A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 3

A COLLECTION OF HANDOUTS ON
SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

CHAPTER 5

Theory and Solution Using Matrix Techniques

1. Fundamental Theory of Systems of ODE's

2. Matrix Technique for Solving Systems with Constant Coefficients

3. Real Distinct Roots

4. Complex Roots

5. Repeated Roots

The general form for a first order linear system of ODE's is given by

$$
\begin{array}{lcccccc}
x_1' & = p_{11}(t)\ x_1 & + p_{12}(t)\ x_2 & + \cdots & + p_{1n}(t)\ x_n & + g_1(t) \\
x_2' & = p_{21}(t)\ x_1 & + p_{22}(t)\ x_2 & + \cdots & + p_{2n}(t)\ x_n & + g_2(t) \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot \\
x_1' & = p_{n1}(t)\ x_1 & + p_{n2}(t)\ x_2 & + \cdots & + p_{nn}(t)\ x_n & + g_n(t)
\end{array}
$$

We choose t as the independent variable so that we can use x's as the components of the "vector-valued" function:

$$\underset{n\times 1}{\vec{x}} = \vec{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$$

(We use the transpose notation to save space.) We make the usual assumption that $p_{ij}(t)$ and $g_i(t)$ are continuous $\forall$ t $\in$ I = (a,b). This system can be rewritten using (the much more compact) matrix notation as:

$$\underset{n\times 1}{\vec{x}'} = \underset{n\times n}{P}\ \underset{n\times 1}{\vec{x}} + \underset{n\times 1}{\vec{g}}(t) \tag{3}$$

where P is the n×n square matrix-valued function

$$
P(t) = \begin{bmatrix}
p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\
p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t)
\end{bmatrix}
$$

and g(t) is the "vector-valued" forcing function

$$\vec{g} = \vec{g}(t) = [g_1(t), g_2(t), \ldots, g_n(t)]^T$$

The usual form for a nonhomogeneous equation requires that we rewrite this as

$$\underset{n\times 1}{\dot{\vec{x}}} - \underset{n\times n}{P}\,\underset{n\times 1}{\vec{x}} = \underset{n\times 1}{\vec{g}}(t) \qquad . \qquad ($$

which we then rewrite as

$$L[\vec{x}] = \vec{g}(t) \qquad \forall\, t \in I = (a,b)$$

where

$$L[\vec{x}(t)] = \underset{n\times 1}{\dot{\vec{x}}} - \underset{n\times n}{P}\,\underset{n\times 1}{\vec{x}} \qquad \forall t\in I=(a,b). \tag{8}$$

Recall that $\mathbf{R}^{n\times n}$ is the set of all real valued n×n matrices. Now let $\mathbf{R}^{n\times n}(I) = \{P=(\, p_{ij}(t)\,):I\to\mathbf{R}^{n\times n}\,)$ and be the set of time varying matrices on the open interval I=(a,b) and $C(\mathbf{R}^{m\times n}(I)) = \{P=(\, p_{ij}(t)\,):I\to\mathbf{R}^{m\times n} :p_{ij}\in C(I)$ for all i and j$\}$ denotes the set of all elements in $\mathbf{R}^{m\times n}(I)$ where all entries are continuous on I. Similarly, $A(\mathbf{R}^{n\times n}(I)) = \{P=(\, p_{ij}(t)\,):I\to\mathbf{R}^{n\times n} :p_{ij}\in A(I)$ for all i and j$\}$ denotes the set of all elements in $\mathbf{R}^{n\times n}(I)$ where all entries are analytic on I, $C(\mathbf{R}^{n}(I)) = \{\, \vec{x} = [\, x_i(t)\,]:I\to\mathbf{R}^{n} :x_i\in C(I)\,\}$, $C^1(\mathbf{R}^{n}(I)) = \{x\ \vec{x} = [\, x_i(t)\,]:I\to\mathbf{R}^{n} :x_i'(t)$ exists for all i and is in $C(I)\,\}$, and $A(\mathbf{R}^{n}(I)) = \{\, \vec{x} = [\, x_i(t)\,]: I\to\mathbf{R}^{n} :x_i(t) \in A(I)\,\}$. Now assume $P(t)\in C(\mathbf{R}^{n\times n}(I))$ and $g(t) \in C(\mathbf{R}^{n}(I))$. Then L is an operator that maps "vector-valued" functions in $C^1(\mathbf{R}^{n}(I))$ to "vector-valued" functions in $C(\mathbf{R}^{n}(I))$. To solve (6) means to find all $\vec{x}$ that map $\vec{g}$. Since algebraically, we can treat a collection of "vector-valued" functions (with the appropriate definitions of vector addition and scalar multiplication) as a vector space, we can view on this operator as mapping one vector space into another. In this case we are mapping $C^1(\mathbf{R}^{n}(I)$; that is, the set of "vector-valued" functions which have a derivatives that is continuous on the interval of validity I = (a,b) into the space $C^1(\mathbf{R}^{n}(I))$ of continuous "vector-valued" function on I. If $P(t)\in A(\mathbf{R}^{n\times n}(I))$, then L maps $A(\mathbf{R}^{n}(I))$ to $A(\mathbf{R}^{n}(I))$.

