A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 4

A COLLECTION OF HANDOUTS ON
PARTIAL DIFFERENTIAL EQUATIONS (PDE's)

CHAPTER 1

Boundary Value Problems (BVP's) for ODE's

1. Review of Second Order Linear Differential Operator
2. Initial Value Problems and Boundary Value Problems
3. Introduction to the Theory of Boundary Value Problems

Handout No. 1 REVIEW OF SECOND ORDER LINEAR DIFFERENTIAL OPERATORS Prof. Moseley

Recall that we defined the general second order linear differential operator L by

$$L[y] = y'' + p(x)y + q(x)y \tag{1}$$

where $p,q \in C(I)$, $I = (a,b)$ in the Classical II context where $L{:}C^2(I)\to C(I)$ and $p,q \in \mathcal{A}(I)$ in the Classical I context where $L{:}\mathcal{A}(I)\to\mathcal{A}(I)$. We also considered the Classical I context where $L{:}H(\mathbf{C})\to H(\mathbf{C})$ and $p,q \in H(\mathbf{C})$. Since we can treat these collections of functions as a vector space, we can view L as mapping one vector space into another.

Also recall the definition of a linear operator from one vector space to another.

DEFINITION (Linear Operator). An operator $L{:}V \to W$ from a vector space V to another vector space W is said to be linear if for all vectors $\bar{x}$ ar $\bar{y}$ in V and all scalars α and β, we have $L(\alpha\bar{x} + \bar{y}) = \bar{x}\ L(\ )\ \bar{y}\ \beta\, L(\ )$.

<u>THEOREM.</u> For all three contexts, L given by (1) is a linear operator with $\dim N(L) = 2$. Hence the general solution of L[y] is $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$ where $B = \{ y_1,\ y_2\}$ is a basis of N(L).

EXERCISES on Review of Second Order Linear Differential Operator

<u>EXERCISE</u> <u>#1</u>.

(a) Compute $L[\varphi] = y'' + y$ if (a) $\varphi(x) = \sin x$, (b) $\varphi(x) = \cos x$, (c) $\varphi(x) = e^x$.

(b) Compute $L[\varphi] = y'' - y$ if (a) $\varphi(x) = \sin x$, (b) $\varphi(x) = e^x$, (c) $\varphi(x) = e^{-x}$.

INITIAL VALUE PROBLEMS AND BOUNDARY VALUE PROBLEMS

The Initial Value Problem (IVP) for the general second order linear o.d.e. is:

$$\text{IVP}\begin{cases}\text{ODE} & y'' + p(x)y' + q(x)y = g(x)\\ \text{IC's} & y(x_0) = y_0,\ y'(x_0) = y'_0\end{cases}$$

where we have chosen Initial Condition's (IC) specifying the values of y and y′ at $x = x_0$. As an example, consider throwing a ball up. In order to uniquely determine the motion of the ball, one must also specify the initial position and the initial velocity. That is, it can be used to model the position (and velocity) of a point mass so that we might use t as our independent variable instead of x.

On0 the other hand (OTOH), the Boundary Value Problem (BVP) for the general second order o.d.e. is:

$$\text{BVP}\begin{cases}\text{ODE} & y'' + p(x)y' + q(x)y = g(x)\\ \text{BC's} & y(x_0) = y_0,\ y'(x_1) = y_1\end{cases}$$

where we have chosen Boundary Conditions (BC's) specifying the values of y at $x = x_0$ and at $x = x_1$. If we throw a ball up, we might wish to specify the initial position and the position at some later time. For example, If we throw the ball up at time t = 0, (so that y(0) = 0), we may wish to specify the time at which it comes down ($y(t_1) = 0$). Alternately, we use the boundary value problem to model steady state conditions for heat flow in a rod where we specify the temperature at both ends of the rod.

Although for elementary problems, the solution technique for both (linear) problems is essentially the same (first obtain the "general solution" for the ODE and they apply the IC's or BC's to obtain the arbitrary constants), the theory of IVP's and the theory of BVP's are quite different. Most theorems involving linear IVP's usually conclude that there exist a unique solution. Whereas theorems for linear BVP's typically reflect the general linear theory and state that there exist no solution, one solution, or an infinite number of solutions.

Handout # 3 INTRODUCTION TO THE THEORY OF BOUNDARY VALUE PROBLEMS Prof. Moseley

Existence and uniqueness theorems for the initial value problem (IVP)

$$\text{IVP} \quad \begin{array}{ll} \text{ODE} & y'' + p(x)y' + q(x)y = 0 \\ \text{IC's} & y(x_0) = y_0, \;\; y'(x_0) = y_1 \end{array}$$

typically state that if the functions p and q are sufficiently "nice" on an open interval I = (a,b) that contains x_0 (i.e. $a < x_0 < b$), then the initial value problem IVP has a unique solution on the open interval I. Thus the set where we look for solutions is the set $C^2(I)$. For example,

<u>THEOREM</u>. If p, q $\in$ C(I), where $x_0 \in I = (a,b)$ (i.e. the functions p and q are continuous on the open interval I = (a,b) which contains the point x_0, $a < x_0 < b$), then the initial value problem IVP has a unique solution in the set $C^2(I)$.

