A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 3

A COLLECTION OF HANDOUTS ON
SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

CHAPTER 1

Vector Valued Functions of One-Variable (Time)

Physical quantities such as velocity and force are considered to be "vectors" since they have magnitude and direction in three dimensional physical space. Hence it is standard to model them as elements in $\mathbf{R}^3$. However, they often vary with time (and space). This leads to consideration of time varying vector-valued functions of the form

$$\vec{v} = \vec{v}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = [x(t), y(t), z(t)]^T \qquad \vec{v} \qquad \in \mathbf{R}$$

where I =(a,b). (We use the transpose notation to save space.)

More generally, if we consider a system with n state variables (e.g., several particles, concentrations of several chemicals in a chemical reactor, or several species in an eco-system), we must consider "vector"-valued functions of time in the form (here the word "vector" refers to the fact that we are considering column vectors or "n-tuples" of functions rather than that they are elements in an abstract vector space):

$$\vec{x} = \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix} \qquad \vec{x}\,\mathbf{R}^n(I) = \mathscr{F}(I,\mathbf{R}^n) = \{ \quad : I \rightarrow \mathbf{R}^n\}$$

where I =(a,b) as well as matrix-valued functions of time the form:

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdot & \cdot & \cdot & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdot & \cdot & \cdot & a_{2n}(t) \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m1}(t) & a_{m2}(t) & \cdot & \cdot & \cdot & a_{mn}(t) \end{bmatrix} \qquad \in \mathbf{R}^{m\times n}(I) = \mathscr{F}$$

since our system or model may also vary with time.

Even more generally, recall the example of time varying vectors. Suppose V is a real vector space (which we think of as a state space). Now let V(I) = {x(t):I→V}=F (I,V) where I = (a,b)⊆**R**. That is, V is the set of all "vector valued" functions on the open interval I. (Thus we allow the state of our system to vary with time.) To make V(I) into a vector space, we must equip it with a set of scalars, vector addition, and scalar multiplication. The set of scalars for V(I) is the same as the scalars for V (i.e.,**R**). Vector addition and scalar multiplication are simply function addition and scalar multiplication of a function. To avoid introducing to much notation, the engineering convention of using the same symbol for the function and the dependent variable will be used (i.e., instead of y=f(x), we use y=y(x)). Hence instead of $\vec{x} = \vec{f}(t)$, for a function in V(I), we use $\vec{x}$ $\vec{x}$ (t). The context will explain whe$\vec{x}$er is a vector in V or a function in V(I).

1) If $\vec{x}$ $\vec{y}$ $\in$V(I), then we de$\vec{x}$ ne$\vec{y}$ + point$\vec{x}$ s$\vec{y}$ s , ($\vec{x}$ + $\vec{y}$ (t) = (t) + (t).

2) If $\vec{x}$ $\in$V(I) and α is a scalar, then we define $\vec{x}$)(t) $\in$V(I) pointwise $\vec{x}$ (α $\vec{x}$) = α (t).

The proof that V(I) is a vector space is left to the exercises. We use the notation V(t) instead of V(I), when, for a math model, the interval of validity is unknown and hence part of the problem. Since V is a real vector space, so is V(t). V(t) can then be embedded in a complex vector space as described above.

To define limits and hence derivatives in an abstract time varying vector space, we need more structure. Suppose V is an **inner product space**. Then it will have an induced **norm** which induces a **metric** and hence a **topology**. Since the real numbers are a field with absolute value, the definition of the limit of a "vector" valued function of t, $\vec{x}(t)$, as t approaches t_0, $\lim_{t\to t_0} \vec{x}(t)$, can be defined Then the derivative as the limit of the difference quotient can be defined, $\lim_{t\to t_0} \dfrac{x(t)-x(t_0)}{t-t_0}$. For $\mathbf{R}^n$ we have an inner product. Also, there are several way define a norm on $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$. Rather than carry out this long process, for $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$ it is much simpler to just define the derivative componentwise. For time varying vectors in $\mathbf{R}^n$ this leads to the subspaces

$$\mathcal{A}(\mathbf{R}^n(I)) = \mathcal{A}(I,\mathbf{R}^n)=\{\vec{x} = [x_1,...,x_n]^T:I\to\mathbf{R}^n \,|\, x_i \text{ is analytic}\}\subseteq C(\mathbf{R}^n(I)) = C(I,\mathbf{R}^n)$$
$$=\{\vec{x} = [x_1,...,x_n]^T:I\to\mathbf{R}^n \,|\, x_i \text{is continuous}\}\subseteq\mathbf{R}^n(I) = \mathscr{F}(I,\mathbf{R}^n) \vec{x} :I\to\mathbf{R}^3\}$$

We leave it to future study to show that the usual inner product on $\mathbf{R}^n$ and any norm on $\mathbf{R}^{m\times n}$ will in fact result in componentwise differentiation.