We now review and apply the linear theory previously developed and applied to second order linear ODE's and higher order linear ODE's to first order linear systems of ODE's, that is to the operator L given in (8) above. We begin by reviewing the definition of a linear operator. A function or mapping T from one vector space V to another vector space W is often called an **operator**. We write T:V→ W. If we wish to think geometrically rather than algebraically we might call T a **transformation**.

<u>DEFINITION #1</u>. An operator T:V → W is said to be **linear** if $\forall$ x-y $\in$V and $\forall$ scalars α,β we have

$$T(\alpha\vec{x} + \beta\vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y}). \tag{9}$$

Recall that we can divide our check that T is a linear operator into two steps by using the following theorem.

<u>THEOREM #1</u>. An operator T is linear if and only if the following two properties hold:

i) $\vec{x}, \vec{y} \in V \Rightarrow T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$

ii) α a scalar and $\vec{x} \in V \Rightarrow T(\alpha \vec{x}) = \alpha T(\vec{x})$.

THEOREM #2. Assume that $P(t) \in C^1(\mathbf{R}^{n\times n}(I))$ where $I = (a,b)$. Then L defined by (8) is a linear operator from $C^1(\mathbf{R}^n(I))$ to $C(\mathbf{R}^n(I))$ and the null space of L is a subset of $C^1(\mathbf{R}^n(I))$. If $P(t) \in A(\mathbf{R}^{n\times n}(I))$, then L defined by (8) is a linear operator from $A(\mathbf{R}^n(I))$ to $A(\mathbf{R}^n(I))$ and $N(L) \subseteq A(\mathbf{R}^n(I))$. In either case, dim N(L) = n. Hence the solution of the homogeneous equation

$$L[\vec{x}] = 0 \qquad \forall x \in I = (a,b) = \text{interval of validity} \tag{12}$$

has the form

$$\vec{x}(t) = c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t) = \sum_{i=1}^{n} c_i\vec{x}_i(t)$$

where $\{\vec{x}_1, \cdots, \vec{x}_n\}$ is a basis for N(L) and c_i, $i = 1, \ldots, n$ are arbitrary constants.

Since we know that the dimension of the null space N(L) is n, if we have a set of n solutions to the homogeneous equation (1), to show that it is a basis of the null space N(L), it is sufficient to show that it is a linearly independent set. As applied to $C^1(\mathbf{R}^n(I))$, $C(\mathbf{R}^n(I))$, and $A(\mathbf{R}^n(I))$, the definition of linear independence is as follows:

DEFINITION #2. The set $S = \{x_1, x_2, x_3, \ldots, x_n\} \subseteq C^1(\mathbf{R}^n(I))$ is said to be linearly independent on I = (a,b) if the only solution to $\exists\, c_1, c_2, c_3, \ldots, c_n \in \mathbf{R}$ such that

$$c_1 \vec{x}_1 \quad + \vec{x}_2 \quad + \cdot \vec{x}_n\, c_n \quad = 0 \quad \forall t \in \mathbf{R} \tag{14}$$

is the trivial solution, $c_1 = c_2 = \cdots c_n = 0$. Otherwise S is linearly dependent on the interval I.

The following theorem is in some sense just a restatement of the definition.

THEOREM #3. The set $S = \{x_1, x_2, x_3, \ldots, x_n\} \subseteq C^1(\mathbf{R}^n(I))$ is linearly independent on I = (a,b) if and only if one "vector"-valued function can be written as a linear combination of the other "vector"-valued functions.

PROCEDURE. To show that $S = \{\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots \vec{x}_n\}$ is linear independent it is standard to assume (8) and try to show $c_1 = c_2 = c_3 = \ldots = c_n = 0$.

