A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND
NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED
MATHEMATICIANS

DE CLASS NOTES 3

A COLLECTION OF HANDOUTS ON
SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

CHAPTER 4

Introduction to Systems of ODE's

and Solution Using Elimination

Systems of O.D.E's arise naturally in applications including mass - spring systems, electric circuits, and ecology models such as predator-prey models. Since we have already seen examples of mass-spring systems and electric circuits, we take a brief look at a predator-prey model. Original work in this area was done by Alfred J. Lotka (1880-1949), a biophysicist, and Vito Voltera (1860-1940), an applied analyst.

Predator-Prey Model

$H(t)$ = Prey   (Hunted) $\begin{pmatrix} \text{Rabbits} \\ \text{Foxes} \end{pmatrix}$ $\begin{pmatrix} \text{Sunfish} \\ \text{Bass} \end{pmatrix}$ $\begin{pmatrix} \text{Fish} \\ \text{Sharks} \end{pmatrix}$
$P(t)$ = Predator (Hunter)
Ecosystem   Woods   Pond   Ocean

The nonlinear model developed by Lotka and Voltera is:

$$\frac{dH}{dt} = aH - \alpha HP$$

ODEs

$$\frac{dP}{dt} = -cP + \gamma HP$$

IC's $H(0) = H_0,$ $P(0) = P_0$

The positive constants a, α, c, and γ can be based on empirical observations of the particular species that are being modeled. These equations are nonlinear and nonlinear equations are generally more complicated than linear ones. To keep the model simple and solvable by techniques we already know, we use a "linearized" model. Obviously it does not model "reality" as well as the nonlinear model, but it has some interesting properties. However, use of the results of this model is not recommended.

Linearized Model

$$\frac{dH}{dt} = aH - \alpha P$$

ODEs

$$\frac{dP}{dt} = -cP + \gamma H$$

IC's $H(0) = H_0,$ $P(0) = P_0$

Although the behavior exhibited by the "linearized" model may carry over to the nonlinear model, no such claim is made. Investigation of the nonlinear model and how well the

linearized and the nonlinear model actually models nature is left to you, if you are interested.

The skills concerning ODE systems that you are to master are.
1) Learn to solve an ODE system by elimination, that is, by eliminating one variable to obtain a single second order scalar equation which can be solved by techniques that you already know. This requires that you compute an "equivalent" second order scalar equation.
2) Learn to use matrix notation to rewrite a system of scalar ODEs as a "vector" equation.
3) Learn to convert a second order scalar equation into an "equivalent" first order system which you can write as either a scalar system or a "vector" equation.
4) Learn to solve a "vector" ODE system by using a vector (or matrix) technique.

We do 2) immediately. 1) is in keeping with our policy of applying known techniques to new problems. 4) is covered by homework problems. 3) makes further use of our review and preview of linear theory.

WRITING TWO SCALAR EQUATIONS AS ONE "VECTOR' EQUATION

The linearized scalar predator-prey system

$$\frac{dH}{dt} = a H - \alpha P$$

$$\frac{dP}{dt} = - c P + \gamma H$$

can be written as a "vector" equation as

$$\begin{bmatrix} \dfrac{dH}{dt} \\ \dfrac{dP}{dt} \end{bmatrix} = \begin{bmatrix} a_1 & -b_1 \\ b_2 & -a_2 \end{bmatrix} \begin{bmatrix} P \\ H \end{bmatrix}$$

More generally, we consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Letting $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\dot{\vec{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$, $P = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}$ $\underset{2x1}{\dot{\vec{x}}} = \underset{2x2}{P} \underset{2x1}{\vec{x}}$

$\underset{2x1}{\dot{\vec{x}}} - \underset{2x2}{P} \underset{2x1}{\vec{x}} = \vec{0}$ which is a first order linear homogeneous system.

WRITING A SECOND ORDER EQUATION AS A FIRST ORDER SYSTEM.