Although the problems look similar and the technique for solution is similar (i.e. first find the general solution of the ode and then apply the initial conditions or boundary conditions, BC's, to obtain the two arbitrary constants), theorems on the existence and uniqueness of solutions to the boundary value problem (BVP)

$$\text{BVP} \quad \begin{array}{ll} \text{ODE} & y'' + p(x)y' + q(x)y = 0 \\ \text{BC's} & y(x_0) = y_0, \;\; y(x_1) = y_1 \end{array}$$

are theoretically quite different. The fundamental theorem is similar to the fundamental theorem for the solution of $A\bar{x} = \bar{b}$. Since we have boundary conditions, the set where we look for solutions is also different. Let I = (a,b), $\bar{I}$ = [a,b], and ($\bar{I}$) = $\bar{I}$ y: →**R** such that f is continuous}. We now extend the definition of $C^2(I)$ to be $C^2(I)$ = {y: $\bar{I}$ →**R** such that its restriction to I is in $C^2(I)$}. Now let V = $C^2(I) \cap C(\bar{I})$ = {$\bar{I}$ →**R** such that y $\bar{I}$ ( ) and y restricted to I is in $C^2(I)$ } so that L now maps V to W = C(I). Thus the derivative of y need not exist at the end points.

<u>THEOREM</u>. Suppose that p(x) = 0 and q(x) is continuous on I = $[x_0,x_1]$. There are three possibilities:

1. The BVP has no solution in V.
2. The BVP has exactly one solution in V.
3. The BVP has an infinite number of solutions in V.

Unlike the IVP, the "niceness" of p and q are not sufficient for the existence and uniqueness of the solution to the BVP. We give an example of each possibility.

EXAMPLE 1.

ODE $y'' + y = 0$
BVP
BC's $y(0) = 0,\ y(\pi/2) = 0.$

Solution. It is obvious that y = 0 is a solution so existence is not a problem. This is because the ODE and the BC's are homogeneous; hence the problem is homogeneous. The only question is: Are we in Case 2 or Case 3? The general solution to the ODE is $y = C_1 \sin(x) + C_2 \cos(x)$.
Applying the BC's we obtain:

$0 = C_1 \sin(0) + C_2 \cos(0) = C_1(0) + C_2(1) = C_2,$

$0 = C_1 \sin(\pi/2) + C_2 \cos(\pi/2) = C_1(1) + C_2(0) = C_1.$

Hence the only solution is y = 0 and we are in Case 2.

EXAMPLE 2.

ODE $y'' + y = 0$
BVP
BC's $y(0) = 0,\ y(\pi) = 0.$

Solution. Note that Example 1 and Example 2 look very similar. Again, it is obvious that y = 0 is a solution so existence is not a problem. This problem also homogeneous since the ODE and the BC's are homogeneous. Again the only question is: Are we in Case 2 or Case 3? The general solution to the ODE is again $y = C_1 \sin(x) + C_2 \cos(x)$. However, applying the BC's we obtain:

$0 = C_1 \sin(0) + C_2 \cos(0) = C_1(0) + C_2(1) = C_2,$

$0 = C_1 \sin(\pi) + C_2 \cos(\pi) = C_1(0) + C_2(1) = C_2.$

Hence only C_2 is zero so that the solution is $y = C_1 \sin(x)$. Since C_1 is arbitrary, this time we are in Case 3.

EXAMPLE 3.

ODE $y'' + y = 0$
BVP
BC's $y(0) = 0,\ y(\pi) = 1.$

Solution. Note that this example looks very similar to Example 1 and Example 2. This time it is obvious that y = 0 is not a solution since y = 0 will not satisfy the second BC. Existence is not assured since the problem is not homogeneous. As before, the general solution to the ODE is again $y = C_1 \sin(x) + C_2 \cos(x)$. However, applying the BC's this time we obtain:

$0 = C_1 \sin(0) + C_2 \cos(0) = C_1(0) + C_2(1) = C_2$,

$1 = C_1 \sin(\pi) + C_2 \cos(\pi) = C_1(0) + C_2(1) = C_2$.

Since C_2 can not be zero and one at the same time, there is no solution and this time we are in Case 1.

EXERCISES on Introduction to the Theory of Boundary Value Problems

EXERCISE #1. $y'' + y = 0$, $y(0) = 0$, $y(\pi/2) = 0$.

EXERCISE #2. $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 0$.

EXERCISE #3. $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 1$.