DEFINITION #3. If $\vec{x}_1, \dots, \vec{x}_n \in \vec{C}^1(I)$, $\vec{x}_i = [x_{i1}, \dots, x_{jn}]^T$, where $I = (a,b)$, and

then

$$W(\vec{x}_1, \dots, \vec{x}_n; t) = W(t) = \begin{vmatrix} x_{1_1} & \cdot & \cdot & \cdot & x_{n_1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ x_{1_n} & \cdot & \cdot & \cdot & x_{n_n} \end{vmatrix}$$

is called the Wronski determinant or the Wronskian of $\vec{x}_1, \dots, \vec{x}_n$ at the point t.

THEOREM #4. The null space N(L) of the operator defined by L in (1) above has dimension n. Hence the solution of the homogeneous equation

$$L[\vec{x}] = 0 \qquad \forall\, x \in I = (a,b) = \text{interval of validity} \qquad (16)$$

has the form

$$\vec{x}(t) = c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t) = \sum_{i=1}^{n} c_i\vec{x}_i(t)$$

where $\{\vec{x}_1, \dots, \vec{x}_n\}$ is a basis for N(L) and c_i, i = 1, ... , n are arbitrary constants. Given a set of n solutions to L[y] = 0, to show that they are linearly independent solutions, it is sufficient to compute the Wronskian.

$$W(\vec{x}_1, \dots, \vec{x}_n; t) = W(t) = \begin{vmatrix} x_{1_1} & \cdot & \cdot & \cdot & x_{n_1} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ x_{1_n} & \cdot & \cdot & \cdot & x_{n_n} \end{vmatrix}$$

where $\vec{x}_i = [x_{i1}, \dots, x_{jn}]^T$, $i = 1, \dots, n$ and show that it is not equal to zero on the interval validity.

THEOREM #5. The nonhomogeneous equation

$$L[\vec{x}] = \vec{g}(t) \qquad \forall\, t \in I = (a,b) = \text{interval of validity} \tag{19}$$

has at least one solution if the function g is contained in the **range space** of L, R(L). If this is the case then the general solution of (19) is of the form

$$\vec{x}(t) = \vec{x}_p(t) + \vec{x}_h(t) \tag{20}$$

where $\vec{x}_p$ is a particular (i.e. any specific) solution to (19) and y_h is the general (e.g. a formula for all) solutions of (16). Since N(L) is finite dimensional with dimension n we have

$$\vec{x}(t) = \vec{x}_p(t) + \sum_{i=1}^{n} c_i \vec{x}_i(t).$$

where $B = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is a basis of the null space N(L).

EXERCISES on Fundamental Theory of Systems of ODE'S

EXERCISE #1

(a) Compute $L[\vec{\varphi}]$ if the operator L is defined by $L[\vec{x}] = x'' - P(t)x$ w

$$P(t) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \varphi = \begin{bmatrix} 4e^{t} \\ \sin t \end{bmatrix}$$

(b) Compute $L[\varphi]$ if the operator L is defined by $L[x] = x'' - P(t)x$ where

$$P(t) = \begin{bmatrix} t & 2 \\ e^{t} & 4 \end{bmatrix} \quad \text{and} \quad \varphi = \begin{bmatrix} 2e^{t} \\ \cos t \end{bmatrix}$$

EXERCISE #2. Directly using the Definition (DUD) or by using Theorem 1, prove that the following operators L[y] which map the vector space $C^1(\mathbf{R}^n(I))$ (the set of "vector-valued" function which have a derivative that is continuous on the interval of validity I) into the space $C(\mathbf{R}^n(I))$ of continuous "vector-valued" functions on I are linear.

(a) $L[\vec{x}(t)] = \underset{n\times 1}{\dot{\vec{x}}} - \underset{n\times n}{P}\,\underset{n\times 1}{\vec{x}}$ $\qquad P(t) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ here

(b) $L[\vec{x}(t)] = \underset{n\times 1}{\dot{\vec{x}}} - \underset{n\times n\, n\times 1}{P\,\vec{x}}$ $\qquad P(t) = \begin{bmatrix} t & 2 \\ e^t & 4 \end{bmatrix}$ re

EXERCISE #3. Determine (and prove your answer directly using the definition (DUD), if the following sets are ℓ.i. or ℓ.d in $C^1(\mathbf{R}^n(I)$;

(a) $S = \{ [\, e^t, \sin t, 3t^2]^T [e^{3t}, \sin t, 3t^2]^T \}$ (b) $S = \{ [3e^t, 3 \sin t, 6t^2]^T\ [e^t, \sin t\,, 3t^2]^T \}$

