A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 3

A COLLECTION OF HANDOUTS ON
SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

CHAPTER 3

Matrix Eigenvalue Problems

1. Eigenvalue Problems for Matrices

2. Hermitian Matrices

3. Basis of Eigenvectors

For a given square matrix, the nonzero vectors $\vec{x}$ and scalars λ such that $A\vec{x} = \lambda\vec{x}$ are special. For example

$$\underset{2x2}{\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}} \underset{2x1}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = 2 \underset{2x1}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \tag{1}$$

$$\underset{2x2}{A} \quad \underset{2x1}{\vec{x}} = \lambda \underset{2x1}{\vec{x}}$$

The proportionality constant λ is called an **eigenvalue** of A and $\vec{x}$ is an **eigenvector** of A associated with the eigenvalue λ. Since we can write (1) as the homogeneous equation.

$$(A - \lambda I)\vec{x} = \vec{0} . \tag{2}$$

the eigenvalues are exactly the solutions of the polynomial equation

$$\det (A - \lambda I) = 0 . \tag{3}$$

which makes the matrix $A-\lambda I$ singular. Recall that (2) has only the trivial solution $\vec{x} = 0$ if $\det(A - \lambda) \neq 0$ so that $A-\lambda I$ is nonsingular. If $\det(A - \lambda_1 I) = 0$, then there are an infinite number of solutions to (2) and that they form a subspace of $\mathbf{R}^n$ (called the **eigenspace** associated with λ_1). By definition, an eigenvector for a given eigenvalue is any nontrivial solution in its eigenspace. Unfortunately, since this conflicts with the formal definition of an eigenvector, for a given eigenvalue $\lambda = \lambda_1$, a basis of this subspace are often referred to as the eigenvectors associated with λ_1. However, like the concept of linear independence, once you understand it, the bad semantics are not a problem.

Solving the eigenvalue problem $A\vec{x} = \lambda\vec{x}$ means finding all eigenvalues and basis sets for their associated eigenspaces. Unfortunately (especially among engineers), this is called finding the eigenvectors. Interestingly, an eigenvalue problem is different from solving $A\vec{x} = \vec{b}$. If the entries in A and $\vec{b}$ are in a field $\mathbf{K}$, then, since only a finite number of field operations are needed for the solution process (Gauss elimination), all solutions will be in $\mathbf{K}^n$. However, the solution of $A\vec{x} = \lambda\vec{x}$ requires the roots 0f of (3) which, even if the entries in A are integers, may be unreal. Hence to solve an eigenvalue problem we must be able to obtain the roots of an n^{th} degree polynomial. Hence we really must use the field $\mathbf{C}$. Eigenvectors in $\mathbf{R}^3$ may have a fgeometric interpretation (e.g., in the design of rotors for motors). However, we are interested in solving $A\vec{x} = \lambda\vec{x}$ in $\mathbf{R}^n$ to be able to use it as part of the solution process for solving a system of linear homogeneous ODE's.

EXAMPLE #1. Solve the eigenvalue problem $A\vec{x} = \lambda\vec{x}$ if $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

Step 1. Find the eigenvalues of A.

Hence

$$p(\lambda) \triangleq \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = 3 - 4\lambda + \lambda^2 + 1$$

$$= \lambda^2 - 4\lambda + 4$$
$$= (\lambda - 2)^2$$

$p(\lambda) = 0 \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2,2$ (repeated real root). We say that the eigenvalue $\lambda = 2$ has **algebraic multiplicity** two. (If $p(\lambda) = (\lambda - \lambda_1)^{n1} \cdots (\lambda - \lambda_i)^{ni} \cdots (\lambda - \lambda_k)^{nk}$, then the algebraic multiplicity of λ_i is n_i.)

Step 2. Find the eigenvectors (i.e., find a basis set for the eigenspace associated with each eigenvalue).

We first solve $(A - 2I)\,\vec{x} = \vec{0}$ where $\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$.

To solve $(A - 2I)\,\vec{x} = \vec{0}$ we use Gauss Elimination to reduce A - 2I. It need not be augmented by the RHS since the problem is homogeneous. The RHS is $\vec{0}$ and any elementary row operation will leave it zero.

$$R_1 + R_2 \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow -x_1 - x_2 = 0 \quad \Rightarrow \quad x_1 = -x_2$$

Hence $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is the general solution of $(A - 2I)\vec{x} = \vec{0}$. Thus there are an infinite number of solutions, all scalar multiples of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. However, all solutions are scalar multiples of the vector $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (we could have chosen $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as well). The dimension of the eigenspace is one. In general, the dimension of the eigenspace is called the **geometric multiplicity** of the eigenvalue.

THEOREM. The algebraic multiplicity of an eigenvalue is always bigger than or equal to the geometric multiplicity.

The "nice" case is when they are equal. Unfortunately, that is not the case in this example.

We record the information in a Table.

TABLE

e-values	Associated e-vectors
$\lambda_1 = \lambda_2 = 2$	$\vec{\mathbf{x}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
(or $\lambda_1 = 0, 0$)	
(This indicates that the algebraic multiplicity is two)	(Unfortunately, since there is only one vector in the basis set for the eigenspace so that the geometric multiplicity is only one.)

EXAMPLE #2. Solve the eigenvalue problem $A\vec{x} = \lambda\vec{x}$ if $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Step 1. Find the eigenvalues of A

$$A - \lambda I = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}. \text{ Hence } p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$p(\lambda) = 0 \Rightarrow \lambda^2+1 = 0 \Rightarrow \lambda = \pm i$.