Although this may move us further away from an application, a second order scalar ODE may be written as a first order system.

Example. Write $\ddot{z} + 3\dot{z} - z = g(t)$ $z + 3z - 2z = g(t)$ with $\dot{z}$'s $z(0) = 4$ and   (0) =5 as a fir order system.

Solution: Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $x = z$ $\dot{z}$ d $y = $ $\dot{x}$ so $\dot{z}$ at $=$ $= y$.

From the equation, $\ddot{z} = -\dot{z} + 2z + g(t)$. Since $\dot{y} =$ we have $y = -3y + 2x + g(t)$
Hence our system is

$\dot{x} = y$

$\dot{y} = 2x - 3y + g(t)$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

EXAMPLE #1. Solve $\dot{\vec{x}} + A\vec{x} = \vec{g}(t)$

where $A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$ $\vec{g}(t) = \begin{bmatrix} 0 \\ -\alpha \end{bmatrix}$ and .

That is, solve $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\alpha \end{bmatrix}$ First order linear Nonhomo

Writing this vector equation as two scalar equations we obtain

    a) $\dot{x}_1 = x_1 - 2x_2$ $x_1(0) = 2$ (1)

    b) $\dot{x}_2 = x_1 - x_2 - \alpha$ (harvesting) $x_2(0) = 1$ (2)

We eliminate x_2

         STATEMENT   REASON

ODE $\ddot{x}_1 = \dot{x}_1 - 2\dot{x}_2$ Derivative of Eq. a)

        $= \dot{x}_1 - 2(x_1 - x_2 - \alpha)$ Substitute into Eq. b)

        $= \dot{x}_1 - 2x_1 + 2x_2 + 2\alpha$ Algebra

        $= \dot{x}_1 - 2x_1 + (-\dot{x}_1 + x_1) + 2\alpha$ Solve for x_2 in Eq. a) and substitute.

        $\ddot{x}_1 + x_1 = 2\alpha$ Algebra

IC. $x_1(0) = 2$ Given I.C.

        $\dot{x}_1(0) = x_1(0) - 2x_2(0)$ From Eq. a)

           $= 2 - 2(1) = 0$ From given IC's

Equivalent $\ddot{x}_1 + x_1 = 2\alpha$

second order IVP $x_1(0) = 2$, $\dot{x}_1(0) = 0$

Homogeneous Equation. Auxiliary Equation: $r^2 + 1 = 0 \Rightarrow r = \pm i$

             $\Rightarrow x_{1_c} = c_1 \sin t + c_2 \cos t$

Particular Solutions of Nonhomogeneous

1) $x_{1_p} = A$

2) $\dot{x}_{1_p} = 0$

3) $\ddot{x}_{1_p} = 0$

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$\ddot{x}_{1_p} + \dot{x}_{1_p} = A = 2\alpha \Rightarrow x_{1_p} = 2\alpha$

$\Rightarrow x_1 = c_1 \sin t + c_2 \cos t + 2\alpha$

$\dot{x}_1 = c_1 \cos t - c_2 \sin t$

Apply IC.

$2 = c_2 + 2\alpha \qquad \Rightarrow c_1 = 0 , c_2 = 2 - 2\alpha.$

$\left.\begin{array}{c} \\ \\ \end{array}\right\}$

$0 = c_1$

$\Rightarrow x_1 = (2 - 2\alpha) \cos t + 2\alpha$

$\dot{x}_1 = -(2 - 2\alpha) \sin t$

<u>Case 1</u> $\alpha = 0 \Rightarrow x_1 = 2 \cos t$

Case 2 $\alpha = 1 \Rightarrow x = 2$

Case 3 $\alpha = 3 \Rightarrow x_1 = -4 \cos t + 6$

From Eq. a) (Solved for x_2)

$x_2 = \dfrac{1}{2}(\dot{x}_1 + x_1)$

$= \dfrac{1}{2}(-(2 - 2\alpha) \sin t + (2 - 2\alpha) \cos t)2\alpha$

$= (1 - \alpha)(\cos t - \sin t) + \alpha$

$= \sqrt{2}(1 - \alpha) \cos(t - \pi/4)$

Step 1. **UNDERSTAND THE APPLICATION AREA**.