Hint: Since (14) must hold $\forall\, t \in \mathbf{R}$, as your first try, pick several (distinct) values of t to show (if possible) that $c_1 = c_2 = c_3 = ... = c_n = 0$. If (14) must hold $\forall\, t \in \mathbf{R}$ then it must hold for any particular value of t. If this is not possible find $c_1, c_2, c_3, \dots, c_n$ not all zero such that (14) holds $\forall\, t \in \mathbf{R}$. Exhibiting (8) with these values provide conclusive evidence that $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \dots \vec{x}_n\}$ is linearly dependent.

EXERCISE #4. Compute the Wronskian $W(x_1, \dots, x_n; t)$ of the following:

(a) $[e^t, \sin t, 3t^2]^T, [e^{3t}, \sin t, 3t^2]^T, [e^t, e^{-t}, \sin t]$
(b) $[\sin t, \cos t, t], [e^{at}, e^{bt}, 0], [\sin t, 0, 0]$

Recall the homogeneous equation:

$$L[\vec{x}] = 0. \qquad (1)$$

where L is a linear operator of the form

$$L[\vec{x}] = \vec{x}' - P(t)\vec{x} \qquad \forall t \in I = (a,b)$$

We consider the special case where P is a constant matrix. For convenience of notation, we consider:

$$\vec{x}' = A\vec{x} \qquad \forall t \in R \qquad (3)$$

where

$$A = \underset{n\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}} = (a_{ij})$$

Note that since the coefficient matrix is constant, it is continuous for all $t \in \mathbf{R}$ and the interval of validity is the entire real line. Also L maps $\mathcal{A}(\mathbf{R}^n(I)$ back to $\mathcal{A}(\mathbf{R}^n(I)$. Applying a previous theorem we obtain:

THEOREM. Let $S = \{\vec{x}_1, \cdots, \vec{x}_n\}$ be a set of solutions to the homogeneous equation (3). Th the following are equivalent (i.e. they happen at the same time).

a. The set S is linearly independent. (This is sufficient for S to be a basis of the null space N(L) of the linear operator L[y] since the dimension of N(L) is n.)

b. $W(\vec{x}, \ldots, \vec{x}_2, t) \neq 0 \qquad \forall\, t \in R.$

c. All solutions of (3) can be written in the form

$$\vec{x}(x) = c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t) = \sum_{i=1}^{n} c_i\vec{x}_i(t)$$

where c_i,i=1,...,n are arbitrary constants. That is, since S is a basis of N(L) it is a spanning set for N(L) and hence every "vector-valued" function in N(L) can be written as a linear combination of the "vector-valued" functions in S.

We note that by this theorem we have reduced the problem of finding the general solution of the homogeneous equation (3) to the finding of n linearly independent functions $\vec{x}_1,\ldots,\vec{x}_2$. We now develop a technique for solving first order linear homogeneous systems with **constant coefficients** (I.E., P(t) = A is a constant matrix) by finding the "vector-valued" functions $\vec{x}_1,\ldots,\vec{x}_n$.

We "guess" that there may be solutions to (3) of the form

$$\vec{x}(t) = e^{rt}\vec{\xi} \qquad (5)$$

where r is a constant and $\vec{\xi} = [\xi_i,\ldots,\xi_n]^T$ is a constant vector, both to be determined later.

We attempt to determine r and $\vec{\xi}$ by substituting x into the ODE and obtaining a condition on r and $\vec{\xi}$ in order that (3) have a solution of the form (5). Using our standard technique for substituting into a linear equation we obtain:

−A)	$\vec{x}(t)$	$= e^{rt}\ \vec{\xi}$
I)	$\vec{x}'(t)$	$= r\,e^{rt}\ \vec{\xi}$

$$\vec{x}' - A\vec{x} = re^{rt}\vec{\xi} - A(e^{rt}\vec{\xi}) = \vec{0} \qquad \forall t\in R$$

We need to first check that indeed computing the derivative yields $\vec{x}'(t) = re^{rt}\vec{\xi}$

PROOF.

