A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 2

A COLLECTION OF HANDOUTS ON SCALAR LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODE"s)

CHAPTER 7

Definition and Properties of Laplace Transforms

Read the introduction and Section 6.1 of Chapter 6 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.). Pay particular attention to Examples 1-6 pages 294-297.

REVIEW OF IMPROPER INTEGRALS. The Laplace transform is defined as an improper integral. Hence we begin with a brief review of improper integrals.

DEFINITION #1. $\int_{t=c}^{t=\infty} f(t)dt = \lim_{A\to\infty} \int_{t=c}^{t=A} f(t)dt$ provided the limit exists.

EXAMPLE #1. Compute $\int_{t=1}^{t=\infty} \frac{1}{t}\,dt$

Solution. $\int_{t=1}^{t=\infty} \frac{1}{t}\,dt = \lim_{A\to\infty} \int_{t=1}^{t=A} \frac{1}{t}dt = \lim_{A\to\infty} (\ln t)\Big|_{t=1}^{t=A} = \lim_{A\to\infty} (\ln A - \ln 1)$

$= \lim_{A\to\infty} (\ln A) = \infty$ (Increases without bound and hence the limit does not exist)

EXAMPLE #2. Compute $\int_{t=1}^{t=\infty} \frac{1}{t^p}\,dt$ where p > 1.

Solution. $\int_{t=1}^{t=\infty} \frac{1}{t^p}\,dt = \lim_{A\to\infty} \int_{t=1}^{t=A} \frac{1}{t^p}dt = \lim_{A\to\infty} \int_{t=1}^{t=A} t^{-p}\,dt = \lim_{A\to\infty} (t^{1-p} / (1-p))\Big|_{t=1}^{t=A}$

$= \lim_{A\to\infty} \left(\frac{A^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}\right) = \lim_{A\to\infty} \left(\frac{A^{1-p}}{1-p} + \frac{1}{p-1}\right) = \frac{1}{p-1}$

Recall that these improper integrals, together with the integral test, imply the divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the convergence of the p series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ if p > 1.

DEFINITION #2 (Laplace Transform). Let I = [0,∞) and f:I→ **R**. Then the Laplace Transform of f(t) is the function F(s)

$$\mathcal{L}\{f(t)\} = F(s) = \int_{t=0}^{t=\infty} f(t)\,e^{-st}\,dt \qquad (1)$$

provided the improper integral exist. (Sufficient conditions for the Laplace Transform to exist are given below.)

Since s is arbitrary, the Laplace transform maps a given function f(t) in the function space we will denote by **T** (time domain) to the function F(s) in the function space of all Laplace transforms which we denote by **F** (complex frequency domain).

EXAMPLE#1 Compute $\mathcal{L}\{f(t)\} = F(s)$ where f(t) = 0 for all t in $[0,\infty)$.

$$\mathcal{L}\{f(t)\} = F(s) = \int_{t=0}^{t=\infty} f(t)\, e^{-st}\, dt = \int_{t=0}^{t=\infty} 0\, e^{-st}\, dt = \lim_{A\to\infty} \int_{t=0}^{t=A} 0\, dt = \lim_{A\to\infty} 0 = 0.$$

EXAMPLE#2 Compute $\mathcal{L}\{f(t)\} = F(s)$ where $f(t) = e^{at}$ for all t in $[0,\infty)$.

Solution. We wish to compute $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{e^{at}\}$.

$$\mathcal{L}\{e^{at}\} = F(s) = \int_{t=0}^{t=\infty} e^{at}\, e^{-st}\, dt = \int_{t=0}^{t=\infty} e^{at-st}\, dt = \int_{t=0}^{t=\infty} e^{(a-s)t}\, dt = \lim_{A\to\infty} \int_{t=0}^{t=A} e^{(a-s)t}\, dt$$

$$= \lim_{A\to\infty} e^{(a-s)t} / (a-s) \Big|_{t=0}^{t=A} = \lim_{A\to\infty} \left[\frac{e^{(a-s)A}}{a-s} - \frac{e^{(a-s)0}}{a-s} \right] = \frac{1}{s-a} \quad \text{if } a - s < 0 \quad (s > a)$$

We begin a table of Laplace Transforms similar to having a table of antiderivatives.

