A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 2

A COLLECTION OF HANDOUTS ON SCALAR LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODE"s)

CHAPTER 8

One-to-One Functions and

One-to-One Linear Operators

Like the concept of a set, the concept of a function can be considered to be a primitive notion (actually, instead of the concept of a set). To develop an intuitive understanding of the concept, we do not worry about the subtleties of logic and set theory but simply give a working definition. Later we give the definition of (the graph of) a function as a set.

Recall the informal definition of a function as a rule of correspondence.

DEFINITION #1. A **function** is a rule of correspondence which assigns to each element in a first set (called the **domain** of the function) exactly one element in a second set (called the **co-domain** of the function).

To define a function, we must first define the two sets A (the domain) and B (the co-domain) before giving the rule of correspondence. Thus these two sets are part of this definition of a function. To distinguish the graphs of functions from other curves, we say that a function is well defined provided one has clearly specified exactly one element in the co-domain for each element of the domain ($x^2 + y^2 = 1$ does not define a function since a vertical line crosses its graph in two places). Often we denote the function or rule by f. If x is any element in the domain then y = f(x) indicates the element in the co-domain that the rule associates with x. We use the notations $f: A \to B$ and $A \xrightarrow{f} B$ to indicate that A is the domain and B is the co-domain of the function f. We also denote the domain of f by D_f or by D(f). The graph of $f:A\to B$ is the set $G = \{ (x,f(x)) \in AxB: x \in A \}$. ($AxB = \{(x,y): x \in A \text{ and } y \in B\}$ is the Cartesian product of A and B.) To provide more rigor, some texts define a function to be its graph. This provides a definition of a function as a set. However, to develop our intuition for properties of functions, we will continue to use the definition of a function as a rule of correspondence from one set to another.

Although we can consider functions from any set to any other set, we are particularly interested in functions from the real numbers **R** to the real numbers **R**. Many, but not all functions are defined by algebraic formulas. Examples are polynomials (e.g. $f(x) = mx + b$ and $f(x) = ax^2 + bx + c$) and rational functions ($f(x) = p(x)/q(x)$ where p and q are polynomials. Familiar examples of functions not defined by algebraic formulas are the trigonometric functions (e.g. f(x) = sin x and f(x) = cos x). It is expected that you should have some familiarity with these functions, particularly when the rule of correspondence is defined by an algebraic formula. In fact, consideration of familiar examples should help motivate interest in understanding fundamental concepts for functions.

DEFINITION #2. Let $f: X \to Y$. Then the range of f is the set $R_f = \{y \in Y: \exists\, x \in X \text{ s.t. } f(x) = y\}$.

Informally, the domain of a function can be described as the set of things that get mapped and the range as the set of things that get mapped into. That is, the range is the set of things in the co-

domain for which there exist an element in the domain that gets mapped into those things. If $A \subseteq X$ where $f{:}X \to Y$, then by definition, the <u>image of A</u> is the set
$f(A) = \{ y \in Y : \exists\, x \in X \text{ s.t. } f(x) = y \}$. Hence the range R_f is the image of the domain.

An <u>algebraic formula</u> or <u>algebraic expression</u> always defines a clear rule of correspondence between the domain **R** (or a subset of **R**) and the co-domain **R** of a <u>real valued function of a real variable</u> using the <u>binary operations</u> of addition and multiplication (and subtraction and division, but we define these as the <u>inverse operations</u> of addition and multiplication rather than as fundamental operations). The algorithm for evaluation of the formula (or expression) is clearly specified using parentheses (), brackets [], and braces {} as well as standard conventions to establish the order in which the operations are to be carried out.

More generally we consider a <u>polynomials</u> of degree n, $p{:}\mathbf{R} \to \mathbf{R}$ defined by
$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n$ where $a_0,...,a_n$ are constants. These constants may be zero except that we require $a_n \neq 0$. $D_f = \mathbf{R}$, but R(f) depends on the constants $a_0,...,a_n \neq 0$. Sometimes the domain of a real valued function of a real variable is not the entire set of real numbers **R** (e.g. $f(x) = \sqrt{x}$ or $f(x) = 1/x$). <u>Rational functions</u> are quotients of polynomials, $f(x) = p(x) / q(x)$ where p and q are polynomials. The domain as well as the range of a rational function depends on the constants in the polynomials, $D(f) = \{x \in \mathbf{R}: q(x) \neq 0\}$. We may write