STATEMENT	REASON
$\vec{x}'(t) = \frac{d\vec{x}}{dt}$	definition of notation
$= \frac{d(e^{rt}\vec{\xi})}{dt}$	definition of $\vec{x}$

$= d(e^{rt}[\xi_1, \xi_1, \ldots, \xi_1]^T) / dt$ — definition of $\vec{\xi}$

$= d([e^{rt}\xi_1, e^{rt}\xi_2, \ldots, e^{rt}\xi_n]^T) / dt$ — definition of scalar mult'n

$= [d(e^{rt}\xi_1)/dt, d(e^{rt}\xi_2)/dt, \ldots, d(e^{rt}\xi_n)/dt]^T$ — definition of derivative of a matrix.

$= [re^{rt}\xi_1, re^{rt}\xi_2, \ldots, re^{rt}\xi_n]$ — Theorems from calculus

$= re^{rt}[\xi_1, \xi_2, \ldots, \xi_n]$ — definition of scalar multiplication

$= r\ e^{rt}\ \vec{\xi}$ — definition of $\vec{\xi}$

Next we show that

$$A(e^{rt}\vec{\xi}) = e^{rt}A\vec{\xi} \tag{6}$$

PROOF.

<u>STATEMENT</u>	<u>REASON</u>
$\underset{n\times n}{A}\ (\underset{\substack{\text{scalar}\\ n\times 1}}{e^{rt}}\ \underset{n\times 1}{\vec{\xi}}) = A\left(e^{rt}\begin{bmatrix}\xi_1\\ \xi_2\\ \cdot\\ \cdot\\ \cdot\\ \xi_n\end{bmatrix}\right)$	Definition of $\vec{\xi}$
$= \underset{x\times n}{A}\left(\underset{n\times 1}{\begin{bmatrix}e^{rt}\ \xi_1\\ e^{rt}\ \xi_2\\ \cdot\\ \cdot\\ \cdot\\ e^{rt}\ \xi_n\end{bmatrix}}\right)$	Definition of scalar multiplication.

$$= \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} e^{rt}\,\xi_1 \\ e^{rt}\,\xi_2 \\ \cdot \\ \cdot \\ \cdot \\ e^{rt}\,\xi_n \end{bmatrix}}_{n\times 1}$$

Definition of A

$$= \underbrace{\begin{bmatrix} a_{11}e^{rt}\,\xi_1 + a_{12}\,e^{rt}\,\xi_2 + \cdots a_{1n}\,e^{rt}\,\xi_n \\ a_{21}e^{rt}\,\xi_1 + a_{22}\,e^{rt}\,\xi_2 + \cdots a_{2n}\,e^{rt}\,\xi_n \\ \cdot \quad \cdot \quad \cdots \quad \cdot \\ \cdot \quad \cdot \quad \cdots \quad \cdot \\ \cdot \quad \cdot \quad \cdots \quad \cdot \\ a_{n1}e^{rt}\,\xi_1 + a_{n2}\,e^{rt}\,\xi_2 \quad \cdots a_{nn}\,e^{rt}\,\xi_n \end{bmatrix}}_{n\times 1}$$

Def'n of matrix mult

$$= \underbrace{\begin{bmatrix} e^{rt}\,(a_{11}\,\xi_1 + a_{12}\,\xi_2 + \cdots + a_{1n}\,\xi_n) \\ e^{rt}\,(a_{21}\,\xi_1 + a_{22}\,\xi_2 + \cdots + a_{2n}\,\xi_n) \\ \cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot \\ e^{rt}\,(a_{n1}\,\xi_1 + a_{n2}\,\xi_2 + \cdots + a_{nn}\,\xi_n) \end{bmatrix}}_{n\times 1}$$

Scalar arithmetic.

$$= e^{rt} \begin{bmatrix} (a_{11}\xi_1 + a_{12}\xi_2 + \cdots + a_{1n}\xi_n) \\ (a_{21}\xi_2 + a_{22}\xi_2 + \cdots + a_{2n}\xi_n) \\ \cdot \quad \cdot \quad \cdots \quad \cdot \\ \cdot \quad \cdot \quad \cdots \quad \cdot \\ \cdot \quad \cdot \quad \cdots \quad \cdot \\ (a_{n1}\xi_1 + a_{n2}\xi_2 + \cdots + a_{nn}\xi_n) \end{bmatrix}_{n\times 1}$$ Scalar arithmetic.

$$= e^{rt} \underset{n\times n}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_n \end{bmatrix}}$$ Definition of matrix multiplication.

$$= \underset{\text{scalar}}{e^{rt}} \; \underset{n\times n}{A} \; \underset{n\times 1}{\vec{\xi}} \quad \text{scalar mult.}$$ Definition of A and $\vec{\xi}$