PARTIAL TABLE OF LAPLACE TRANSFORMS

Time Domain f(t)	(Complex) Frequency Domain F(s)	
0	0	s > 0
e^{at}	1/(s-a)	s > a

SUFFICIENT CONDITIONS FOR THE IMPROPER INTEGRAL TO EXIST. To insure that the improper integral $\int_{t=0}^{t=\infty} f(t)\, e^{-st}\, dt$ exists, we must first insure that the proper integral $\int_{t=0}^{t=A} f(t)\, e^{-st}\, dt$ exists for all positive real numbers A. A good start is to recall that continuous functions are integrable:

THEOREM. Let f be a real valued function of a real variable whose domain includes the closed interval I = $[\alpha,\beta]$. Suppose that f is continuous on I. Then the Riemann Integrals

$$\int_{t=\alpha}^{t=\beta} f(t)\,dt \quad \text{and} \quad \int_{t=\alpha}^{t=\beta} f(t)\,e^{-st}\,dt \tag{4}$$

both exist. (Since products of continuous functions are continuous, if f is continuous on $[\alpha,\beta]$, then so is $f(t)\,e^{-st}$ for all values of s.)

However, we would like to consider a larger class of functions where the Riemann integral exists.

DEFINITION #3. A function f is said to be **piecewise continuous** on a closed finite interval $I = [\alpha,\beta]$ if the interval can be partitioned by a finite number of points

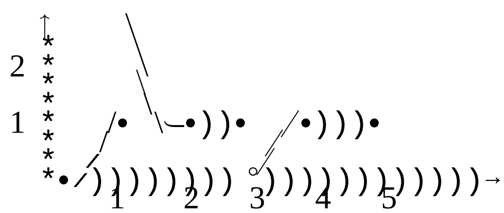
$$\alpha = t_0 < t_1 < t_2 < \ldots < t_n = \beta,$$

so that:

1. F is continuous on each of the open subintervals $I_n = (t_{i-1}, t_i)$ i = 1, 2, ...,n.
2. f approaches a finite limit as the end points of each subinterval are approached from within the subinterval; that is the limits: $\lim_{t \to t_{i-1}+} f(t)$, $\lim_{t \to t_i -} f(t)$, i = 1, 2, ..., n, all exist.

EXAMPLE. Let $f:I\to\mathbf{R}$ where I=[0,5] be defined as follows:

$$f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ (t-1)^{-1} & 1 < t \le 2 \\ 1 & 2 < t \le 3 \\ t-3 & 3 < t \le 4 \\ 1 & 4 < t \le 5 \end{cases}$$



CLASS EXERCISE True or False

______ 1. f is continuous on (1,3].

______ 2. f is continuous on [0,3].

______ 3. f is continuous on [3,5].

______ 4. f is piecewise continuous on [2,5].

______ 5. f is piecewise continuous on [1,3].

______ 6. f is piecewise continuous on [0,3].

THEOREM #2. Sums and products of piecewise continuous functions are piecewise continuous.

Also scalar multiples of piecewise continuous functions are also piecewise continuous functions.

THEOREM #3. The set of piecewise continuous function on the closed interval $\bar{I} = [\alpha,\beta]$, denoted by $PC(\bar{I}) = PC[\alpha,\beta]$, is a subspace of the vector space $F(\bar{I})$ of all real valued functions on the closed interval $\bar{I}$.

In mathematical (and physical) problems the value of a piecewise continuous function at the points of discontinuity usually does not matter and we usually do not wish to distinguish between two piecewise continuous functions that differ only at their points of discontinuity. One way to eliminate the distinction between two piecewise continuous functions that differ only at their points of discontinuity is to use the concept of an **equivalence class**. (Look up the definition of an **equivalence relation** in a modern algebra text.) All piecewise continuous functions that differ only at their points of discontinuity are said to be in the same equivalence class and we perform Laplace transformations on equivalence classes of functions. An easier way conceptually is to simply specify a unique function in each equivalence class which we place in a new set we call $PC_a[\alpha,\beta]$ (a is for average). Let $f\in PC[\alpha,\beta]$. If f is discontinuous at α, then redefine $f(\alpha)$ as $f(\alpha) = \lim_{t\to\alpha+} f(t)$. Similarly, if f is discontinuous at β, redefine $f(\beta)$ as $f(\beta) = \lim_{t\to\beta-} f(t)$. At any other point of discontinuity let $f(x) = [f(x-) + f(x+)]/2$ where $f(t-) = \lim_{h\to t-} f(h)$ and $f(x+) = \lim_{h\to t+} f(t)$. Since the new f only differs from the old f at the points of discontinuity, they are in the same equivalence class. (There is at most one continuos function in each equivalence class.) Let $PC_a[\alpha,\beta]$ be the set of all such functions (a stands for average value). We see that we have exactly one function from each equivalence class. Also, if we add such functions together, we get such a function. Such functions are also closed under scalar multiplication. Hence they form a subspace of $PC[\alpha,\beta]$, but each having unique values at the points of discontinuity.