$$\begin{array}{l} \mathbf{R} \\ \cup| \\ f{:}D_f \to \mathbf{R} \end{array}$$

to indicate that the domain of f is a subset of **R**. However, in an informal discussion, we may sometimes indulge in an abuse of notation (i.e. something that is technically incorrect but we do it anyway) and write $f{:}\mathbf{R} \to \mathbf{R}$ even when the domain of f is not the entire set of real numbers (e.g. $f(x) = 1/x$). Often the domain of a function is well known or the context makes it clear and it is not given explicitly. However, when writing proofs, it is essential that the domain (and co-domain) be given explicitly and correctly.

All functions are not defined by algebraic formulas. There are functions where the rule of correspondence is not so simple. Recall the trigonometric functions sine, cosine and tangent. Although we can evaluate these function <u>exactly</u> for some values of x (e.g. $\pi/6$, $\pi/4$,...) using our knowledge of trigonometry, more often we find <u>approximate</u> values of these functions using a table or a calculator.

Once a function has been defined (including the domain and co-domain as well as specifying the rule of correspondence), we can consider properties. A function is <u>one-to-one</u> (1-1) if every element in the domain gets mapped to a different element in the co-domain. A function is <u>onto</u> if every element in the co-domain has an element in the domain which maps into it. The formal definition of one-to-one is the <u>contrapositive</u> of the above informal definition. (The contrapositive of a <u>statement</u> is equivalent to the statement, but is stated with <u>negations</u>.) The formal definition of onto is stated in terms of the range.

<u>DEFINITION #3</u>. Let $f{:}\ X \to Y$. The function f is said to be **<u>one</u>-<u>to</u>-<u>one</u>** (<u>injective</u>) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. (This is the contrapositive of the <u>statement</u>, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. That is, distinct values of x in X get mapped to distinct values of y in Y. Recall that the negation of a negative is the positive.) The function f is said to be **<u>onto</u>** (<u>surjective</u>) if $R(f) = Y$. (That is, f is onto if the range is the entire co-domain.) If f is both 1-1 and onto, then it

is said to be bijective or to form a **one-to-one correspondence** between the domain and the co-domain. (If f is 1-1, then it always forms a 1-1 correspondence between its domain and its range.) The identity function (denoted by I) from a set X to itself is the function i_X:X → X defined by $i_X(x) = x \ \forall x \in X$.

We consider these concepts using the examples of functions previously considered. We first ask you to use your intuition to determine whether given functions are one-to-one and onto. Later, we ask for precise arguments (i.e. proofs) to validate your intuition.

EXERCISES on Review of Introductory Function Concepts

EXERCISE #1. Let f(x) = 3. Find f(2), f(4), and f(x+1). Give the domain of f.

EXERCISE #2. Let f(x) = 3x + 5. Find f(2), f(4), and $f(x^2+1)$. Find D_f and R_f.

EXERCISE #3. Let $p(x) = x^2 + 3x - 6$. Find p(2), p(4), and $p(x^3+1)$. Find D_p and R_p by first sketching the graph of this function (i.e. the graph of the curve $y = x^2 + 3x - 6$) and answering the following questions: Which way does the parabola open? Where is its vertex? What is the set that gets mapped by p(x)? For what values of y is there an x such that y=p(x)? Foe what values of y are there two values of x which get mapped in that value of y? For what value of y is there exactly one value of x which gets mapped into that value of y? For what value of y is there no values of x which gets mapped into that value of y?

EXERCISE #4. Let $f(x) = 1/x^2$. Find f(2), f(4), and $f(x^3+1)$. Find D_f and R_f. Hint: First sketch the graph of this function (i.e. the graph of the curve $y = 1/x^2$).

EXERCISE #5. Give the domains and ranges of sin(x), cos(x), and tan(x). Evaluate each of these functions for x = π/6, π/4, π/3, π/2, 1, 2, 0.52, and 0.12345. Give exact values if possible. Otherwise, give approximate values.

EXERCISE #6. Let f(x) = 1, p(x) = 3x + 2, g(x) = 1/x, and h(x) = sin(x) with their "natural" domains. Determine if the following are true or false. To help you decide, first sketch the graphs of the functions. Then find their domains and their ranges.

