A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 3

A COLLECTION OF HANDOUTS ON
SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

CHAPTER 2

Introduction To Autonomous Systems

1. The General Autonomous System

2. The Solution Process for the General Autonomous System

Statics and Dynamics-I THE GENERAL AUTONOMOUS SYSTEM
Handout #1 Prof. Moseley

Suppose V is a real vector space which we think of as a state space. Now let V(I) = {x(t):I→V}=$\mathcal{F}$ (I,V) where I = (a,b)⊆**R**. That is, V is the set of all "vector valued" functions on the open interval I. We allow the state of our system to vary with time. To make V(I) into a vector space, we must equip it with a set of scalars, vector addition, and scalar multiplication. The set of scalars for V(I) is the same as the scalars for V (i.e.,**R**). Vector addition and scalar multiplication are simply function addition and scalar multiplication of a function. To avoid introducing to much notation, the engineering convention of using the same symbol for the function and the dependent variable will be used (i.e., instead of y = f(x), we use y = y(x)). Hence instead of $\vec{u} = \vec{f}(t)$, for a function in V(I), $\vec{u} = \vec{u}(t)$. The context will explain whether $\vec{u}$ is a vector in V or a function in V(I).

1) If $\vec{u}$, $\vec{v}$ ∈V(I), then we define $\vec{u} + \vec{v}$ pointwise as $(\vec{u} + \vec{v})(t) = \vec{u}(t) + \vec{v}(t)$.

2) If $\vec{u}$ ∈V(I) and α is a scalar, then we define $(\alpha\vec{u})(t)$ ∈V(I) pointwise as $(\alpha\vec{u})(t) = \alpha(\vec{u}(t))$.

We use the notation V(t) instead of V(I), when, for a math model, the interval of validity is unknown and hence part of the problem. Since V is a real vector space, so is V(t). V(t) can then be embedded in a complex vector space as previously described.

To define limits and hence derivatives in an abstract time varying vector space, we need more structure. Suppose V is an **inner product space**. Then it will have an induced **norm** which induces a **metric** and hence a **topology**. Since the real numbers are a field with absolute value, the definition of the limit of a "vector" valued function of t, $\vec{u}(t)$, as t approaches t_0, $\lim_{t\to t_0} \vec{u}(t)$, can be defined Then the derivative as the limit of the difference quotient can be defined, $\lim_{t\to t_0} \frac{\vec{u}(t) - \vec{u}(t_0)}{t - t_0}$. We assume that V is a Hilbert Space (i.e. a complete inner product space with R as the scalars. The key idea is that V may finite dimensional (i.e., $\mathbf{R}^n$) or infinite dimensional (which is more complicated). Now let T:V→V be a (possibly linear) operator. Then the following problem makes sense:

ODE : $\frac{d\vec{u}}{dt} + T(\vec{u}) = \vec{b} + \vec{g}(t)$.

IC: $\vec{u}(t_0) = \vec{u}_0$

Our general strategy for solving this problem is as before: First find the general solution of the ODE and then apply the initial condition to obtain the unique soultion of the IVP. However, our strategy for obtaining the general solution to the ODE does not involve a sequence of equivalent problem; it uses the linear theory. Thus we will consider the form of the general solution as dictated by the linear theory and then determine strategies for finding these terms.

We think of the operator T as defining the system. It is **autonomous** as it does not depend on time. We give two examples of autonomous systems. These will involve different notation.

EXAMPLE #1. Discrete. (e.g., Circuits or Springs) Let V = $\mathbf{R}^{n+2}$,
$\vec{x} = [x_0, x_1, x_2, \ldots, x_n x_{n+1}]^T$ $\vec{b} = [0, b_1, b_2, \ldots, b_n, 0]^T$,
$\vec{g}(t) = [0, g_1(t), g_2(t), \ldots, g_n(t), 0]^T$ $\vec{x}$ $\in$V(t) = { $\in \mathbf{R}^{n+2}$(t): $x_0 = 0$, x_n
= [A ($\mathbf{R}$,$\mathbf{R}$)]^n and A$\in \mathbf{R}^{n+2xn+2}$ where the first and last rows of A are all zeros.. T($\vec{x}$) = $\vec{x}$, T:
$\mathbf{R}^{n+2} \to \mathbf{R}^{n+2}$.

