A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 2

A COLLECTION OF HANDOUTS ON SCALAR LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODE"s)

CHAPTER 9

Inverse Transform and

Solution to the Initial Value Problem

We wish to establish a subspace of **T** where the Laplace Transform is a one-to-one mapping. This means that different functions f(t) get mapped (or transformed) into different functions F(s). Specifically, if $f \neq g$, then $\mathcal{L}\{f\} \neq \mathcal{L}\{g\}$. The contrapositive of this statement, which is equivalent, is that if $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, then f = g. This is the standard definition of one-to-one. For a linear mapping between vector spaces, there is an easy test to determine if a mapping (like the Laplace Transform) is or is not one-to-one (1-1).

<u>THEOREM.</u> If T is a linear operator from the vector space V to the vector space W and its null space N_T is $\{\vec{0}\}$, then T is a one-to-one mapping; that is, if $\vec{x}, \vec{y} \in V$ and $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$.

<u>PROOF:</u> We start by proving the following:

<u>Lemma.</u> If N_T is $\{\vec{0}\}$ and $T(\vec{x}) = \vec{0}$, then $\vec{x} = \vec{0}$.

<u>Proof of lemma:</u> We begin by recalling the definition of the null space N_T.

<u>DEFINITION</u> (Null Space). The null space is the set of all vectors $\vec{x}$ that satisfy the linear homogeneous equation $T(\vec{x}) = \vec{0}$; that is $N_T = \{\vec{x} \in V: T(\vec{x}) = \vec{0}\}$.

Thus if $T(\vec{x}) = \vec{0}$, then by the definition of N_T we have that $\vec{x} \in N_T$. However, we also assume that $N_T = \{\vec{0}\}$. Hence since $\vec{x} \in N_T$, we have that $\vec{x} = \vec{0}$.

QED for lemma.

Rest of proof of theorem (For variety, this time we write the proof in paragraph form): Now assume that $\vec{x}, \vec{y} \in V$ and $T(\vec{x}) = T(\vec{y})$. Hence we have that $T(\vec{x}) - T(\vec{y}) = \vec{0}$. Hence since T is a linear operator we have that $T(\vec{x} - \vec{y}) = \vec{0}$. Since we also are assuming that $N_T = \{\vec{0}\}$, we have by the lemma that $\vec{x} - \vec{y} = \vec{0}$. Hence $\vec{x} = \vec{y}$. Hence T is a one-to-one mapping.

QED for Theorem.

Recall that $C[0,\infty)$ denotes the set of continuous functions on $[0.\infty)$.

LERCH'S THEOREM. If $f,g \in C[0,\infty)$, and $\mathcal{L}\{ f(t) \}(s) = \mathcal{L}\{ g(t) \}(s)\ \forall\ s > a$, then $f(t) = g(t)\ \forall\ t \geq 0$.

Lerch's theorem says that the Laplace transform is a one-to-one mapping on the vector space $C[0,\infty)$ and hence has only the zero function in its null space. Different functions in $C[0,\infty)\cap$**T** are mapped to different functions in the space of transforms **F**. Thus our table gives the unique transform for each of the continuous functions in the table. Although it is not true that $\mathcal{L}$ is one-to-one on all functions in **T**, if we "throw out" the "null functions", it is one-to-one. For example, $\mathcal{L}$ is one-to-one on $\mathbf{T}_{pcexp} = PC_a[0,\infty) \cap E_{xp}$, the set of all piecewise continuous functions of exponential order where we have taken the average value at the points of discontinuity. Since $\mathcal{L}$ is a one-to-one mapping on $\mathbf{T}_{pcexp}$, its inverse mapping exists. (Similar to limiting the domain of sin(x) so that we can define $\mathrm{Sin}^{-1}(x)$, but the sets we are mapping from and to are sets of functions, not sets of numbers.) We denote the inverse Laplace Transform of a function F(s) by $f(t) = \mathcal{L}^{-1}\{ F(s) \}$.