____ 1. f is one-to-one. ____ 2. f is onto. ____ 3. p is one-to-one.

____ 4. p is onto. ____ 5. g is one-to-one. ____ 6. g is onto.

____ 7. h is one-to-one ____ 8. h is onto.

Handout #2 VERIFYING ELEMENTARY PROPERTIES OF FUNCTIONS Prof. Moseley

We now wish to review the techniques needed to provide clear precise arguments to prove that functions are one-to-one and onto if this is the case, and to prove that they are not, if this is the case. We do this by considering examples first and then more general cases (usually in the exercises). We begin by reviewing the definitions of one-to-one and onto.

DEFINITION #1. Let $f: X \to Y$. The function f is said to be **one-to-one** (injective) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. (This is the contrapositive of the statement, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. That is, distinct values of x in X get mapped to distinct values of y in Y. Recall that the negation of a negative is the positive.) The function f is said to be **onto** (surjective) if $R(f) = Y$. (That is, f is onto if the range is the entire co-domain.) If f is both 1-1 and onto, then it is said to be bijective or to form a **one-to-one correspondence** between the domain and the co-domain. (If f is 1-1, then it always forms a 1-1 correspondence between its domain and its range.) The identity function (denoted by I) from a set X to itself is the function $i_X:X \to X$ defined by $i_X(x) = x \ \ \forall x \in X$.

EXAMPLE #1. Let f be the constant function 3; that is, let $f:\mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3$. This function maps everything into one number, 3, and is "obviously" not 1-1 or onto. $D_f = \mathbf{R}$ and $R_f = \{3\}$. To indicate the kind of logic used to prove that something is not true, we provide proofs of these "obvious" facts. We begin by giving a clear and complete statement of what we are proving (including an explicit statement of the domain and co-domain of the function). Although most texts reserve the word theorem for important facts that are proved, in this section, we will use this word for any fact that is proved.

THEOREM #1. The function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3$ is not one-to-one (i.e. f is not injective).

Proof. We prove that $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3$ is not one-to-one (injective) by finding a counter example to the statement that f is one-to-one. That is, we find two different real numbers x_1 and x_2 in D_f such that $f(x_1) = f(x_2)$. Let $x_1 = 1$ and $x_2 = 2$. Then $x_1, x_2 \in D_f = \mathbf{R}$ and $x_1 \neq x_2$. Furthermore, $f(x_1) = 3$ and $f(x_2) = 3$ by the definition of f. Hence $f(x_1) = f(x_2)$. We have found $x_1, x_2 \in D_f$ such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$. Since the existence of such a pair contradicts the definition of one-to-one, we have that f is not one-to-one.

QED (the proof is complete)

Since everything gets mapped into 3, it may at first appear that giving specific numbers x_1 and x_2 such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$ does not add a lot toward showing that f is not 1-1. However, we emphasize that the specific case of considering $x_1 = 1$ and $x_2 = 2$ provides a specific **counter example** to the statement $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Hence it shows that this statement is not true and hence that f is not 1-1. It also may appear that our proof is repetitive. It is and mathematicians love to be terse. However, initially, it is better to err on the side of being too wordy and repetitive, than to err on the side of not saying enough and being unclear. Clarity is much more important than brevity. On the other hand (OTOH), to much repetition can lead to confusion and hence decrease clarity. This, like an English composition, calls for judgement on the part of the writer. Explain clearly and completely and, to the extent possible, in your own words. As a rule of thumb, you should end your proof by summarizing what you have proved, with the last clause being the conclusion of the proof. This is sometimes implied, but usually you should put it in explicitly.

We now prove that f(x) = 3 is <u>not</u> onto. Note that whether this is true or not depends on the assumed co-domain of f. Again we begin by giving a clear and complete statement of what it is we plan to prove.

<u>THEOREM #2</u>. The function $f:\mathbf{R} \to \mathbf{R}$ defined by f(x) = 3 is not onto.

<u>PROOF</u>. First recall the definition of what it means for a function to be onto (surjective). $f:\mathbf{R} \to \mathbf{R}$ is onto if $R_f = \mathbf{R}$. From set theory, to show that these two sets are equal, we need to show that $R_f \subseteq \mathbf{R}$ and $R_f \supseteq \mathbf{R}$. However, by definition, the range is always a subset of the co-domain. Hence to show that f is onto it would remain to show that $R_f \supseteq \mathbf{R}$ or $\mathbf{R} \subseteq R_f$. We could do this by assuming that $y \in \mathbf{R}$ (i.e. y is an arbitrary number in the co-domain **R**) and showing that $y \in R_f$. But $y \in R_f$ means that there exist $x \in D_f = \mathbf{R}$ s. t. f(x) = y. Summarizing, to show that $R_f = \mathbf{R}$, we wish to show that $\forall\, y \in \mathbf{R}$ $\exists$ (there exists) an $x \in D_f = \mathbf{R}$ s.t. f(x) = y. However, what we wish to show is that f is <u>not</u> onto.

