}) : k \in N\}$

$B_{PC^1_{f}([-\ell,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in N\} \cup \{\sin(\frac{k\pi}{\ell}) : k \in N\}$ $PC^1_{ffs}([-\ell,\ell],\mathbf{R};\ell)$ a **Hamel basis** of

function if $PC^1_{f}([-\ell,\ell],\mathbf{R};\ell)$ can be written as a "infinite linear combination" of functions in

$B_{PC^1_{tk}([-\ell,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in \mathbf{N}\} \cup \{\sin(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$, we call

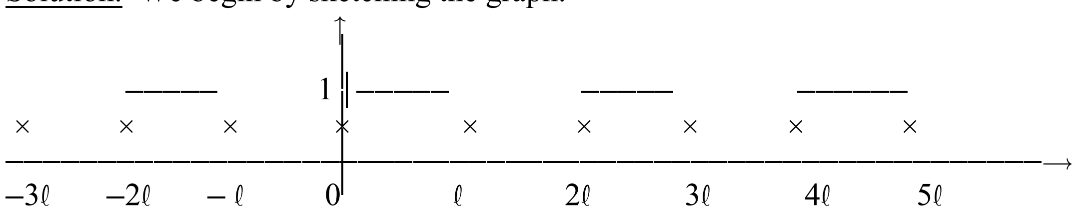
$B_{PC^1_{tk}([-\ell,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in \mathbf{N}\} \cup \{\sin(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ $PC^1_{tk}([-\ell,\ell],\mathbf{R};\ell)$ a **Schauder basi**

EXAMPLE. Find the Fourier series for f(x) if f is periodic of period 2ℓ and

$$f(x) = \begin{cases} 0 \text{ for } -\ell < x < 0 \\ 1 \text{ for } 0 < x < \ell \end{cases}$$

Sketch the graph for several periods (i.e. four: one between $-\ell$ and ℓ, two to the right and one to the left). Indicate in an appropriate manner the function to which the Fourier series converges. Note that values for the function are not given at the points of discontinuity. Explain why.

Solution. We begin by sketching the graph.



The Fourier series will converge to the function at points of continuity. It will converge to (f(x+) + f(x−)/2 (i.e. to the average of the limits from the right and left) at points of discontinuity. These are indicated by an ×. The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(\frac{m\pi}{\ell}x) + b_m \sin(\frac{m\pi}{\ell})$$

where

$$a_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(\frac{m\pi}{\ell}x)dx \qquad \text{for} \quad m = 0, 1, 2, 3, \ ...$$

$$b_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(\frac{m\pi}{\ell}x)dx \qquad \text{for} \quad m = 1, 2, 3, \ ... \ .$$

Always compute a_0 separately.

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(\frac{(0)\pi}{\ell}x)dx \qquad \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx \qquad \frac{1}{\ell}\int_{-\ell}^{0} f(x)\,dx \quad \frac{1}{\ell}\int_{0}^{\ell} f(x)\,dx \qquad =$$

$$= \frac{1}{\ell}\int_{-\ell}^{0} 0\ dx \quad \frac{1}{\ell}\int_{0}^{\ell} 1\ dx \quad \frac{1}{\ell}x \Big|_{x=0}^{x=\ell} \quad \frac{1}{\ell} \qquad = \qquad (\ell - 0) = 1$$

Hence $a_0/2 = 1/2$.

Next we compute a_m for m = 1, 2, 3, ...

$$a_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(\frac{m\pi}{\ell}x)dx \quad \frac{1}{\ell}\int_{-\ell}^{0} f(x)\cos(\frac{m\pi}{\ell}x)dx \quad \frac{1}{\ell}\int_{0}^{\ell} f(x)\cos(\frac{m\pi}{\ell}x)dx \qquad +$$

$$= \frac{1}{\ell}\int_{-\ell}^{0} (0)\cos(\frac{m\pi}{\ell}x)dx \quad \frac{1}{\ell}\int_{0}^{\ell} (1)\cos(\frac{m\pi}{\ell}x)dx \quad \frac{1}{\ell}\int_{0}^{\ell} \cos(\frac{m\pi}{\ell}x)dx \qquad =$$

$$= \frac{1}{\ell}\frac{\sin(\frac{m\pi}{\ell}x)}{\frac{mx}{\ell}} \Big|_{x=0}^{x=\ell} \quad \frac{1}{m\pi}[\sin(\frac{m\pi}{\ell}\ell) - \sin(\frac{m\pi}{\ell}0)]$$

$$= \frac{1}{m\pi} \qquad \sin(m\pi) = 0$$

Hence $a_m = 0$ for m = 1, 2, 3, ...