THEOREM #4. Let f be a real valued function of a real variable whose domain includes the closed interval $[\alpha,\beta]$. Suppose that f is piecewise continuous on I. Then the Riemann Integrals $\int_{t=\alpha}^{t=\beta} f(t)\,dt$ and $\int_{t=\alpha}^{t=\beta} f(t)\,e^{-st}\,dt$ both exist. Also, if $f\in PC_a[\alpha,\beta]$, then $\int_{t=\alpha}^{t=x} f(t)\,dt = 0$ for all $x\in[\alpha,\beta]$ implies that $f(t) = 0$ for all $t\in[\alpha,\beta]$; that is, the zero function is the only function in $PC_a[\alpha,\beta]$ with this property. (Such functions are called **null functions** and we wish the zero function to be the only null function in our function spaces.)

DEFINITION #4. If the domain of a function is $[0,\infty)$, and $f\in PC[0,A]\ \forall\ A > 0$, then we say f is **piecewise continuous** on $[0,\infty)$. If for all $A>0$, $f\in PC_a[0,A]$, then $f\in PC_a[0,\infty)$.

THEOREM #5. The set of piecewise continuous functions on $I = [0,\infty)$, denoted by $PC[0,\infty)$ is a subspace of the set of all real valued functions on I. $PC_a[0,\infty)$ is a subspace of $PC[0,\infty)$.

To insure that the Laplace transform exists, we not only need for the proper integral on [0,A] to exist for all A, we need for the function $|f(t)\,e^{-st}|$ to grow sufficiently slowly so that the improper integral exists.

DEFINITION #5. A function $f:[0,\infty)\to\mathbf{R}$ is said to be **of exponential order** if there exist constants K, a, and M such that

$$|f(t)| \le Ke^{at} \qquad \forall\, t \ge M.$$

THEOREM #6. Sums and products of functions of exponential order, are of exponential order. Note also that scalar multiples of functions of exponential order are of exponential order. In fact,

THEOREM #7. The set of functions of exponential order, denoted by E_{xp}, is a subspace of the set $\mathscr{F}(I,\mathbf{R})$ of all real valued functions on $I = [0,\infty)$.

THEOREM #8. Suppose that

1) f is piecewise continuous on $[0,\infty)$; that is, f is piecewise continuous on the interval $[0,A]$ $\forall\, A > 0$ and
2) f is of exponential order so that there exist K, a, and M such that $|f(t)| \le Ke^{at}$ $\forall\, t \ge M$,

then the Laplace transform $\mathscr{L}\{f(t)\} = F(s) = \int_{t=0}^{t=\infty} f(t)\,e^{-st}\,dt$ exists with domain (a,∞). i.e. for $s > a$.

THEOREM #9. $PC[0,\infty) \cap E_{xp}$ is a subspace of $\mathbf{T}$. So is $\mathbf{T}_{pcexp} = PC_a[0,\infty) \cap E_{xp}$.

EXAMPLE #3 Compute $\mathscr{L}\{f(t)\} = F(s)$ where $f(t) = 1$ for $t \ge 0$.

Solution. We wish to compute $\mathscr{L}\{f(t)\} = F(s) = \mathscr{L}\{1\}$.

$$\mathscr{L}\{e^{at}\} = \int_{t=0}^{t=\infty} f(t)\,e^{-st}\,dt = \lim_{A\to\infty}\int_{t=0}^{t=A} e^{-st}\,dt = \lim_{A\to\infty} e^{-st}/(-s)\Big|_{t=0}^{t=A} = \lim_{A\to\infty}\left[\frac{e^{-sA}}{-s} - \frac{e^{-s(0)}}{-s}\right]$$

$$= \lim_{A\to\infty}\left[\frac{e^{-sA}}{-s} + \frac{1}{s}\right] = \frac{1}{s} \quad \text{if } -s < 0 \quad (s > 0)$$

We continue our table of Laplace Transforms.

PARTIAL TABLE OF LAPLACE TRANSFORMS

Time Domain $f(t)$	(Complex) Frequency Domain $F(s)$	
0	0	$s > 0$
e^{at}	$1/(s-a)$	$s > a$
1	$1/s$	$s > 0$

Note that the new entry is just a special case of the old entry, e^{at}, with $a = 0$.

EXAMPLE #4 Compute $\mathcal{L}\{f(t)\} = F(s)$ where $f(t) = \sin(at)$ for $t \geq 0$.