All we have to do to show that f is <u>not</u> onto is to find a $y_1 \in \mathbf{R}$ (the co-domain) s.t. y_1 is not in the range which we know to be $R_f = \{3\}$. Thus we wish to find $y_1 \neq 3$ and show that there exist no $x \in D_f = \mathbf{R}$ s.t. $f(x) = y_1$. Let $y_1 = 4 \in \mathbf{R}$ (the co-domain). We subscript y to emphasize that we have picked a specific $y \in \mathbf{R}$. To show that there is no number $x \in \mathbf{R}$ s.t. $f(x) = y_1 = 4$, we assume that such an x exists and reach a <u>contradiction</u>. This shows that the statement of existence is false and hence that no such x exists. Suppose that there is an $x \in D_f = \mathbf{R}$ s.t. $f(x) = y_1 = 4$. But by the definition of f we have $f(x) = 3\ \forall\, x \in D_f = \mathbf{R}$. Since $f(x) = 3 \neq 4 = y_1$, we have reached a <u>contradiction</u> and the assumption of the existence of such an x must be false. Hence no such x exits and f is not onto.

QED (the proof is complete)

Again, it may appear that our proof is repetitive. It is, and mathematicians love to be terse. Again, it is initially better to err on the side of being too wordy and repetitive, than to err on the side of not saying enough and being unclear. Again, clarity is much more important than brevity. Again we also note that to much repetition can lead to confusion and hence decrease clarity. Again, we note that, like any English composition, this calls for judgement on the part of the writer. Again, you are encouraged to explain clearly and completely and, to the extent possible, in your own words. Again, as a rule of thumb, you should end your proof by summarizing what you have proved, with the last clause being the conclusion of the proof. Although this is sometimes implied, you should initially put it in explicitly.

Again we note that the standard format for showing two sets are equal does not appear to be appropriate here since we are showing that two sets are <u>not</u> equal. (Recall that the standard STATEMENT/REASON format can always be used to show sets are equal.) As with any

English theme, the form in which the logic flows most easily depends on the problem and on personal style. Again we note that the range is always a subset of the co-domain. Hence to show that a function is not onto, we need only to show that there is an element in the co-domain for which there is no element in the domain that maps into that element in the co-domain.

Next we consider the linear functions $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = m x + b$ where $m \neq 0$ and b are constants. A sketch of any line $y = mx + b$ where $m \neq 0$ "shows" (but does not prove) that these functions are 1-1 and onto. Proofs that functions are indeed 1-1 and onto take a different path then those proving that they are not. Follow the logic carefully to be sure you can reproduce it in the future. As always, we provide a clear and complete statement of what we plan to prove. We begin with a specific function and leave the generalization to the exercises.

THEOREM #3. The function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x + 2$ is one-to-one.

PROOF. We first note that by definition $D_f = \mathbf{R}$, and that this is the natural domain of f; that is, we can substitute any real number into the algebraic formula $3x + 2$ and compute the value of f at this value of x. To show that f is 1-1, we proceed directly using the definition (we refer to this as the DUD method). Hence we review what it means for a function to be 1-1. f is 1-1 if the assumption $f(x_1) = f(x_2)$ where x_1 and x_2 are any two elements in D_f implies that $x_1 = x_2$. Hence we choose two arbitrary elements x_1 and x_2 in $D_f = \mathbf{R}$ and assume $f(x_1) = f(x_2)$. To show f is 1-1, it remains to show that this implies that we must have $x_1 = x_2$. To arrive at the logical implication of the statement $f(x_1) = f(x_2)$, we use the STATEMENT/REASON format along with assumed algebraic properties of **R**. (We will investigate these properties in more detail later.)

Summarizing, to prove that f is 1-1, let $x_1, x_2 \in D_f = \mathbf{R}$ and assume $f(x_1) = f(x_2)$. If we can show that $x_1 = x_2$, then f is 1-1.