<u>DEFINITION #1</u>. If $\vec{x}$ and A are as given in (2) and (3), th $\dfrac{dA}{dt} = A' = \dot{A} = \left[\dfrac{da_{ij}}{dt}\right]$

$$\frac{d\vec{x}}{dt} = \vec{x}\,' = \dot{\vec{x}} = \left[\frac{dx_i}{dt}\right] .$$

That is, for “vectors” and matrices we compute derivatives (and integrals) componentwise. We state one theorem on the properties of derivatives of vector and matrix valued functions.

<u>THEOREM #1</u>. Let A,B $\in$ $\mathbf{R}^{m\times n}(I)$, $\bar{x}$ $\bar{y}$ $\in$ $\mathbf{R}^n(I)$, and c $\in$ $\mathbf{R}$. Assuming all derivatives exist,

$$\frac{d(cA)}{dt} = c\frac{dA}{dt}, \qquad \frac{d(c\bar{x})}{dt} = c\frac{d\bar{x}}{dt},$$

$$\frac{d(A+B)}{dt} = \frac{dA}{dt} + \frac{dB}{dt} \qquad \frac{d(\bar{x}+\bar{y})}{dt} = \frac{d\bar{x}}{dt} + \frac{d\bar{y}}{dt},$$

and $\frac{d(AB)}{dt} = A\frac{dB}{dt} + \frac{dA}{dt}B$.

Handout #2 LINEAR INDEPENDENCE OF VECTOR VALUED FUNCTIONS OF ONE VARIABLE (TIME) Prof. Moseley

It is important that you understand the definition of linear independence in an abstract vector space.

DEFINITION #1. Let V be a vector space. A finite set of vectors $S = \{\vec{x}_1,\ldots,\vec{x}_k\} \subseteq V$ is **linearly independent** (ℓ.i.) if the only set of scalars $c_1, c_2, \ldots, c_k$ which satisfy the (homogeneous) vector equation

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{0}$$

is $c_1 = c_2 = \cdots = c_n = 0$; that is, (1) has only the trivial solution. If there is a set of scalars not all zero satisfying (1) then S is **linearly dependent** (ℓ.d.).

DEFINITION #2. Let $\vec{f}_1, \ldots, \vec{f}_k \in \mathscr{F}(I,\mathbf{R}^n)$ where $I = (a,b)$. Now let $J = (c,d) \subseteq (a,b)$ and for $i = 1,\ldots,k$, denote the restriction of $\vec{f}_i$ to J by the same symbol. Then we say that $S = \{\vec{f}_1, \ldots, \vec{f}_k\} \subseteq \mathscr{F}(J,\mathbf{R}^n) \subseteq \mathscr{F}(I,\mathbf{R}^n)$ is **linearly independent** on J if S is linearly independent as a subset of $\mathscr{F}(J,\mathbf{R}^n)$. Otherwise S is **linearly dependent** on J.

Applying Definitions #1 and 2 to a set of k functions in the function space $C^1(I,\mathbf{R}^n) = \{\vec{x} = [x_1,\ldots,x_n]^T : I \to \mathbf{R}^n : \frac{dx_i}{dt}$ exists and is continuous$\}$ we obtain:

THEOREM #1. The set $S = \{\vec{f}_1, \ldots, \vec{f}_k\} \subseteq C^1(I,\mathbf{R}^n)$ where $I = (a,b)$ is **linearly independent** on I if (and only if) the only solution to the equation

$$c_1\vec{f}_1(t) + \cdots + c_k\vec{f}_k(t) = 0 \qquad \forall\, t \in I \tag{1}$$

is the trivial solution $c_1 = c_2 = \cdots = c_k = 0$ (i.e., S is a **linearly independent set** in the vector space $C^1(I,\mathbf{R}^n)$). If there exists $c_1, c_2, \cdots, c_n \in \mathbf{R}$, not all zero, such that (1) holds, (i.e, there exists a nontrivial solution) then S is **linearly dependent** on I (i.e., S is a **linearly dependent set** in the vector space $C^1(I,\mathbf{R}^n)$ which is a subspace of $\mathscr{F}(I,\mathbf{R}^n)$).

Often people abuse the definition and say the functions in S are linearly independent or linearly dependent on I rather than the set S is linearly independent or dependent. Since it is in general use, this abuse is permissible, but not encouraged as it can be confusing. Note that Eq. (1) is really an infinite number of equations in the two unknowns c_1 and c_2, one for each value of x in the interval I. Four theorems are useful.

THEOREM #2. If a finite set $S \subseteq C^1(I,\mathbf{R}^n)$ where $I = (a,b)$ contains the zero function, then S is linearly dependent on I.

<u>THEOREM #3</u>. If f is not the zero function, then $S = \{\bar{f}\} \subseteq C^1(I,\mathbf{R}^n)$ is linearly independent.