Step 2. Find the eigenvectors (i.e., find a basis set for the eigenspace associated with each eigenvalue).

We first solve $(A - iI)\,\vec{x} = \vec{0}$ where $A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$.

To solve $(A - iI)\,\vec{x} = \vec{0}$ we use Gauss Elimination to reduce A - iI. It need not be augmented by the RHS since the problem is homogeneous. The RHS is $\vec{0}$ and any elementary row operation will leave it zero.

$$R_2 - \mathbf{i}R_1 \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \Rightarrow \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow -i\,x_1 - x_2 = 0 \quad \Rightarrow \quad x_2 = -i\,x_2$$

Hence $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is the general solution of $(A - iI)\vec{x} = \vec{0}$. Thus there are an infinite number of solutions, all scalar multiples of the vector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

If A has all real entries, then $p(\lambda) = \det(A - \lambda I)$ has all real coefficients so that unreal roots of $(A - \lambda I)\vec{x} = \vec{0}$ come in complex conjugate pairs. Furthermore, since A is real, if $\lambda = \mu+i\nu$ is an eigenvalue with eigenvector $\vec{\xi} = \vec{a} + i\vec{b}$, then $\vec{\xi} = \vec{a} - i\vec{b}$ is an eigenvalue for $\lambda = \mu-i\nu$. Hence $\begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector associated with $-i$. We record the information in a Table.

TABLE

e-values	Associated e-vectors
$\lambda_1 = i$	$\vec{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$
$\lambda_2 = -i$	$\vec{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

DEFINITION #1. A square matrix $\underset{n\text{x}n}{A} \in \mathbf{C}^{n\times n}$ is called Hermitian (or self-adjoint) if $A = A^*$ where

$\mathbf{A}^* = \overline{\mathbf{A}}^{\mathbf{T}}$ (i.e. the transpose of the complex conjugate of A).

THEOREM #1. If $A\in\mathbf{C}^{n\times n}$ is Hermitian (self-adjoint), then

1. All eigenvalues are real.
2. There exist a full set of n linearly independent eigenvectors for A which form a basis for $\mathbf{C}^n$ (if A is real, the eigenvectors can be chosen to be real and hence form a basis for $\mathbf{R}^n$).
3. If $\vec{x}$ and $\vec{y}$ are eigenvectors for different eigenvalues, then they are orthogonal (perpendicular), so that $(\vec{x},\vec{y}) = \overline{\vec{x}}^{\,T}\vec{y} = \vec{0}$
4. If a given eigenvalue λ has more than one linearly independent eigenvector associated with it (we say λ has geometric multiplicity m) then they can be chosen to be orthogonal.

COROLLARY #1. If $\underset{m\text{x}n}{A} \in\mathbf{R}^{n\times n} \subseteq \mathbf{C}^{n\times n} = \mathbf{R}^{n\times n}+i\mathbf{R}^{n\times n}$ is real and symmetric, it is Hermitian (self-adjoint) and its eigenvalues are real. Also, it has a full set of eigenvectors which may be chosen to be orthogonal.

In an appropriate setting, the real matrices associated with linear circuits and systems of linear springs are symmetric and hence have real eigenvalues. They could also be positive or negative definite indicating exponential decay. Nonreal eigenvalues may indicate pure oscillations or damped oscillations.

Let T:$\mathbf{R}^n \to \mathbf{R}^n$ be defined by T($\vec{x}$) = A $\vec{x}$. A matrix is nice if it has a full set of n linearly independent eigenvectors that form a basis of $\mathbf{R}^n$. Recall that B = $\{\hat{e}_1,\ldots,\hat{e}_n\}$ where $\hat{e}_i = \underbrace{[0,\ldots,0,1,0,\ldots,0]^T}_{\longleftarrow\cdots\, i^{th}\text{ slot}\cdots\longrightarrow}$ is a basis of $\mathbf{R}^{n.}$ If $\vec{x} = [x_1, x_2, \ldots, x_n] \in \mathbf{R}^n$, then x_1, x_2, ..., x_n are the coordinates of $\vec{x}$ with respect to this basis. If $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are a a full set of n linearly independent eigenvectors that form a basis of $\mathbf{R}^n$, then the coordinates of $\vec{x}$ with respect to this basis can be obtained from the set of linear algebraic equations

$$c_1\vec{x}_1 + \cdots + c_n x_n = \vec{x} \ .$$

Recall that if A is Hermitian (self-adjoint) then it has a full set of eigenvectors that are orthogonal. The coordinates of an orthogonal basis are particularly easy to obtain. Let B = $\{\vec{x}_1, \cdots, x_n\}$ be an orthogonal basis. so that any vector $\vec{x}$ can be written as

$$\vec{x} = c_1\vec{x}_1 + \cdots + c_n x_n \ . \ . \tag{1}$$

We obtain the coordinate c_j by taking the dot product of (1) with $\vec{x}_j$.

$$(x_j, \vec{x}) = (x_j, c_1\vec{x}_1 + \cdots + c_n x_n) \ =. \ c_{(1}(\vec{x}_j, \vec{x}_1) + \cdots + c_n(\vec{x}_j x_n) \quad = c_{(1}(\vec{x}_j, \vec{x}_j)$$

Hence $c_j = \dfrac{(x_j, x)}{\|x_j\|^2}$. We might call these the Fourier Series coefficients for this basis.