STATEMENT	REASON
$f(x_1) = f(x_2)$	Assumed (i.e. by hypothesis)
$3(x_1) + 2 = 3(x_2) + 2$	Definition of the function f
$(3(x_1) + 2) + (-2) = (3(x_2) + 2) + (-2)$	Property of equality (Equals added to equals are equal) and a property of real numbers (existence of the additive inverse of 2 which we denote by -2)
$3(x_1) + (2 + (-2)) = 3(x_2) + (2 + (-2))$	Associativity of addition of real numbers
$3(x_1) + (0) = 3(x_2) + (0)$	Definition of additive inverse of a real number
$3(x_1) = 3(x_2)$	Definition of the additive identity (i.e. 0)for the
$3^{-1} (3(x_1)) = 3^{-1} (3(x_2))$	Property of equality (Equals multiplied by equals are equal) and a property of real numbers (existence of the multiplicative inverse of 3 which we denote by 3^{-1}).
$(3^{-1} 3) x_1 = (3^{-1} 3) x_2$	Associativity of multiplication of real numbers.
$(1) x_1 = (1) x_2$	Definition of the multiplicative inverse of a real number
$x_1 = x_2$	Definition of multiplicative identity for real number

Summarizing, since we have proved that the assumption $f(x_1) = f(x_2)$ where x_1 and x_2 are any two elements in D_f implies that $x_1 = x_2$, we see that by the definition of what it means for a function to be 1-1, that f is indeed one-to-one.

Q.E.D. (the proof is complete)

We now prove that a linear function is onto (if it is not parallel to the x axis). We also continue to analyze of the algebraic properties of **R**. We begin with a specific function and leave the general case to the exercises.

THEOREM #4. The function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x + 2$ is onto.

PROOF. We first note that by definition $D_f = \mathbf{R}$, and that this is the natural domain of f; that is, we can substitute any real number into the algebraic formula $3x + 2$ and compute the value of f at this value of x. To show that f is onto, we proceed directly using the definition (i.e. we use the DUD method). Hence we review what it means for a function to be onto. f is onto if $R_f = \mathbf{R}$. From set theory, to show that these two sets are equal (a picture of the graph is not sufficient), we need to show that $R_f \subseteq \mathbf{R}$ and $R_f \supseteq \mathbf{R}$. However, by definition, the range is always a subset of the co-domain which in this problem is **R**. Hence it remains to show that $R_f \supseteq \mathbf{R}$ or $\mathbf{R} \subseteq R_f$. We do this by assuming that $y \in \mathbf{R}$ (i.e. y is an arbitrary number in the co-domain **R**) and showing that $y \in R_f$. But $y \in R_f$ means that there exist $x \in D_f = \mathbf{R}$ s.t. $f(x) = y$. Summarizing, we have shown that $R_f = \mathbf{R}$ if we can show that $\forall\, y \in \mathbf{R}\ \exists$ (there exists) an $x \in D_f = \mathbf{R}$ s.t. $f(x) = y$. (The easiest way to do this is to find and exhibit the element (number) x that gets mapped into y and to prove that this x indeed gets mapped into y. The equation $y = 3x + 2$ is easily solved for x in terms of y: $x = (y - 2)/3 = (y+(-2))(3^{-1})$. We think of 3^{-1} as the multiplicative inverse of 3, rather than as 1/3 and $y-2 = y+(-2)$ as adding the additive inverse of 2 to y, rather than as subtracting 2 from y.) Let $x = (y+(-2))(3^{-1})$ where 3^{-1} denotes the multiplicative inverse of 3 and -2 denotes the additive inverse of b. This is possible (i.e. 3^{-1} and -2 exist) all elements of **R** have multiplicative inverses except 0 and all elements in **R** have additive inverses. Now using the algebraic properties of real numbers, we prove that f(x) is indeed y. We use a STATEMENT/REASON format and justify each and every step in the computation using known properties of real numbers.

Summarizing, we can prove that f is onto if we can show that $\forall\, y_1 \in \mathbf{R}\ \exists$ an $x_1 \in D_f = \mathbf{R}$ s.t. $f(x_1) = y_1$. (We subscript y and x to emphasize that we are picking a particular, but unspecified y and we will define a particular x in terms of the y picked.) Let $y_1 \in \mathbf{R}$. Now define $x_1 = (y_1+(-2))(3^{-1})$. It remains to show that $f(x_1) = y_1$. We use the STATEMENT/REASON format.