Solution. We wish to compute $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{\sin(at)\}$.

$$\mathcal{L}\{\sin(at)\} = \int_{t=0}^{t=\infty} [\sin(at)]\, e^{-st} dt = \lim_{A\to\infty} \int_{t=0}^{t=A} [\sin(at)]\, e^{-st}\, dt$$

We first compute the antiderivative (or look it up in a table)

Computation of the Antiderivative Use integration by parts. Let

$$I = \int \sin(at)\, e^{-st}\, dt = \frac{e^{-st}\sin at}{-s} - \int \frac{e^{-st} a\cos at}{-s}\, dt = \frac{e^{-st}\sin at}{-s} + \int \frac{e^{-st} a\cos at}{s}\, dt$$

$u = \sin(at)$ $dv = e^{-st}$

$du = a\cos(at)$ $v = e^{-st}/(-s)$

But

$$\int \cos(at)\, e^{-st}\, dt = \frac{e^{-st}\cos at}{-s} - \int \frac{-e^{-st} a\sin at}{-s}\, dt = \frac{e^{-st}\cos at}{-s} - \int \frac{e^{-st} a\sin at}{s}\, dt$$

$u = \cos(at)$ $dv = e^{-st}$

$du = -a\sin(at)$ $v = e^{-st}/(-s)$

Hence

$$I = \frac{e^{-st}\sin at}{-s} + \frac{a}{s}\int \cos(at)\, e^{-st}\, dt = \frac{e^{-st}\sin at}{-s} + \frac{a}{s}\left[\frac{e^{-st}\cos at}{-s} - \int \frac{e^{-st} a\sin at}{s}\, dt \right]$$

$$= \frac{e^{-st}\sin at}{-s} + \frac{a}{s}\left[\frac{e^{-st}\cos at}{-s} - \frac{a}{s} I \right] = \frac{e^{-st}\sin at}{-s} - \frac{e^{-st}\cos at}{s^2} - \frac{a^2}{s^2} I$$

Hence

$$I + \frac{a^2}{s^2} I = -\frac{e^{-st}\sin at}{s} - \frac{e^{-st}\cos at}{s^2} \text{ so that } \frac{s^2 + a^2}{s^2} I = -\frac{e^{-st}\sin at}{s} - \frac{e^{-st}\cos at}{s^2},$$

$(s^2 + a^2)\, I = -s\, e^{-st} \sin(at) - a\, e^{-st} \cos(at)$ and hence $I = \dfrac{-s\, e^{-st}\sin at - a\, e^{-st}\cos at}{s^2 + a^2}$.

Back to computing the Laplace Transform

$$F(s) = \mathcal{L}\{\sin(at)\} = \lim_{A\to\infty} \left. \frac{-\ s\, e^{-st}\sin at\ -\ a\, e^{-st}\cos at}{s^2 + a^2} \right|_{t=0}^{t=A}$$

$$= \lim_{A\to\infty}\left[\frac{-\ s\, e^{-sA}\sin aA\ -\ a\, e^{-sA}\cos aA}{s^2 + a^2} - \frac{-\ s\, e^{-s(0)}\sin a(0)\ -\ a\, e^{-s(0)}\cos a(0)}{s^2 + a^2}\right]$$

$$= \lim_{A\to\infty}\left[\frac{-\ s\, e^{-sA}\cos aA\ -\ a\, e^{-sA}\cos aA}{s^2 + a^2} + \frac{a}{s^2 + a^2}\right] = \frac{a}{s^2 + a^2} \qquad s > 0.$$

We continue our table of Laplace Transforms.

PARTIAL TABLE OF LAPLACE TRANSFORMS

Time Domain	(Complex) Frequency Domain	
f(t)	F(s)	
0	0	$s > 0$
e^{at}	1/(s-a)	$s > a$
1	1/s	$s > 0$
sin(at)	$\frac{a}{s^2 + a^2}$	$s > 0$

EXERCISES on Computation of the Laplace Transform Using the Definition

EXERCISE#1. Provide a more rigorous statement of Theorem #2 for functions defined on the closed interval I = [α,β].

EXERCISE #2. Provide a more rigorous statement of Theorem #6 by including specifics.