ODE : $L[\vec{x}] = \frac{d\vec{x}}{dt} - A\vec{x} = \vec{b} + \vec{g}(t)$.
BC: $x_0 = T_1$, $x_{n+1} = T_2$
IC: $\vec{x}(t_0) = \vec{x}_0$

EXAMPLE #2. Continuum.(Heat conduction in a rod.) Let V = {u$\in H$(I,$\mathbf{R}$):u(0) = 0, u(ℓ) = 0},u,

b(x), g(x,t)$\in$V(t) = A ($\mathbf{R}$,H(I,$\mathbf{R}$)) = A (I$\times\mathbf{R}$,$\mathbf{R}$)$\cap$C($\bar{I}$ $\times\mathbf{R}$, $\mathbf{R}$). T(u(x)) = $-\frac{d^2u}{dt^2}$, T:V$\to$V.

ODE : $L[u] = \frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial t^2} = b(x) + g(x,t)$.
BC: u(0,t) = T_1, u(ℓ,t) = T_2
IC: u(x,t) = u_0(x)

Statics and Dynamics-I THE SOLUTION PROCESS
Handout #2 FOR THE GENERAL AUTONOMOUS SYSTEM Prof. Moseley

We consider a solution strategy for the problem:

ODE : $L[\vec{u}] = \frac{d\vec{u}}{dt} + T(\vec{u}) = \vec{b} + \vec{g}(t)$.

IC: $\vec{u}(t_0) = \vec{u}_0$

The state space V is a (real) Hilbert space. T:V→V is a linear operator. In applications T may not be linear and we might linearize at an equilibrium point.

<u>THEOREM</u>. If T:V→V is linear, then L:V(t)→V(t) is linear.

Since L is linear, we apply the linear theory to obtain the form of the general solution to the ODE as

$$\vec{u}(t) = \vec{u}_c(t) + \vec{u}_{p_b}(t) + \vec{u}_{p_g}(t)$$

where 1. $\vec{u}_c(t)$ is the general solution of the homogeneous equ $L[\vec{u}] = \frac{d\vec{u}}{dt} + T(\vec{u}) = \vec{0}$

2. $\vec{u}_{p_b}(t)$ is any particular soluti $L[\vec{u}] = \frac{d\vec{u}}{dt} + T(\vec{u}) = \vec{b}$

3. $\vec{u}_{p_g}(t)$ is any particular solut $L[\vec{u}] = \frac{d\vec{u}}{dt} + T(\vec{u}) = \vec{g}(t)$

We consider general strategies for finding $\vec{u}_c(t)$ $\vec{u}_{p_b}(t)$ $\vec{u}_{p_g}(t)$, and .

1. To find solutions of the homogeneous equation $L[\vec{u}] = \vec{0}$ $\vec{u}(t) = \vec{\xi} e^{-rt}$.

Substituting into the ODE we obtain $L[\vec{\xi} e^{-rt}] = \frac{d\vec{\xi} e^{-rt}}{dt} + T(\vec{\xi} e^{-rt}) = \vec{0}$ so that

$-re^{-rt}\vec{\xi} + e^{-rt}T(\vec{\xi}) = \vec{0}$ $T(\vec{\xi}) = r\vec{\xi}$ and .

Hence we are interested in the spectral (eigenvalue) problem.

2. Since the forcing function $\vec{b}$ is independent of time, to find a particular solution of the nonhomogeneous equation $L[\vec{u}] = \frac{d\vec{u}}{dt} + T(\vec{u}) = \vec{b}$ $\vec{u}_{p_b}(t) = \vec{u}_e$ assume is th solution and hence is independent of time. Hence $\vec{u}_e$ is the unique solution ($T(\vec{u}) = \vec{b}$,

3. For some $\vec{g}(t)$'s we may be able to guess the $\vec{u}_{p_g}(t)$.