STATEMENT	REASON
$f(x_1) = (\, 3(\, (y_1+(-2))3^{-1}\,)\,) + 2$	Definition of valuation of the function f at the point x_1.
$= 3(\, (3^{-1})(y_1 + (-2))\,) + 2$	Multiplication of real numbers is commutative.
$= (\, (3)(3^{-1})\,)(y_1 + (-2)\,) + 2$	Multiplication of real numbers is associative.
$= (\, (1)(y_1 + (-2)) + 2$	Definition of the multiplicative inverse of 3 as 3^{-1}.

$= ((y_1 + (-2)) + 2$ Definition of 1 as the multiplicative identity.
$= y_1 + ((-2) + 2)$ Addition of real numbers is associative.
$= y_1 + (0)$ Definition of -2 as the additive inverse of 2.
$= y_1$ Definition of 0 as the additive identity element.

Summarizing, since we have proved that $R_f = \mathbf{R}$, we see that by the definition of what it means for a function to be onto, that f is indeed onto.

Q.E.D. (the proof is complete)

EXERCISES on Verifying Elementary Properties for Functions

EXERCISE #1. Prove Theorems #5 and #6. Begin by giving a clear and complete statement of what it is you are going to prove (i.e. copy down the theorem). Mimic the proof given above. For the first "theorem" replace 3 by 5. For the second, use your judgement. Hint: What is the correct domain of $f(x) = 1/x^2$? What is the co-domain? As you write each proof, think about the logic that drives the proof.

THEOREM #5. The function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = 5$ is not one-to-one.

THEOREM #6. The function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1/x^2$ is not one-to-one.

We wish to generalize to say that all constant functions are not 1-1. As the statements we prove become more general, the term theorem becomes more appropriate.

EXERCISE #2. Prove Theorem #7. Begin by giving a clear and complete statement of what it is you are going to prove (i.e. by copying down the theorem). Next mimic the proof given above (i.e. replace 3 by c). However, the generalization to an arbitrary constant function requires some rethinking and some rewriting. Don't be afraid to rewrite a proof after you have a first draft. (You often do this with English themes.) The difficulty with generalizing to an arbitrary constant function is that to provide a specific counter example to the statement that f is 1-1 may appear to require that we know specifically what c is. However, some thought should convince you that this is not the case. Now write the proof of this theorem in your own words.

THEOREM #7. The function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = c$ where c is an arbitrary constant is not one-to-one.

EXERCISE #3. Prove Theorems #8 and #9. Begin by giving a clear and complete statement of what it is you are going to prove (i.e. copy down the theorem). Mimic the proofs given above. For the first "theorem" replace 3 by 5. For the second, use your judgement. Hint: What is the correct domain of $f(x) = 1/x^2$? What is the co-domain? As you write the proof, think about the logic that drives the proof.

THEOREM #8. The function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = 5$ is not onto.

THEOREM #9. The function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1/x^2$ is not onto.

We wish to generalize to all constant functions. Again, as the statements we prove become more general, the term theorem becomes more appropriate.

EXERCISE #4. Prove Theorem #10. Begin by giving a clear and complete statement of what it is you are going to prove (i.e. by copying down the theorem). Next mimic the proof given above (i.e. replace 3 by c). However, the generalization to an arbitrary constant function requires some rethinking and some rewriting. Don't be afraid to rewrite a proof after you have a first draft. (You often do this with English themes.) The difficulty with generalizing to an arbitrary constant function is that to provide a specific counter example to the statement that f is onto seems to require that we know specifically what c is. This probably didn't appear to be as necessary in your previous proof that $f(x) = c$ is not 1-1 as it does in the current proof that f is not onto. Some thought should convince you that we need not know specifically what c is, if we can be assured that there are other numbers in **R** other than c. This comes from our knowledge of **R** (the set of real numbers is very large and certainly contains numbers other than c, no matter what c is). Hence, by our assumed knowledge of **R**, we can assume that no matter what c is, that we can pick $y_1 \neq c$. We subscript the y we pick to emphasize that we have picked a specific $y \in \mathbf{R}$ (y_1 is a specific, but unknown, number, since c is specific but unknown). All we know and all we need to know is that $y_1 \neq c$. As before, to show that there is no number x s.t. $f(x) = y_1$, we assume that such an x exists and reach a contradiction. Suppose that there is an $x \in D_f = \mathbf{R}$ s.t. $f(x) = y_1$. But by the definition of f we have $f(x) = c \neq y_1$. Now write the proof of this theorem in your own words.