THEOREM. The inverse Laplace Transform is linear; that is,
$\mathcal{L}^{-1}\{ c_1 F(s) + c_2 G(s) \} = c_1 \mathcal{L}^{-1}\{ F(s) \} + c_2 \mathcal{L}^{-1}\{ G(s) \}$.

PROOF. Let $F(s) = \mathcal{L}\{ f(t) \}$ and $G(s) = \mathcal{L}\{ g(t) \}$ so that, since $\mathcal{L}$ is one-to-one we have $f(t) = \mathcal{L}^{-1}\{ F(s) \}$ and $g(t) = \mathcal{L}^{-1}\{ G(s) \}$. Then by the linearity of $\mathcal{L}$ we have

$$\mathcal{L}\{ c_1 f(t) + c_2 g(t) \} = c_1 \mathcal{L}\{ f(t) \} + c_2 \mathcal{L}\{ g(t) \}.$$

Since $\mathcal{L}$ is one-to-one, we have

$$\mathcal{L}^{-1}\{ c_1 \mathcal{L}\{ f(t) \} + c_2 \mathcal{L}\{ g(t) \} \} = c_1 f(t) + c_2 g(t)$$

Rewriting this equation we have

$$\mathcal{L}^{-1}\{ c_1 F(s) + c_2 G(s) \} = c_1 \mathcal{L}^{-1}\{ F(s) \} + c_2 \mathcal{L}^{-1}\{ G(s) \}$$

as required.

Q.E.D.

EXAMPLE #3. If $F(s) = \dfrac{s}{s^2 - 2s + 3}$, find $f(t) = \mathcal{L}^{-1}\{ F(s) \}$.

Solution.

$$\frac{s}{s^2 - 2s + 3} = \frac{s}{s^2 - 2s + 1 + 3 - 1} = \frac{s}{(s-1)^2 + 2}$$

From the table we recall that

$$\mathcal{L}\{ e^{at} \sin(\omega t) \} = \frac{\omega}{(s-a)^2 + \omega^2}, \qquad \mathcal{L}\{ e^{at} \cos(\omega t) \} = \frac{s-a}{(s-a)^2 + \omega^2}$$

So
$$\frac{s}{s^2 - 2s + 3} = \frac{s}{s^2 - 2s + 1 + 3 - 1} = \frac{s-1+1}{(s-1)^2 + 2} = \frac{s-1}{(s-1)^2 + 2} + \frac{1}{(s-1)^2 + 2}$$
$$= \frac{s-1}{(s-1)^2 + 2} + \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{(s-1)^2 + 2}$$

Hence $\mathcal{L}^{-1}\{ \dfrac{s}{s^2 - 2s + 3} \} = \mathcal{L}^{-1}\{ \dfrac{s-1}{(s-1)^2 + 2} + \dfrac{1}{\sqrt{2}} \dfrac{\sqrt{2}}{(s-1)^2 + 2} \}$

$$= \mathcal{L}^{-1}\{ \frac{s-1}{(s-1)^2 + 2} \} + \frac{1}{\sqrt{2}} \mathcal{L}^{-1}\{ \frac{\sqrt{2}}{(s-1)^2 + 2} \}$$