Laplace Transforms-1 PARTIAL TABLE OF ANTIDERIVATIVES
Handout # 2 (INDEFINITE) INTEGRALS Professor Moseley

1. $\int x[\sin(ax)]\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) + C$

2. $\int x[\cos(ax)]\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$

3. $\int x^2[\sin(ax)]\,dx = \frac{2x}{a^2}\sin(ax) - \frac{a^2x^2-2}{a^3}\cos(ax) + C$

4. $\int x^2[\cos(ax)]\,dx = \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2-2}{a^3}\sin(ax) + C$

5. $\int [\sin^2(ax)]\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) + C$

6. $\int [\cos^2(ax)]\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax) + C$

7. $\int [\sin(ax)[\cos(bx)]\,dx = \frac{1}{2a}\sin^2(ax) + C$

8. $\int [\sin(ax)][\sin(bx)]\,dx = \frac{\sin[(a-b)x]}{2(a-b)} - \frac{\sin[(a+b)x]}{2(a+b} + C \qquad a^2 \neq b^2$

9. $\int [\cos(ax)][\cos(bx)]\,dx = \frac{\sin[(a-b)x]}{2(a-b)} + \frac{\sin[(a+b)x]}{2(a+b} + C \qquad a^2 \neq b^2$

10. $\int [\sin(ax)][\cos(bx)]\,dx = -\frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)} + C \qquad a^2 \neq b^2$

Read the introduction and Section 6.1 of Chapter 6 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.) again. Pay particular attention to Examples 1-6 pages 294-297. Pay special attention to the paragraph after Example 6, in particular Equation (5).

The set $\mathbf{T}$ = { f:[0,∞)→$\mathbf{R}$ * f has a Laplace transform } is a subspace of the function space $\mathscr{F}([0,\infty),\mathbf{R})$ vector space and hence is a vector space in it's own right. We have seen that the set $\mathbf{T}_{pcexp} = PC[0,\infty) \cap E_{xp}$ = { f ∈ $\mathbf{T}$: f is piecewise continuous on [0,∞) and of exponential order } is a subspace of $\mathbf{T}$ and hence a vector space in it's own right.

<u>THEOREM #1</u>. The Laplace transform

$$\mathfrak{L}\{f(t)\} = \int_{t=1}^{t=\infty} f(t)\, e^{-st}\, dt = F(s) \qquad (1)$$

is a linear operator acting on the vector space $\mathbf{T}$

<u>Proof:</u> To verify that $\mathfrak{L}$ is a linear operator from $\mathbf{T}$ to $\mathbf{F}$, we first state the definition of a linear operator.

<u>Definition</u>. An operator T:V → W (which maps the vector space V to the vector space W) is said to be <u>linear</u> if ∀ $\vec{x}, \vec{y} \in V$ and ∀ scalars α,β we have

$$T(\alpha\, \vec{x} + \beta\, \vec{y}) = \alpha\, T(\vec{x}) + \beta\, T(\vec{y}). \qquad (2)$$

Applying (2) to $\mathfrak{L}$ we see that to show that it is a linear operator, we wish to verify the identity:

$$\mathfrak{L}\{c_1\, f_1(t) + c_2 f_2(t)\} = c_1\, \mathfrak{L}\{f_1(t)\} + c_2\, \mathfrak{L}\{f_2(t)\} \qquad \forall\, c_1, c_2 \in \mathbf{R}, \forall\, f_1, f_2 \in \mathbf{T} \qquad (3)$$

We use the standard format for proving identities. Let $c_1, c_2 \in \mathbf{R}$ and $f_1, f_2 \in \mathbf{T}$

<u>STATEMENT</u>.	<u>REASON</u>.
$\mathfrak{L}\{c_1 f_1(t)+c_2 f_2(t)\} = \int_{t=0}^{t=\infty} [c_1 f_1(x) + c_2 f_2(x)] e^{-st} dx$	Def'n. of Laplace Trans.
$= \lim_{A\to\infty} \int_{t=0}^{t=A} [c_1 f_1(x) + c_2 f_2(x)] e^{-st} dx$	Def'n. of Improper Integral
$= \lim_{A\to\infty} \left[c_1 \int_{t=0}^{t=A} [f_1(x)] e^{-st} dx + c_2 \int_{t=0}^{t=A} [f_2(x)] e^{-st} dx \right]$	Property of Riemann Integral

$$= \lim_{A\to\infty} c_1 \int_{t=0}^{t=A} [f_1(x)]e^{-st}dx + \lim_{A\to\infty} c_2 \int_{t=0}^{t=A} [f_2(x)]e^{-st}dx$$ Property of Limits

$$= c_1 \int_{t=0}^{t=\infty} [f_1(x)]e^{-st}dx + c_2 \int_{t=0}^{t=\infty} [f_2(x)]e^{-st}dx$$ Def'n. of Improper Integral

$$= c_1 \mathcal{L}\{ f_1(t) \} + c_2 \mathcal{L}\{f_2(t)\}$$ Def'n.of Laplace transform

QED.