THEOREM #10. The function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = c$ where c is an arbitrary constant is not onto.

EXERCISE #5. Prove Theorem #11. Begin by explaining why we may assume $D_f = \mathbf{R}$. Then proceed directly using the definition (i.e. using DUD). In this exercise, you need not write reasons for each step in the computation of $f(x_1)$. However, you should try to justify each step in your head.

THEOREM #11. The linear function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = 5\,x + 4$ is onto.

EXERCISE #6. Prove that the linear function $f(x) = m\,x + b$ where $m \neq 0$ and b are constants is onto. Begin by writing (as a theorem) a clear and complete statement of what it is that you are going to prove. Then begin the proof by explaining why we may assume $D_f = \mathbf{R}$. Then proceed directly using the definition (i.e. using DUD). To show that $R(f) = \mathbf{R}$ (a picture of the graph is not sufficient), you must show that for any element (number) $y \in \mathbf{R}$, $\exists$ (there exist) an element $x \in D_f = \mathbf{R}$, such that $y = f(x)$. (The easiest way to do this is to find and exhibit the element (number) x that gets mapped into y and to prove that this x indeed gets mapped into y. The difficulty is that x and y are arbitrary. However, the equation $y = mx + b$ is easily solved for x in terms of y, $x = (y - b)/m = (y+(-b))m^{-1}$, provided $m \neq 0$, as we have assumed. We think of m^{-1} as the multiplicative inverse of m, rather than as 1/m and $y-b = y+(-b)$ as adding the additive inverse of b to y, rather than as subtracting b from y.) Let y_1 be an arbitrary, but fixed, element in **R** (which we will show to be R_f). Now let $x_1 = (y_1+(-b))m^{-1}$ where m^{-1} denotes the multiplicative inverse of m and -b denotes the additive inverse of b. This is possible (i.e. m^{-1} and -b exist) since we have assumed that $m \neq 0$ and all other elements in **R** have multiplicative inverses and all elements in **R** have additive inverses. Now rewrite the entire proof in your own words. In this exercise, give the best reason that you can for each step in the computation of $f(x_1)$ using the STATEMENT/REASON format.

EXERCISE #7. Determine if the function $f:\mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1/x$ is 1-1 and onto and explain why or why not. First find D(f). Is it all of **R**? If not, replace $f:\mathbf{R} \to \mathbf{R}$ with $f:D(f) \to \mathbf{R}$, where D(f) = ? f is 1-1 if all of the elements in D(f) get mapped to different numbers. This can be proved by letting $x_1, x_2 \in D(f)$ and showing that $f(x_1) = f(x_2)$ implies $x_1 = x_2$. That f is not 1-1 can be proved by finding and displaying two elements (numbers) $x_1, x_2 \in D_f$ that get mapped into the same element (number), that is, such that $f(x_1) = f(x_2)$. This can be proved by evaluating $f(x_1)$ and $f(x_2)$ using the format previously described. To determine if f is onto, first find R(f). The function f is onto if $R(f) = \mathbf{R}$. Since we are dealing with a real valued function of a real variable, we always have $R(f) \subseteq \mathbf{R}$. Hence to show that f is onto it is often only necessary to show that $\mathbf{R} \subseteq R_f$. That is, given an arbitrary $y \in \mathbf{R}$, we show (usually by computing it first and then verifying that it maps where we say that it does) that $\exists\, x \in D_f$ s.t. $f(x) = y$. To show that f is not onto, first find and display a $y \in \mathbf{R}$ such that y is not in R_f. Then show that y is not in R_f by assuming that it is, that is, by assuming that $\exists\, x \in D_f$ such that $y = f(x)$ and then reaching a contradiction (e.g. 1=0). This shows that the assumption that there exists x such that $y = f(x)$ was bad and hence no such x exists.

Handout #3 LINEAR OPERATORS Prof. Moseley

A function or map T from one vector space V to another vector space W is often call an <u>operator</u>. If we wish to think geometrically rather than algebraically we might call A a <u>transformation</u>.