1. We accept the Liebowitz scale as a measure of social anxiety.
   Total is 0- 2(3)(24)= 144   (or Average 0-3)
        Interpretation:
        50-65   0.735294117- 0.955882352   Moderate social phobia
        65-80   0.955882352- 1.323529412   Marked social phobia
        80-95   1.323529412- 1.397058824   Severe social phobia
        >95   >1.397058824   Very severe social phobia
2. We believe that a normal person has some measure, say $u_n>0$, of social anxiety. That is a
   normal person will have some social anxiety. (For computational convenience, we assume
   $u_n =1$ instead of say 0.5)
3. We are looking for a treatment strategy for a temporary stress that causes one anxiety level to
   increase. We wish to know the "best" treatment strategy (e.g., drug dosage).

Step 2. **DEVELOPMENT OF MODEL**.

We wish to develop a model that can be used to evaluate treatment strategies (e.g., drug dosage).

Nomenclature: u_1= Current dosage of drug (e.g., Paxel)
                       u_2 = Social Anxiety of Patient (or average for class)
                       $u_1(0) = u_1^0$ = Initial dosage of drug.
                       $u_2(0) = u_2^0$ = Initial social anxiety.
                       u_n = Normal anxiety level
                       u_0= Initial anxiety level that we wish to bring back to normal
                       a_{11}, a_{12}, a_{21}, a_{12} model parameters, see below.

MODEL:

Scalar Form: $$\frac{du_1}{dt} = - a_{11} u_1 + a_{12} (u_2 - u_n)$$

                      $$\frac{du_2}{dt} = - a_{21} u_1 - a_{22} (u_2 - u_n)$$

                      $u_1(0) = 0$ $u_2(0) = u_0$

Vector Form: $$\frac{d\bar{u}}{dt} = A\bar{u} + \bar{g}_b$$
                      $$\bar{u}(0) = \bar{u}_0$$

where

$$A_{2\times2} = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}, \quad \vec{u}_{2\times1} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \vec{g}_b_{2\times1} = \begin{bmatrix} -a_{12}u_n \\ -a_{22}u_n \end{bmatrix}, \quad \vec{u}_0_{2\times1} = \begin{bmatrix} 0 \\ u_0 \end{bmatrix},$$

Analysis of model:

A. a_{11} and a_{12} are treatment strategy parameters.
  1. $a_{11} > 0$ so that $-a_{11} < 0$ since if the dosage is high (and social anxiety is low) we would
   like to decrease the dosage.
  2. $a_{12} > 0$ since if $u_2 > u_n$, we increase the dosage.
B. a_{21} and a_{22} are physiological constants for the paient (or the general population).
  1. $a_{21} > 0$ so that $-a_{21} < 0$. a_{21} is the effectiveness of the drug; that is, the rate at which
   the drug can control anxiety. If u_1 increases, since $-a_{21} < 0$, u_2 will go down.
  2. $a_{22} > 0$ so that $-a_{22} < 0$. We expect that even if no drug is given, the bodies on
   defenses will cause the anxiety to go down.

Steps 3 and 4. **SOLUTION OF MODEL and INTERPRETATION OF RESULTS**.