Proceeding to b_m, for m = 1, 2, 3, ... we obtain

$$b_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(\frac{m\pi}{\ell}x)dx \quad \frac{1}{\ell}\int_{-\ell}^{0} f(x)\sin(\frac{m\pi}{\ell}x)dx \quad \frac{1}{\ell}\int_{0}^{\ell} f(x)\sin(\frac{m\pi}{\ell}x)dx \qquad +$$

$$= \frac{1}{\ell}\int_{-\ell}^{0} (0)\sin(\frac{m\pi}{\ell}x)dx \quad \frac{1}{\ell}\int_{0}^{\ell} (1)\sin(\frac{m\pi}{\ell}x)dx \quad \frac{1}{\ell}\int_{0}^{\ell} \sin(\frac{m\pi}{\ell}x)dx \qquad =$$

$$= \frac{1}{\ell}\frac{-\cos(\frac{m\pi}{\ell}x)}{\frac{mx}{\ell}} \Big|_{x=0}^{x=\ell} \quad \frac{1}{m\pi}[-\cos(\frac{m\pi}{\ell}\ell) - [-\cos(\frac{m\pi}{\ell}0)]\,]$$

$$= \frac{1}{m\pi} \qquad [1 - \cos(m\frac{1}{m\pi} \qquad [1 - \begin{cases} 0 & \text{if m is even} \\ \frac{2}{m\pi} & \text{if m is odd} \end{cases}$$

We summarize in a table (Be sure you always list your Fourier coefficients in a table.).)

<u>TABLE.</u>

$a_0 / 2 = 1/2$

$a_m = 0$ for $m = 1, 2, 3, \ldots$

$$b_m = \begin{cases} 0 & \text{if } m = 2,4,6,\ldots \\ \dfrac{2}{m\pi} & \text{if } m = 1,3,5,\ldots \end{cases}$$

We now write the Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(\frac{m\pi}{\ell}x) + b_m \sin(\frac{m\pi}{\ell})$$

$$= \frac{1}{2} + \sum_{m=1}^{\infty} (0)\cos(\frac{m\pi}{\ell}x) + b_m \sin(\frac{m\pi}{\ell}) \qquad \frac{1}{2} + \sum_{m=1}^{\infty} b_m \sin(\frac{m\pi}{\ell}) \qquad =$$

$$= \frac{1}{2} + \sum_{m\,odd} b_m \sin(\frac{m\pi}{\ell}) \quad + \sum_{m\ even} b_m \sin(\frac{m\pi}{\ell})$$

$$= \frac{1}{2} + \sum_{m\,odd} \frac{2}{m\pi} \sin(\frac{m\pi}{\ell}) \quad + \sum_{m\ even} (0) \sin(\frac{m\pi}{\ell}) = \frac{1}{2} + \sum_{m\,odd} \frac{2}{m\pi} \sin(\frac{m\pi}{\ell}) \qquad =$$

Letting $m = 2k + 1$ for $k = 0, 1, 2, 3, \ldots$ so that

$k =$	0	1	2	3	4	$\cdots$
$m = 2k + 1 =$	1	3	5	7	9	$\cdots$

Hence

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin(\frac{(2k+1)\pi}{\ell})$$

$$= \frac{1}{2} \quad \frac{2}{\pi}\sin(\frac{\pi}{\ell}) \quad \frac{2}{3\pi}\sin(\frac{3\pi}{\ell}) \quad \frac{2}{5\pi}\sin(\frac{5\pi}{\ell}) \quad \frac{2}{7\pi}\sin(\frac{7\pi}{\ell}) \quad \frac{2}{9\pi}\sin(\frac{9\pi}{\ell}) \qquad +$$