<u>THEOREM #4</u>. Let $S = \{\bar{f}, \bar{g}\} \subseteq C^1(I,\mathbf{R}^n)$ where $I = (a,b)$. If either $\bar{f}$ or $\bar{g}$ is the zero "vector" in $C^1(I,\mathbf{R}^n)$ (i.e., is zero on I), then S is linearly dependent on I.

<u>THEOREM #5</u>. Let $S = \{\bar{f}, \bar{g}\} \subseteq C^1(I,\mathbf{R}^n)$ where $I = (a,b)$ and suppose neither $\bar{f}$ or $\bar{g}$ is the zero function. Then S is linearly dependent if and only if one function is a scalar multiple of the other (on I).

<u>PROCEDURE</u>. To show that $S = \{\bar{f}_1, \ldots, \bar{f}_k\}$ is linearly independent it is standard to assume (1) and try to show $c_1 = c_2 = c_3 = \ldots = c_k = 0$. If this can not be done, to show that S is linearly dependent, it is mandatory that a nontrivial solution to (1) be exhibited.

<u>EXAMPLE #1</u>. Determine (using DUD) if $S = \{ [e^t, \sin t, 3t^2]^T, [e^{3t}, \sin t, 3t^2]^T \}$ is linearly independent. (We prefer column vectors, but use the transpose notation to save space.)

Proof. (This is not a yes-no question). We assume

$$c_1 \begin{bmatrix} e^t \\ \sin t \\ t^2 \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ \sin t \\ 3t^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \forall\ t \in \mathbf{R}.$$

and try to solve. Note that, in this context, the zero vector is the zero function for all three components defined by

$$\bar{0}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \forall\ t \in \mathbf{R}. \tag{3}$$

The one "vector equation" (2) can be written as the three scalar equations

$$\begin{aligned} c_1\ e^t &+ c_2\ e^{3t} = 0 \\ c_1\ \sin t &+ c_2\ \sin t = 0 \\ c_1\ t^2 &+ c_2\ 3t^2 = 0 \end{aligned} \qquad .\ \forall\ t \in \mathbf{R} \tag{4}$$

Since these equations must hold $\forall\ t \in \mathbf{R}$, it is really an infinite number of algebraic equations (there are an infinite number of values of t) in the two unknowns c_1 and c_2. Intuitively, unless we are very lucky, the two unknowns, c_1 and c_2, can not satisfy an infinite number of equations . To show this, we simply select two equations (i.e. values of t) that are "independent". Choosing t = 0 and t = 1 (so as to make the algebra easy) we obtain.

$$\begin{aligned} c_1 \; e^0 \quad &+ \quad c_2 \; e^{3(0)} &= 0 \\ c_1 \; \sin 0 \quad &+ \quad c_2 \; \sin 0 &= 0 \\ c_1 \; (0)^2 \quad &+ \quad c_2 \; 3\,(0)^2 &= 0 \end{aligned} \tag{5}$$

$$\begin{aligned} c_1 \; e^1 \quad &+ \quad c_2 \; e^{3(1)} &= 0 \\ c_1 \; \sin (1) \quad &+ \quad c_2 \; \sin (1) &= 0 \\ c_1 \; (1)^2 \quad &+ \quad c_2 \; 3\,(1)^2 &= 0 \end{aligned} \tag{6}$$

or if we simplify

$$\begin{aligned} c_1 \quad &+ \quad c_2 &= 0 \\ 0 \quad &+ \quad 0 &= 0 \\ 0 \quad &+ \quad 0 &= 0 \end{aligned} \tag{7}$$

$$\begin{aligned} c_1 \; e \quad &+ \quad c_2 \; e^3 &= 0 \\ c_1 \; \sin (1) \quad &+ \quad c_2 \; \sin (1) &= 0 \\ c_1 \quad &+ \quad c_2 \; 3 &= 0 \end{aligned} \tag{8}$$

Note that the second and third equation in the first set yield 0 = 0. This is obviously true, but is not helpful in showing $c_1 = c_2 = 0$. Also, we can divide by e in the first equation in the second set and by $\sin(1) \neq 0$ in the second equation in the second set. Ignoring 0=0 we obtain:

$$\begin{aligned} c_1 \quad &+ \quad c_2 &= 0 \\ c_1 \quad &+ \quad c_2 \; e^2 &= 0 \\ c_1 \quad &+ \quad c_2 &= 0 \\ c_1 \quad &+ \quad c_2 \; 3 &= 0 \end{aligned} \tag{9}$$

Hence for S to be linearly independent, we need only show that these equations imply $c_1 = c_2 = 0$. We could use Gauss elimination, but for relatively simple equations examination may be faster. Note that the first and third are the same and yield $c_2 = -c_1$. Substituting into the last equation we obtain $c_1 + 3\,(-c_1) = 0$. Hence $-2\,c_1 = 0$. Thus $c_1 = 0$ and hence $c_2 = -c_1 = 0$. Since we have proved that the only solution to the "vector" equation (2) is the trivial solution $c_1 = c_2 = 0$, the set S is linearly independent.