Thus if f(t) is a linear combination of functions: $f(t) = c_1 f_1(t) + c_2 f_2(t)$, then its Laplace transform F(s) is a linear combination of transforms:

$F(s) = c_1 F_1(s) + c_2 F_2(s)$ where $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{f_1(t)\} = F_1(s)$, and $\mathcal{L}\{f_2(t)\} = F_2(s)$.

We can use linearity to compute the transform of linear combinations of functions in our table.

EXAMPLE #1 Compute $\mathcal{L}\{f(t)\} = F(s)$ where $f(t) = 3\,e^{3t} + 5\sin(10t)$ for $t \geq 0$.

Solution. We wish to compute $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{ 3\,e^{3t} + 5\sin(10t)\}$.

$$\mathcal{L}\{ 3\,e^{3t} + 5\sin(10t) \} = 3\,\mathcal{L}\{ e^{3t} \} + 5\,\mathcal{L}\{\sin(10t)\} = 3\,\frac{1}{s-3} + 5\,\frac{10}{s^2+100}.$$

When dealing with Laplace transforms, it is usually not necessary or desirable to find a common denominator. However, simple arithmetic is expected: $F(s) = \dfrac{3}{s-3} + \dfrac{50}{s^2+100}$.

EXERCISES on Properties of Laplace Transforms: Linearity

EXERCISES. Using the table computed so far, find $F(s) = \mathcal{L}\{f(t)\}(s)$ if

a) $f(t) = 4\,e^{-3t} + 5$
b) $f(t) = 4\,e^{-3t} + 5\,e^{4t} + 3.5$
c) $f(t) = 28\,e^{8t} + 9\,e^{7t} + 3\sin(3t)$

Laplace Transforms-1 OTHER PROPERTIES OF
Handout No. 4 THE LAPLACE TRANSFORM Professor Moseley

Read Section 6.2 - 6.3 of Chapter 6 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.). Pay particular attention to Theorem 6.2.1 on page 300, the Corollary on page 300, Theorem 6.3.1 on page 311, and Theorem 6.3.2 on page 313.

<u>THEOREM #1</u> Suppose $f \in \mathbf{T}_{pcexp}$; that is, f is piecewise continuous on $[0,A]$ $\forall$ $A>0$, and of exponential order so that $\exists$ T,M and σ such that $*f(t)* < Me^{\sigma t}$ $\forall$ $t>T$. Then each of the following properties holds:

1) $\mathcal{L}\{\, t\, f(t)\, \} = -\dfrac{d}{ds}\, \mathcal{L}\{\, f(t)\, \}(s), \quad \forall\ s > \sigma,$

2) $\mathcal{L}\{\, e^{at} f(r)\, \} = \mathcal{L}\{\, f(t)\, \}(s-a), \quad \forall\ s > \sigma,$ Shifting property,

3) $\mathcal{L}\{\, \displaystyle\int_{r=0}^{r=t} f(r)dr\, \} = \dfrac{1}{s}\, \mathcal{L}\{\, f(t)\, \}(s), \quad \forall\ s > \sigma,$

If in addition, f is continuous and f' piecewise continuous on $[0,\infty)$,then

4) $\mathcal{L}\{\, f'(t)\, \} = s\, \mathcal{L}\{\, f(t)\, \} - f(0), \quad \forall\ s > \sigma.$

More generally, if f, f', ... , $f^{(n-1)}$, are continuous on $[0,\infty)$ and of exponential order, and $f^{(n)}$ is piecewise continuous on $[0,\infty)$, then

5) $\mathcal{L}\{\, f^{(n)}(t)\, \} = s^n\, \mathcal{L}\{\, f(t)\, \}\ )\ s^{n-1} f(0)\ \cdots\ )\ s\, f^{(n-2)}(0)\ )\ f^{(n-1)}(0), \quad \forall\ s > \sigma.$

Specifically

6) $\mathcal{L}\{\, f''(t)\, \} = s^2\, \mathcal{L}\{\, f(t)\, \} - s\, f(0) - f'(0), \quad \forall\ s > \sigma.$

<u>Proof</u>. These are all identities and can be verified using the standard format. We prove only 1). 2) through 6) are left as exercises. For 1) we start with the right hand side (RHS)