Handout #4 EVEN AND ODD FUNCTIONS ` Professor Moseley

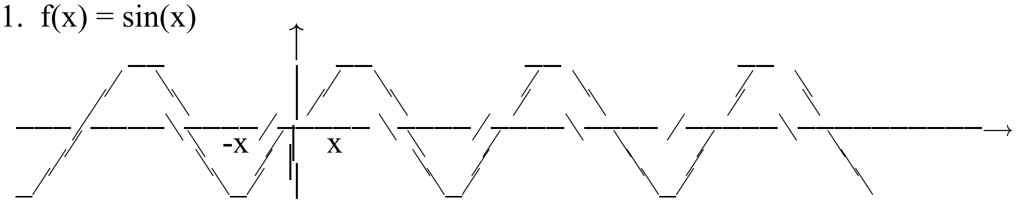
Read Section 10.3 of Chapter 10 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.) again. Also read Section 10.4. Pay particular attention to Examples 1-2 on pages 567-569.

DEFINITION. A function $f:\mathbf{R} \to \mathbf{R}$ is even if $\forall\, x \in \mathbf{R}$ we have $f(-x) = f(x)$.
f is odd if $\forall\, x \in \mathbf{R}$, $f(-x) = -f(x)$.

We can check this property either graphically or analytically (or both).

EXAMPLES.

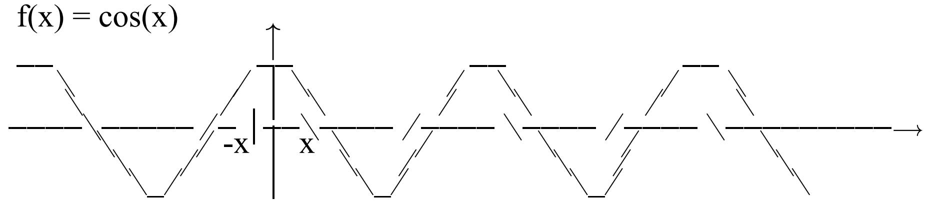
1. $f(x) = \sin(x)$



STATEMENT	REASON
$f(-x) = \sin(-x)$	Definition of the function f
$= -\sin(x)$	Trig identity
$= -f(x)$	Definition of the function f

Hence $f(x) = \sin(x)$ is an odd function

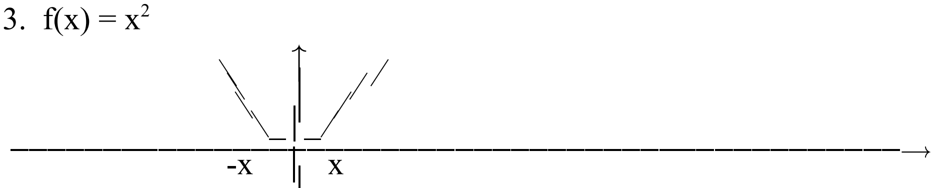
2. $f(x) = \cos(x)$



STATEMENT	REASON
$f(-x) = \cos(-x)$	Definition of the function f
$= \cos(x)$	Trig identity
$= f(x)$	Definition of the function f

Hence $f(x) = \cos(x)$ is an even function

3. $f(x) = x^2$

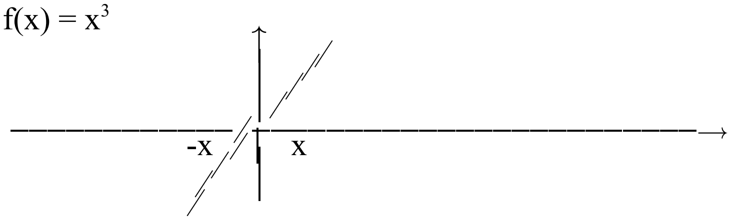


STATEMENT REASON

$f(-x) = (-x)^2$	Definition of the function f
$= x^2$	(Theorem from) Algebra
$= f(x)$	Definition of the function f

Hence $f(x) = x^2$ is an even function

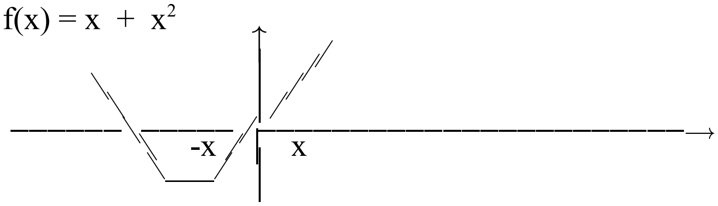
4. $f(x) = x^3$



STATEMENT	REASON
$f(-x) = (-x)^3$	Definition of the function f
$= - x^3$	(Theorem from) Algebra
$= f(x)$	Definition of the function f

