A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 4

A COLLECTION OF HANDOUTS ON
PARTIAL DIFFERENTIAL EQUATIONS (PDE's)

CHAPTER 2

Eigenvalue Problems (EVP's) for ODE's

1. Eigen Value Problems for Second Order Linear ODE's

2. Basis of Eigenvectors

FOR SECOND ORDER LINEAR ODE'S

We consider the Eigenvalue Problem (EVP) for Second Order Linear ODE's (it is a special case of a more general EVP known as a regular **Sturm-Liouville problem**):

EVP
- ODE $-(p(x)y')' - q(x)y = \lambda y$
- IC's $y(x_0) = 0, \; y(x_1) = 0.$

We assume $p \in C^1(I) \cap C(\bar{I})$, $q \in C(\bar{I})$, and $p(x) > 0 \;\forall x \in \bar{I}$ (e.g. $p(x) = 1$), where $I = (x_0, x_1)$. Note that if λ is given, then this is a homogenous boundary value problem (BVP). We can rewrite the problem as

EVP
- ODE $p(x)\,y'' + p'(x)y' + (q(x) + \lambda)y = 0$
- IC's $y(x_0) = 0, \; y(x_1) = 0.$

The theory of this EVP is similar to that of the matrix EVP

$$Ax = \lambda x \qquad \text{or} \qquad (A - x\lambda I)x = 0.$$

Since for fixed λ we have a homogenous BVP, the "problem" EVP always has at least one solution, namely $y = 0$, no matter what the value of λ. However, the Eigen Value Problem (EVP) is not simply to solve the BVP for a given λ. By definition, the EVP requires you to find all of the values of λ such that the BVP problem defined by that value of λ has at least one nontrivial solution. However, remember that if it has one nontrivial solution, then by linearity, it will have an infinite number of solutions (all scalar multiples of solutions are solutions). In fact, the solution set will be a nontrivial subspace of $C^2(I) \cap C(\bar{I})$ where $I = (x_0, x_1)$. Like matrix EVP"s, often this subspace will be one dimensional; that is, all of the eigen vectors (we will call them eigen functions since now our vector space is a function space) are multiples of a (non-unique) function which is a basis for the eigen space.

<u>THEOREM</u> <u>#1</u>. The eigen value problem EVP defined above is **self-adjoint**. (We could also say that the operator in the eigenvalue problem is self-adjoint but you must remember that the operator includes the boundary conditions as well as the differential operator.)

The important thing to remember is that since the problem is self-adjoint, the eigen values are real. Also the eigen functions can be chosen to be real.

If the ODE has constant coefficients, the procedure for solving an EVP is similar to the procedure for solving a BVP and IVP for second order ODE's: Since λ is a constant we first find the general solution to the ODE and then apply the boundary conditions. We illustrate with an example.

EXAMPLE #1. Solve the eigenvalue problem

$$\text{EVP}\quad \begin{cases} \text{ODE} & y'' + \lambda y = 0 \\ \text{IC's} & y(0) = 0,\ \ y(1) = 0. \end{cases}$$

Solution. Since the problem (or operator which defines the problem) is self-adjoint, the eigen values are all real. The general solution of the ODE depends on three cases.

Case 1. $\lambda < 0$. In this case letting $y = e^{rx}$ yields the auxiliary equation $r^2 + \lambda = 0$ or (since $-\lambda > 0$) $r = \pm\sqrt{-\lambda}$. Hence we obtain $y = C_1$ $\sqrt{-\lambda}$ x) $\sqrt{-\lambda}$ exp(− x).

Case 2. $\lambda = 0$. In this case letting $y = e^{rx}$ yields the auxiliary equation $r^2 + \lambda = 0$ or (since $\lambda > 0$) $r = 0,0$. Hence we obtain $y = C_1 x + C_2$.

Case 3. $\lambda > 0$. In this case letting $y = e^{rx}$ yields the auxiliary equation $r^2 + \lambda = 0$ or (since $\lambda > 0$) $r = \pm\sqrt{\lambda}$. Hence we obtain $y = C_1$ $\sqrt{\lambda}$ x) + $\sqrt{\lambda}$ cos(x).

We now apply the boundary conditions in these three cases to determine if there are any (real) values of λ for which the "problem" EVP has nontrivial solutions.

Case 1. $\lambda < 0$. Recall that the general solution of the ODE for this case is $y = C_1$ exp($\sqrt{-\lambda}$ x) + C_2 e $\sqrt{-\lambda}$ x). Applying the boundary conditions yields

$0 = C_1 \exp(\,0\,) + C_2 \exp(\,0\,)$
$0 = C_1$ exp($\sqrt{-\lambda}$) + C_2 e $\sqrt{-\lambda}$)

The first condition is independent of the value of λ and yields $C_2 = -\,C_1$. Substituting into the second equation (which is not independent of λ) yields

C_1 exp($\sqrt{-\lambda}$) − C_1 e $\sqrt{-\lambda}$) = 0.

or

C_1 (exp($\sqrt{-\lambda}$) − e $\sqrt{-\lambda}$) = 0.