$$\frac{du_1}{dt} = -a_{11} u_1 + a_{12}(u_2 - u_n) \qquad a_{11} > 0, \quad a_{12} > 0$$

$$\frac{du_2}{dt} = -a_{21} u_1 - a_{22}(u_2 - u_n) \qquad a_{21} > 0, \quad a_{22} > 0$$

$$u_1(0) = 0 \qquad\qquad u_2(0) = u_0$$

Dosage strategy #1 Constant dosage $(a_{11} = a_{12} = 0)$
  If we assume $a_{11} = a_{12} = 0$, then the ODE's are uncoupled and

$$\frac{du_1}{dt} = 0$$

$$\frac{du_2}{dt} = -a_{21} u_1 - a_{22}(u_2 - u_n) \qquad a_{21} > 0, \quad a_{22} > 0$$

$$u_1(0) = 0 \qquad\qquad u_2(0) = u_0$$

Hence $u_1 =$ constant. However, we do not assume that since $u_1(0) = 0$, that u_1 is always 0. Instead
we assume that there is a jump discontinuity at $t = 0$ to some value u1 to be decided later. It
remains to solve
$$\frac{du_2}{dt} = -a_{21} u_1 - a_{22}(u_2 - u_n) \qquad a_{21} > 0, \quad a_{22} > 0$$

$u_2(0) = u_0$

or

$$\frac{du_2}{dt} + a_{22} u_2 = a_{22} u_n - a_{21} u_1 \qquad a_{21} > 0, \quad a_{22} > 0$$

$u_2(0) = u_0$

Since this problem is now first order linear, we could use an integrating factor. However, we note that it has constant coefficients and the right hand side is "nice". Hence we can (and will) use the techniques we learned for second order ODE's with constant coefficients and "nice" right hand sides.

Homogeneous: $du_2/dt + a_{22} u_2 = 0$.

Auxiliary equation: $r + a_{22} = 0$. Hence $r = -a_{22}$ so that $u_{2c} = c_2 e^{-a_{22}t}$

Particular Solution of Nonhomogeneous: $du_2/dt + a_{22} u_2 = a_{22} u_n - a_{21} u_1$.

Let
$a_{22})$ u_{2p} $= A$.
1) $du_{2p}/dt = 0$.
_____

$du_{2p}/dt + a_{22} u_{2p} = a_{22} A = a_{22} u_n - a_{21} u_1$.

$A = u_n - (a_{21}/a_{22}) u_1$.
$u_2 = u_n - (a_{21}/a_{22}) u_1 + c_2 e^{-a_{22}t}$

We apply the initial condition to obtain c_2.

$u_0 = u_n - (a_{21}/a_{22}) u_1 + c_2 e^{-a_{22}(0)}$
$c_2 = u_0 - u_n + (a_{21}/a_{22}) u_1$

$u_2 = u_n - (a_{21}/a_{22}) u_1 + [u_0 - u_n + (a_{21}/a_{22}) u_1] e^{-a_{22}t}$

Note:

$$\lim_{t \to \infty} u_2 = u_n - (a_{21}/a_{22}) u_1 \neq u_n$$

$\tau = $ Decay time constant $= 1/a_{22}$ ($= \frac{1}{2}$ using the parameters in the specific model that follows).

Suppose $u_2 = u_n$ at $t = t_1$. Then t_1 satisfies

$$u_n = u_n - (a_{21}/ a_{22}) u_1 + [u_0 - u_n + (a_{21}/ a_{22}) u_1] \, e^{-a_{22}t_1}$$

or

$$[u_0 - u_n + (a_{21}/ a_{22}) u_1] \, e^{-a_{22}t_1} = (a_{21}/ a_{22}) u_1$$

$$-a_{22}t_1 = \ln \frac{(a_{21} / a_{22}) u_1}{[u_0 - u_n + (a_{21} / a_{22}) u_1]}$$

$$t_1 = \frac{1}{a_{22}} \frac{[u_0 - u_n + (a_{21} / a_{22}) u_1]}{(a_{21} / a_{22}) u_1}$$

$$t_1 = \frac{1}{a_{22}} \ln [\frac{[u_0 - u_n]}{(a_{21} / a_{22}) u_1} \quad]$$

$$t_1 = \frac{1}{a_{22}} \ln [\frac{a_{22}[u_0 - u_n]}{a_{21} u_1} \quad]$$

General Dosage Strategy for a Specific Model ($a_{11} = 4$, $a_{12} = 1$, $a_{21} = 1$, $a_{22} = 2$)