Hence $f(x) = x^3$ is an odd function

5. $f(x) = x + x^2$



STATEMENT	REASON
$f(-x) = -x + (-x)^2$	Definition of the function f
$= -x + x^2$	(Theorem from) Algebra
$\neq f(x)$ or $-f(x)$	At least so it seems

Hence we believe that $f(x) = x + x^2$ is neither odd nor even but we do not have a proof. How do you think you could construct a proof. Hint: Look at the definition of odd and even and note that the equations must hold $\forall x \in \mathbb{R}$

PROPERTIES OF ODD AND EVEN FUNCTIONS.

THEOREM #1. We state this theorem informally.

1. The sum (or difference) and product (or quotient) of two even functions are even.
2.. The sum (or difference) of two odd functions is odd. However, the product (or quotient) of two odd functions is even.
3. The product of an odd function and an even function is odd.

A formal statement with proof that the sum of two even functions is even follows:

THEOREM #2. Let f:**R**→**R** and g:**R**→**R** be even functions. The h = f· g defined by h(x) = f(x)· g(x) is an even function.

PROOF. Let f:**R**→**R** and g:**R**→**R** be even functions so that f(−x) = f(x) and g(−x) = g(x) ∀ x ∈ **R**. Let h = f·g so that h(x) = f(x) ·g(x) ∀ x ∈ **R**. Then

STATEMENT		REASON
h(−x) = f(−x)· g(−x)	∀ x ∈ **R**	Definition of h as the product of f and g
= f(x) · g(x)	∀ x ∈ **R**	f and g are assumed to be even functions
= h(x)	∀ x ∈ **R**	Definition of h as the product of f and g

Since h(−x) = h(x) ∀ x ∈ **R**, by the definition of what it means for a function to be even, h is even.

QED

EXERCISES on Even and Odd Functions

EXERCISE #1. Give formal statements to the remaining parts of Theorem #1.

EXERCISE #2. Give formal proofs of the statements in the previous exercise.

THEOREM. Let f:**R**→**R** be an even function and g:**R**→**R** be an odd function so that f(−x) = f(x) and g(−x) = −g(x) ∀ x ∈ **R**. Then

1) $\int_{-\ell}^{\ell} f(x)dx$ $\int_{0}^{\ell} f(x)dx$ 2) $\int_{-\ell}^{\ell} g(x)dx$ = 0.

EXERCISE #3. Write a proof of the above theorem using the STATEMENT REASON format.

DEFINITION. Let $f:[0, \ell] \to \mathbf{R}$. We define the even periodic extension f_{epe} as follows:
1. Let $f_e(-x) = f(x)$ for $x \in (-\ell, 0)$ so that $f(-x) = f(x)$ $\forall x \in (-\ell, \ell]$.
2, Let f_{epe} be the periodic extension of f_e, that is,
   $f_{epe}(x + 2n\ell) = f_e(x)$ $\forall x \in (-\ell, \ell]$ and $\forall n \in \mathbf{Z}$.

DEFINITION. Let $f:[0, \ell] \to \mathbf{R}$. We define the Fourier cosine series for f as the Fourier series of the even periodic extension, f_{epe}, of f.

Recall:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(\frac{m\pi}{\ell} x) + b_m \sin(\frac{m\pi}{\ell})$$

where

$$a_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(\frac{m\pi}{\ell}x)dx \qquad \text{for} \quad m = 0, 1, 2, 3, \ldots$$

$$b_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(\frac{m\pi}{\ell}x)dx \qquad \text{for} \quad m = 1, 2, 3, \ldots .$$

Since f_{epe} is an even function, we have

$$a_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f_{epe}(x) \cos(\frac{m\pi}{\ell}x)dx \quad \frac{2}{\ell}\int_{0}^{\ell} f_{epe}(x) \cos(\frac{m\pi}{\ell}x)dx \quad \frac{2}{\ell}\int_{0}^{\ell} f(x) \cos(\frac{m\pi}{\ell}x)dx$$

even × even = even

$$b_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f_{epe}(x) \sin(\frac{m\pi}{\ell}x)dx \qquad = 0$$

even × odd = odd

Summarizing, given $f:[0, \ell]\to\mathbf{R}$, to compute the Fourier cosine series for f given by