Hence (6) becomes

$$r\,e^{rt} \underset{n\times 1}{\vec{\xi}} = e^{rt} \left(\underset{n\times n}{A} \; \underset{x\times 1}{\vec{\xi}} \right).$$
$$\text{scalar mult.} \qquad \text{scalar mult.}$$

Since $e^{rt} \neq 0$, (it appears in each component of the vector equation on both sides of the equation), we obtain the eigen value problem:

$$A\,\xi = r\,\vec{\xi}.$$ Eigen Value Problem (8)

Hence we have changed the "calculus" problem of solving a first order system of ODE's with constant coefficients to an eigen value problem for a matrix. (Hence no anti derivatives need be computed.) Recall that to solve the eigen value problem for a matrix, we must find the zeros (i.e. the eigen values) of the n^{th} degree polynomial $p(r) = \det(A - rI)$. Thus we obtain the auxiliary equation:

$$p(r) = \det(A - rI) = 0 \quad \underline{\text{Auxiliary Equation}} \tag{9}$$

In accordance with the fundamental theorem of algebra, (9) has "n solutions" i.e. the left hand side (LHS) can be factored into

$$(r - r_1)(r - r_2) \dots (r - r_n) = 0. \tag{10}$$

Hence your factoring skills for higher degree polynomial equations are a must. Two or more of the factors in Equation (8) may be the same. This makes the semantics of the standard verbiage sometimes confusing, but once you understand what is meant, there should be no problem. As in second order equations, we speak of the special case of "repeated roots".)

We illustrate the procedure with an example.

EXAMPLE. Solve $\vec{x}' = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$

SOLUTION. Letting $\vec{x}(t) = e^{rx}\vec{\xi}$, we obtain the eigen value problem $A\vec{\xi} = r\vec{\xi}$

Step 1. Solve the eigen value problem.

Step 1a. Find the eigen values.

$$p(r) = \det(A - rI) = \begin{vmatrix} 1-r & 1 \\ 4 & -2-r \end{vmatrix} = (1-r)(-2-r) - (4)(1) = -2 + r + r^2 - 4$$

$$= r^2 + r - 6 = (r+3)(r-2)$$

$p(r) = 0 \Rightarrow r_1 = 2,\ r_2 = -3$

Step 1b. Find the associated eigen vector. (Let $\vec{\xi} = [\xi_1, \xi_2]^T$)

r = 2 $\Rightarrow$ Solve $A\vec{\xi} = 2\vec{\xi}$ $\Rightarrow$ reduce $A - 2I = \begin{bmatrix} 1-2 & 1 \\ 4 & -2-2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}$

$$R_2 - (-4)R_1 \quad \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow -\xi_1 + \xi_2 = 0 \Rightarrow \xi_1 = \xi_2$$

Hence the infinite family of solutions (the vector in the null space of A - 2I) is given by:

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \xi_2 \end{bmatrix} = \xi_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \text{let} \quad \vec{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

r = -3 $\Rightarrow$ Solve $A\vec{\xi} = -3\vec{\xi} \Rightarrow$ reduce $A + 3I = \begin{bmatrix} 1+3 & 1 \\ 4 & -2+3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$

$$R_2 - (1)R_1 \quad \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow 4\xi_1 + \xi_2 = 0 \Rightarrow \xi_1 = -\xi_2/4$$

Hence the infinite family of solutions (the vector in the null space of A - 2I) is given by:

$$\vec{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -1/4\xi_2 \\ \xi_2 \end{bmatrix} = \xi_2 \begin{bmatrix} -1/4 \\ 1 \end{bmatrix} \Rightarrow \text{let } \vec{\xi}_1 = -4 \begin{bmatrix} -1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

TABLE

Eigen Values	Associated Eigen Vectors (Basis of the null space)
$r_1 = 2$	$\vec{\xi}_1 = [1,1]^T$
$r_2 = -3$	$\vec{\xi}_1 = [1,-4]^T$

Step 2. Write the solution to the system of ODEs.

$$\text{Let } \vec{x}_1 = \vec{\xi}_1 e^{r_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}, \qquad \vec{x}_1 = \vec{\xi}_1 e^{r_2 t} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$$

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$$

Hence, in vector form we have the solution as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + c_2 e^{-3t} \\ c_1 e^{2t} - 4c_2 e^{-37} \end{bmatrix}$$

or in scalar form as

$$x_1 = c_1 e^{2t} + c_2 e^{-3t}$$

$$x_2 = c_1 e^{2t} - 4c_2 e^{-37}$$