$$= e^t \cos(\sqrt{2} \surd 2 \ t) + \frac{1}{\sqrt{2}} e^t \sin(\sqrt{2} \ t)$$

EXERCISES on The Inverse Transform

Find $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$ if

1) $F(s) = \dfrac{5}{s+3}$

2) $F(s) = \dfrac{3}{s^2+25}$

3) $F(s) = \dfrac{s-4}{s^2-4}$

4) $F(s) = \dfrac{3}{s}$

5) $F(s) = \dfrac{5}{s^2}$

6) $F(s) = \dfrac{s+1}{s^2+1}$

7) $F(s) = \dfrac{9}{s^2+3s}$

8) $F(s) = \dfrac{4}{s^2+3s+2}$

9) $F(s) = \dfrac{3}{(s+2)^2+4}$

10) $F(s) = \dfrac{3s+4}{s^2+6s+10}$

11) $F(s) = \dfrac{4s+7}{2s^2-12s+22}$

12) $F(s) = \dfrac{-2}{s+16}$

13) $F(s) = \dfrac{3}{s-7} + \dfrac{1}{s^2}$

14) $F(s) = \dfrac{1}{(s-3)^2} - \dfrac{2}{s^2} + \dfrac{1}{s}$

15) $F(s) = \dfrac{4s}{s^2+5} - \dfrac{2}{s^2} + \dfrac{1}{s}$

16) $F(s) = \dfrac{7}{s^2+5} - \dfrac{2}{s^2} + \dfrac{1}{s}$

17) $F(s) = \dfrac{4s+7}{s^2+5} - \dfrac{2}{s^2} + \dfrac{1}{s}$

18) $F(s) = \dfrac{4s}{s^2-14}$

19) $F(s) = \dfrac{-1}{s(s^2+8)}$

EXAMPLE. Using Laplace transforms, solve the IVP $y'' - y' - 2y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution. We transform the problem into the (complex) frequency domain:
For convenience let $\mathcal{L}\{y\} = Y$.

$$\mathcal{L}\{ y'' - y' - 2y \} = \mathcal{L}\{ 0 \}$$
$$\mathcal{L}\{ y'' \} - \mathcal{L}\{ y'\} - 2\,\mathcal{L}\{ y \} = 0$$
$$\mathcal{L}\{ y'' \} - \mathcal{L}\{ y'\} - 2\,\mathcal{L}\{ y \} = 0$$

$$(s^2 Y - s\, y(0) - y'(0)) - (s Y - y(0)) - 2 Y = 0$$
$$(s^2 Y - s (1) - (0)) - (s Y - (1)) - 2 Y = 0$$
$$s^2 Y - s - s Y + 1 - 2 Y = 0$$
$$s^2 Y - s Y - 2 Y = s - 1$$
$$(s^2 - s - 2) Y = s - 1$$

$$Y = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

where we may use partial fractions to obtain A and B. Multiplying by $(s-2)(s+1)$ we obtain

$$s - 1 = A(s + 1) + B(S - 2) = A s + A + B s - 2 B$$

Equating coefficients

s^1) $1 = A + B$
s^0) $-1 = A - 2B$

$\Rightarrow\ 2 = 3 B \Rightarrow B = 2/3 \Rightarrow A = 1 - B = 1/3$

$$Y = \frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

$\Rightarrow\quad y(t) = 1/3\, e^{2t} + 2/3\, e^{-t}$

Handout #3 | **TABLE OF LAPLACE TRANSFORMS THAT NEED NOT BE MEMORIZED** | Professor Moseley

$f(T) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$	Domain F(s)
)))))))))))	))))))))))	)))))))))
t^n where n is a positive integer	$\frac{n!}{s^{n+1}}$	s > 0
sinh (at)	$\frac{a}{s^2 - a^2}$	s > *a *
cosh (at)	$\frac{s}{s^2 - a^2}$	s > *a *
e^{at}sin (bt)	$\frac{\omega}{(s-a)^2 + \omega^2}$	s > a
e^{at}cos(bt)	$\frac{s-a}{(s-a)^2 + \omega^2}$	s > a
$t^n e^{at}$ n = positive integer	$\frac{n!}{(s-a)^{n+1}}$	s > a
u(t)	$\frac{1}{s}$	s > 0
u(t - c)	$\frac{e^{-cs}}{s}$	s > 0
$e^{ct}f(t)$	F(s - c)	
f(ct) c > 0	$\frac{1}{c}F(\frac{s}{c})$	
δ(t)	1	
δ(t - c)	e^{-cs}	