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(\frac{m\pi}{\ell} x)$$

compute the Fourier cosine series coefficients using

$$a_m = \frac{2}{\ell}\int_0^{\ell} f(x)\cos(\frac{m\pi}{\ell}x)dx \qquad \text{for} \quad m = 0, 1, 2, 3, \ldots$$

We let $PC^1_{fs,e}(\mathbf{R},\mathbf{R};\ell)$ be the s $PC^1_{fs}(\mathbf{R},\mathbf{R};\ell)$ f containing only even functions. We let $PC^1_{fs,e}([0,\ell],\mathbf{R};\ell)$ be th $PC^1_{fs,e}(\mathbf{R},\mathbf{R};\ell)$ in with their domains restricted to [0,ℓ]. We $PC^1_{ffs,e}([0,\ell],\mathbf{R};\ell)$ be th $PC^1_{fs,e}([0,\ell],\mathbf{R};\ell)$ for which the fourier series is finite. $B_{PC^1_{fs,e}([0,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ is linearly independent (every finite subset is lin independent) and every function in $PC^1_{ffs,e}([0,\ell],\mathbf{R};\ell)$ can be written as a (finite) linear combination of functions in $B_{PC^1_{fs,e}([0,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ $B_{PC^1_{fs,e}([0,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ **Hamel basis** of $PC^1_{ffs,e}([0,\ell],\mathbf{R};\ell)$. Since eve $PC^1_{fs,e}([0,\ell],\mathbf{R};\ell)$ can be writter "infinite linear combination" of functions in $B_{PC^1_{fs,e}([0,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$, w $B_{PC^1_{fs,e}([0,\ell],\mathbf{R};\ell)} = \{1/2\} \cup \{\cos(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ $PC^1_{ffs,e}([0,\ell],\mathbf{R};\ell)$ **chauder basis** of

Similarly, given f:[0, ℓ]→ **R**, to compute the Fourier sine series for f given by

$$f(x) = \sum_{m=1}^{\infty} b_m \sin(\frac{m\pi}{\ell})$$

compute the Fourier sine Series coefficients using

$$b_m = \frac{2}{\ell}\int_0^{\ell} f(x)\sin(\frac{m\pi}{\ell}x)dx \qquad \text{for} \quad m = 1, 2, 3, \ldots .$$

We let $PC^1_{fs,o}(\mathbf{R},\mathbf{R};\ell)$ be the s $PC^1_{fs}(\mathbf{R},\mathbf{R};\ell)$ f containing only odd functions. We let $PC^1_{fs,o}([-\ell,\ell],\mathbf{R};\ell)$ be tl $PC^1_{fs,o}(\mathbf{R},\mathbf{R};\ell)$ in with their domains restricted to [0,ℓ]. $PC^1_{ffs,o}([0,\ell],\mathbf{R};\ell)$ be th $PC^1_{fs,o}([0,\ell],\mathbf{R};\ell)$ for which the fourier series is finite. set $B_{PC^1_{fs,o}([0,\ell],\mathbf{R};\ell)} = \{\sin(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ is linearly independent (every finite subset is linearly inde and every function in $PC^1_{ffs,o}([0,\ell],\mathbf{R};\ell)$ can be written as a (finite) linear combination of functions in $B_{PC^1_{fs,o}([0,\ell],\mathbf{R};\ell)} = \{\sin(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ $B_{PC^1_{fs,o}([0,\ell],\mathbf{R};\ell)} = \{\sin(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ $PC^1_{ffs,o}([0,\ell],\mathbf{R};\ell)$ Since every function if $PC^1_{fs,o}([0,\ell],\mathbf{R};\ell)$ can be written as a "infinite linear combination" of functions in $B_{PC^1_{fs,o}([0,\ell],\mathbf{R};\ell)} = \{\sin(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ $B_{PC^1_{fs,o}([0,\ell],\mathbf{R};\ell)} = \{\sin(\frac{k\pi}{\ell}) : k \in \mathbf{N}\}$ $PC^1_{ffs,o}([0,\ell],\mathbf{R};\ell)$.

EXERCISES on Sine and Cosine Series

EXERCISE #1. Give a formal definition of the odd periodic extension of f where $f:[0,\ell\} \to \mathbb{R}$ where $f(0) = 0$. (Why is this required?)