<u>STATEMENT</u>.	<u>REASON</u>.
$-\dfrac{d}{ds}\, \mathcal{L}\{\, f(t)\, \}(s) = -\dfrac{d}{ds}\displaystyle\int_{t=0}^{t=\infty} f(t)e^{-st}dt \quad \forall\ s > \sigma$	Definition of the Laplace Transform
$= -\dfrac{d}{ds}\displaystyle\lim_{A\to\infty}\int_{t=0}^{t=A} f(t)e^{-st}dt \quad \forall\ s > \sigma$	Definition of an Improper Integral
$= -\displaystyle\lim_{A\to\infty}\dfrac{d}{ds}\int_{t=0}^{t=A} f(t)e^{-st}dt \quad \forall\ s > \sigma$	Property of limits and derivatives from advanced analysis: The limit and derivative can be switched under certain conditions.
$= -\displaystyle\lim_{A\to\infty}\int_{t=0}^{t=A} \dfrac{\partial}{\partial s} f(t)e^{-st}dt \quad \forall\ s > \sigma$	Property of derivatives and integrals from advanced analysis. Derivatives and integrals can be switched under certain conditions. Derivatives becomes partials.

$$= -\lim_{A\to\infty}\int_{t=0}^{t=A}(-t)f(t)e^{-st}dt \quad \forall\, s > \sigma$$ Property of partial derivative

$$= \lim_{A\to\infty}\int_{t=0}^{t=A}t\, f(t)e^{-st}dt \quad \forall\, s > \sigma$$ Properties of integrals and limits

$$= \int_{t=0}^{t=\infty}t\, f(t)e^{-st}dt \quad \forall\, s > \sigma$$ Definition of improper integral

$$= \mathcal{L}\{\, t\, f(t)\,\}$$ Definition of Laplace Transform

Q.E.D.

<u>EXAMPLE #1</u> Compute $\mathcal{L}\{f(t)\} = F(s)$ where $f(t) = t$ for $t \geq 0$.

<u>Solution</u>. We wish to compute $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{\, t\,\}$.

<u>STATEMENT</u>.	<u>REASON</u>.
$\mathcal{L}\{\, t\,\} = \mathcal{L}\{\, t\,(1)\,\}$	Algebra
$= -\dfrac{d}{ds}\,(\,\mathcal{L}\,\{\,1\,\}\,)$	Part 1) of theorem above
$= -\dfrac{d}{ds}(\dfrac{1}{s})\,)$	From the TABLE $\mathcal{L}\,\{\,1\,\} = \dfrac{1}{s}$
$= -\dfrac{d}{ds}\, s^{-1}$	Algebra
$= -\,(-1\,)\, s^{-2}$	Calculus
$= \dfrac{1}{s^2}$	Algebra

Similarly we can compute

t^2	$\dfrac{2}{s^3}$		$s > 0$
t^3	$\dfrac{(3)(2)}{s^4}$	$= \dfrac{3!}{s^4}$	$s > 0$

And by mathematical induction we can prove that

$$t^n \qquad \frac{n!}{s^{n+1}} \qquad s > 0$$

We continue the development of our table of Laplace Transforms.

PARTIAL TABLE OF LAPLACE TRANSFORMS

Time Domain $f(t)$	(Complex) Frequency Domain $F(s)$	
0	0	$s > 0$
e^{at}	$1/(s-a)$	$s > a$
1	$1/s$	$s > 0$
$\sin(at)$	$\dfrac{a}{s^2 + a^2}$	$s > 0$
t	$\dfrac{1}{s^2}$	$s > 0$
t^2	$\dfrac{2}{s^3}$	$s > 0$
t^3	$\dfrac{(3)(2)}{s^4} = \dfrac{3!}{s^4}$	$s > 0$
t^n	$\dfrac{n!}{s^{n+1}}$	$s > 0$

Now by using the linearity of the Laplace Transform, we can compute the transform for any polynomial.

EXAMPLE #2 Compute $\mathcal{L}\{f(t)\} = F(s)$ where $f(t) = 3 + 5t + 7t^2$ for $t \geq 0$.

Solution. We wish to compute $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{3 + 5t + 7t^2\}$.

STATEMENT.	REASON.
$\mathcal{L}\{3 + 5t + 7t^2\} = 3\,\mathcal{L}\{1\} + 5\,\mathcal{L}\{t\} + 7\,\mathcal{L}\{t^2\}$	Linearity of Laplace Transform
$= 3\,\dfrac{1}{s} + 5\,\dfrac{1}{s^2} + 7\,\dfrac{2}{s^3}$	Table of known Laplace Transforms
$= \dfrac{3}{s} + \dfrac{5}{s^2} + \dfrac{14}{s^3}$	Algebra

We can use Theorem #1 to obtain other transform pairs:

EXAMPLE #3 Compute $\mathcal{L}\{f(t)\} = F(s)$ where $f(t) = t\sin(at)$ for $t \geq 0$.