This means $C_1 = 0$ or (exp($\sqrt{-\lambda}$) − e $\sqrt{-\lambda}$) = 0.

If $C_1 = 0$, then $C_2 = -\,C_1 = 0$ so that $y = 0$. That is we get only the trivial solution. There are two ways to handle the case when (exp($\sqrt{-\lambda}$) − e $\sqrt{-\lambda}$) = 0. We do one. The other is left as an exercise. If you remember the definition of the hyperbolic sine function as

$\sinh(x) = (e^x - e^{-x})/2$, then dividing the condition by 2 yields $\sinh(\sqrt{-\lambda}) = 0$. Since (if you remember the properties of the sinh function) the only zero of the sinh function is zero, we obtain $\sqrt{-\lambda} = 0$ so that $-\lambda = 0$ and hence $\lambda = 0$. However, we are considering the case when $\lambda < 0$. Hence for no $\lambda < 0$ are there any eigen values.

EXERCISE. Using only the properties of e^x and ln x show that if
$\exp(\sqrt{-\lambda}) - e^{-\sqrt{-\lambda}}) = 0$, then $\lambda = 0$.

The bottom line is that there are no eigenvalues in Case 1.

Case 2. $\lambda = 0$. Recall that the general solution of the ODE for this case is $y = C_1 x + C_2$. Applying the boundary conditions yields

$$0 = C_1 (0) + C_2,$$
$$0 = C_1 (1) + C_2.$$

The first condition implies $C_2 = 0$. The second condition then implies $C_2 = - C_1 = 0$. Hence $y = 0$. Thus we get only the trivial solution and $\lambda = 0$ is not an eigen values.

Case 3. $\lambda > 0$. Recall that the general solution of the ODE for this case is
$y = C_1 \sin(\sqrt{\lambda}\, x) + C_2 \cos(\sqrt{\lambda}\, x)$. Applying the boundary conditions yields

$$0 = C_1 \sin(0) + C_2 \cos(0)$$
$$0 = C_1 \sin(\sqrt{\lambda}) + C_2 \cos(\sqrt{\lambda})$$

The first condition is independent of the value of λ and yields $C_2 = 0$. Substituting into the second equation (which is not independent of λ) yields

$$C_1 \sin(\sqrt{\lambda}) = 0.$$

This means $C_1 = 0$ or $\sin(\sqrt{\lambda}) = 0$. If $C_1 = 0$, then $y = 0$. That is we get only the trivial solution. Hence our only hope to find any eigen values is if there are values of λ such that the equation

$$\sin(\sqrt{\lambda}) = 0$$

has positive solutions. Hence this equation is called the characteristic equation for this eigen value problem. Recalling from the graph of the sine function that the zeros of the sine function are $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ or $\pm$ n where $n \in \{0\} \cup N$ where $N = \{1,2,3,4,\ldots\}$, we see that the solutions are given by

$\sqrt{\lambda} = n\pi$ or $\lambda = n^2\pi^2$.

Recall that n = 0, ±1, ±2, ±3, ... However, if n = 0, then $\lambda = 0$. But we are in the case where $\lambda > 0$. Hence we must throw this value of n out. Also note that if n is negative we get the same eigen values as we do when n is positive. Before throwing out the negative values of n we must first examine the eigen functions. Recall that since $\sin(\sqrt{\lambda}) = 0$, there is no requirement that $C_1 = 0$. Hence the general solution of the "problem" EVP is $y = C_1 \sin(\sqrt{\lambda}\, x)$ or $y = C_1 \sin(n\pi x)$. Note that if n is negative, then using a trig identity, the minus sign can be taken out and incorporated into the arbitrary constant C_1. Hence the set of eigen functions is just the set of all scalar multiples of $y_n = \sin(n\pi x)$ no matter whether n is positive or negative. Hence we obtain the table explicitly (you should list them this way when solving problems):

TABLE

Eigen Values	Associated Eigen Functions
$\lambda_1 = \pi^2$	$y_1 = \sin(\pi x)$
$\lambda_2 = 4\pi^2$	$y_2 = \sin(2\pi x)$
$\lambda_2 = 9\pi^2$	$y_3 = \sin(3\pi x)$
.	.
.	.
.	.
$\lambda = n^2\pi^2$	$y_n = \sin(n\pi x)$
.	.
.	.
.	.

or more compactly

TABLE

Eigen Values	Associated Eigen Functions
$\lambda = n^2\pi^2$, n = 1, 2, 3, ...	$y_n = \sin(n\pi x)$ n = 1, 2, 3, ...

Note that there are an infinite number of eigen values each with a one dimensional eigen space. The listed functions are the basis functions for the eigen spaces. The hope is that the entire collection will be a basis for the entire space; that is that we have a full set of eigen functions. This will be explained more fully later.