For convenience of computation, assume the physiological constants are $a_{21} = 1$, $a_{12} = 2$. Also for computational convenience assume $u_n = 1$ and $u_0 = 2$. Now choose $a_{11} = 4$, $a_{12} = 1$. Hence our system becomes:

$$\frac{du_1}{dt} = -4 u_1 + (u_2 - 1)$$

$$\frac{du_2}{dt} = -u_1 - 2 (u_2 - 1)$$

$$u_1(0) = 0 \qquad u_2(0) = 2$$

or, using dot notation

$$\dot{u}_1 = -4 u_1 + u_2 - 1$$
$$\dot{u}_2 = -u_1 - 2 u_2 + 2$$

$$u_1(0) = 0 \qquad u_2(0) = 2$$

Solving the first equation for u_2 we obtain

$$u_2 = \dot{u}_1 \quad + 4\, u_1 \; + 1$$

Once we have solved for u_1, substitution into this equation gives u_2. To obtain $u_2(0)$ we use the first equation directly:

$$\dot{u}_1 \quad (0) \; = -\, 4\, u_1(0) \; + \; u_2(0) - 1 = -\, 4\,(0) + 2 \; - \; 1 = 1$$

Taking the derivative of the first equation and substituting the second equation in for u_2, we obtain:

$$
\begin{aligned}
\ddot{u}_1 \quad &= -\dot{u}_1 \qquad + \dot{u}_2 \; - 4\,\dot{u}_1 \quad + (-\,u_1 \; - 2\,u_2 + 2) \\
&= -\, 4\,\dot{u}_1 \qquad -\, u_1 \; - 2\,u_2 + 2 \\
&= -\, 4\,\dot{u}_1 \qquad -\, u_1 \; - \dot{u}_1 \qquad + 4\,u_1 \; + 1) + 2 \\
&= -\, 4\,\dot{u}_1 \qquad -\, u_1 \; - \dot{u}_1 \qquad -\, 8\,u_1 \; - 2 + 2 \\
&= -\, 6\,\dot{u}_1 \qquad -\, 9u_1
\end{aligned}
$$

or

$$\ddot{u}_1 \quad + \dot{u}_1 \qquad + 9u_1 \; = \; 0$$

with the initial conditions $u_1(0) = 0$ and $\dot{u}_1 \quad (0) = 1$
The auxiliary equation is $r^2 + 6\,r + 9 = 0$ or $(r + 3\,)^2 = 0$
Hence the general solution is

$$u_1 = c_1\, e^{-3\,t} \; + c_2\, t\, e^{-3\,t}$$

$$\dot{u}_1 \quad = -\, 3\,c_1\, e^{-3\,t} \; + c_2\, e^{-3\,t} \; - 3\,c_2\, t\, e^{-3\,t}$$

Applying the initial conditions we obtain

$$0 = c_1\, e^{-3\,(0)} \; + c_2(0)\, e^{-3\,(0)}$$
$$1 = -\, 3\,c_1\, e^{-3(0)} \; + c_2\, e^{-3\,(0)} \; - 3\,c_2\,(0)\, e^{-3\,(0)}$$

or

$$c_1 \qquad = 0$$
$$-\, 3\,c_1 \; + c_2 \; = 1$$

Hence $c_1 = 0$ and $c_2 = 1$ so that

$$u_1 = t\, e^{-3t}$$

$$\dot{u}_1 = e^{-3t} - 3\, t\, e^{-3t}$$

$$u_2 = \dot{u}_1 + 4\, u_1 + 1 = e^{-3t} - 3\, t\, e^{-3t} + 4\, t\, e^{-3t} + 1$$
$$= (1 + t)\, e^{-3t} + 1$$

τ = Decay time constant = $1/3$ (Compare to ½ for the constant dosage model).

$$\lim_{t \to \infty} u_2 = 1 = u_n.$$