Solution. We wish to compute $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{t\sin(at)\}$.

STATEMENT. REASON.

$\mathcal{L}\{\, t\sin(at) \,\} = -\dfrac{d}{ds}\left(\mathcal{L}\{\, \sin(at) \,\} \right)$ — Part 1) of theorem above

$= -\dfrac{d}{ds}\left(\dfrac{a}{s^2+a^2}\right)$ — From the TABLE $\mathcal{L}\{\, \sin(at) \,\} = \dfrac{a}{s^2+a^2}$

$= -\dfrac{d}{ds}\left[a\left(s^2+a^2\right)^{-1}\right]$ — Algebra

$= -a(-1)(s^2+a^2)^{-2}(2s)$ — Calculus

$= \dfrac{2as}{\left(s^2+a^2\right)^2}$ — Algebra

EXAMPLE #4 Compute $\mathcal{L}\{f(t)\} = F(s)$ where $f(t) = e^{at}\sin(\omega t)$ for $t \ge 0$.

Solution. We wish to compute $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{\, e^{at}\sin(\omega t) \,\}$.

STATEMENT. REASON.

$\mathcal{L}\{\, e^{at}\sin(\omega t) \,\} = \mathcal{L}\{\, \sin(\omega t) \,\}(\, s - a \,)$ — Part 2) of theorem above

$= \dfrac{\omega}{(s-a)^2+\omega^2}$ — From the TABLE $\mathcal{L}\{\sin(\omega t)\,\} = \dfrac{\omega}{s^2+\omega^2}$

EXAMPLE #5 Compute $\mathcal{L}\{f(t)\} = F(s)$ where $f(t) = \cos(\omega t)$ for $t \ge 0$.

Solution. We wish to compute $\mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{\, \cos(\omega t) \,\}$. We will use property 4): $\mathcal{L}\{\, f'(t) \,\} = s\,\mathcal{L}\{\, f(t) \,\} - f(0)$ with $f(t) = \dfrac{\sin(\omega t)}{\omega}$ so that $f'(t) = \dfrac{d}{dt}\dfrac{\sin(\omega t)}{\omega} = \cos(\omega t)$.

Hence $\mathcal{L}\{\, \cos(\omega t) \,\} = s\ \mathcal{L}\{\dfrac{\sin(\omega t)}{\omega}\} - \dfrac{\sin(\omega\,(0))}{\omega} = \dfrac{s}{\omega}\ \mathcal{L}\{\dfrac{\sin(\omega t)}{\omega}\} = \dfrac{s}{\omega}\,\dfrac{\omega}{s^2+\omega^2}$

$$= \frac{s}{s^2+\omega^2}$$

We continue the development of our table of Laplace Transforms.

PARTIAL TABLE OF LAPLACE TRANSFORMS

Time Domain $f(t)$	(Complex) Frequency Domain $F(s)$	
0	0	$s > 0$
e^{at}	$1/(s-a)$	$s > a$
1	$1/s$	$s > 0$
$\sin(at)$	$\frac{a}{s^2 + a^2}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^2	$\frac{2}{s^3}$	$s > 0$
t^3	$\frac{3!}{s^4}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$t\sin(at)$	$\frac{2as}{\left(s^2 + a^2\right)^2}$	$s > 0$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2 + \omega^2}$	$s > a$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	

Laplace transforms TABLE OF LAPLACE TRANSFORMS
Handout No. 5 TO BE MEMORIZED Professor Moseley

Although it is not reasonable to memorize all Laplace Transform pairs, it is reasonable to memorize the most common ones. Below is a list of the Laplace Transform pairs **you must memorize.**

f(t)	F(S) = $\mathscr{L}\{f(t)\}(s)$	Domain of F(s)
1	$\frac{1}{s}$	s > o
t	$\frac{1}{s^2}$	s > o
t^2	$\frac{2}{s^3}$	s > o
t^n $n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	s > o
e^{at}	$\frac{1}{s-a}$	s > a
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	s > o
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	s > o
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$	s > a
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$	s > a

EXERCISES on Other Properties of the Laplace Transform

EXERCISE #1 Using a Table and Your Knowledge find $F(s) = \mathcal{L}\{f(t)\}(s)$ if

1) $f(t) = 3t^2 + 4t+5$

2) $f(t) = (t+2)^2$

3) $f(t) = \sin\left(3t + \frac{\pi}{6}\right)$ }Hint: Use trig identities

4) $f(t) = \sin^2(3t)$

5) $f(t) = 2te^{3t}$

6) $f(t) = t^2e^{-2t}$

7) $f(t) = e^{-2t} \cos 3t$