<u>DEFINITION</u>. An operator T:V → W is said to be <u>linear</u> if ∀ $\vec{x}$ $\vec{y}$ ∈ V and ∀ scalars α,β we have

$$T(\alpha \vec{x} \quad + \vec{y} \quad) = \vec{x}\, T(\quad \vec{y} + \beta T(\quad). \tag{1}$$

<u>THEOREM</u>. An operator T is linear if and only if the following two properties hold:

i) $\vec{x}$ $\vec{y}$ ε V ⇒ $\vec{x}$ T $\vec{y}$ + $\vec{x}$) = T $\vec{y}$)+ T( ) (2)

ii) α a scalar and $\vec{x}$ εV ⇒ T $\vec{x}$) = $\vec{x}$ T( ). (3)

<u>EXAMPLE #1</u> Let the <u>operator</u> A:$\mathbf{R}^n \to \mathbf{R}^m$ be defined by matrix multiplication by the <u>matrix</u> A; that is, let

$$T(\vec{x} \quad) \; \underset{m\times n}{A} \; \underset{n\times 1}{\vec{x}} \tag{4}$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ ε $\underset{m\times n}{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} & a_{m,1} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix}$

<u>EXAMPLE #2</u> Let the operator D:C^1(a,b) → C(a,b) be defined by

$$D(f) \triangleq \frac{df}{dx} \tag{5}$$

where f ε C^1(a,b) = {f:(a,b) → **R**: $\frac{df}{dx}$ exists and is continuous} and C(a,b) ={f:(ab) → **R**:f is continuous}.

To solve linear ODEs with constant coefficients using Laplace Transforms, we need the fact that the Laplace transform establishes a one-to-one correspondence between a subspace of the time domain **T** (which is a function space and hence a vector space) and the frequency domain **F** (another function space and hence another vector space). The following theorem, which you will be expected to be able to prove, shows that a linear operator is one-to-one if its null space contains only the zero vector.

<u>THEOREM.</u> If T is a linear operator from the vector space V to the vector space W and its null space N_T is { $\vec{0}$ }, then T is a one-to-one mapping; that is, $\vec{x}$ $\vec{y}$, ε V a $\vec{x}$ l T($\vec{y}$) = T( ), then $\vec{x}$ $\vec{y}$.

<u>PROOF:</u> We start by proving the following:

<u>Lemma.</u> If N_T is { $\vec{0}$ } and $\vec{x}$ $\vec{0}$ = $\vec{x}$ he $\vec{0}$ = .

<u>Proof of lemma:</u> We begin by recalling the definition of the null space N_T.

<u>DEFINITION</u> (Null Space). The null space is the set of all vectors $\vec{x}$ that satisfy the linear homogeneous equation T($\vec{x}$) $\vec{0}$; that is N $\vec{x}$ = { $\vec{x}$ /: T $\vec{0}$) = }.

We now finish the proof of the Lemma. Thus if T($\vec{x}$) $\vec{0}$, then by the definition of N_T we have that $\vec{x}$ ε N_T. However, we have also assumed that N_T = $\vec{0}$ }. Hence since we have $\vec{x}$ at εN_T and N_T = { $\vec{0}$ } (i.e., the only vector in N_T is the zero vector, we have th $\vec{x}$ $\vec{0}$ = .

Q.E.D. for lemma.

Having finished the proof of the lemma, we now use it to complete the proof of the theorem: To show that T is one-to-one, we assume that $\vec{x}$ $\vec{y}$ ε V an $\vec{x}$ T($\vec{y}$ = T( ) and sh $\vec{x}$ v t $\vec{y}$ t = .
Since this is an identity, we use the STATEMENT/REASON FORMAT. (However, we do not start with one side and go to the other.)

<u>STATEMENT</u>	<u>REASON</u>
T($\vec{x}$) = $\vec{y}$)	Given. (Hypothesis.)
T($\vec{x}$) − $\vec{y}$ $\vec{0}$ = .	Vector algebra in W
T($\vec{x}$ $\vec{y}$ $\vec{0}$ = .	T is a linear operator
$\vec{x}$ $\vec{y}$ $\vec{0}$ = .	The above Lemma
$\vec{x}$ $\vec{y}$.	Vector algebra in V

Since we assumed T($\vec{x}$) = $\vec{y}$) and showed $\vec{x}$ at $\vec{y}$ = , we have that T is a one-to-one mapping QED for the theorem.