A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 2

A COLLECTION OF HANDOUTS ON SCALAR LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODE"s)

CHAPTER 6

Power Series Solutions to Second Order Linear ODE's

Handout No. 1 REVIEW OF LINEAR THEORY Professor Moseley
AND MOTIVATION FOR USING POWER SERIES

Recall that for the remainder of the course that we will not attempt to cover all of the material in the text on a particular topic. Rather you will get only a taste of the topic so that you will have a good start when you see it again in another course.

REVIEW OF LINEAR THEORY FOR ANALYTIC FUNCTIONS. Recall the general second order linear differential operator L defined by

$$L[y] = y'' + p(x)y + q(x)y \tag{1}$$

where p, q $\in A(I)$ and $I = (a,b)$ is the interval of validity. We have considered the **homogeneous** equation $L[y] = 0$:

$$(\ L[y] = \ )\quad y'' + p(x)\,y' + q(x)\,y = 0 \quad \forall\ x \in I \tag{2}$$

and the **nonhomogeneous** equation $L[y] = g \in A(I)$:

$$(\ L[y] = \ )\quad y'' + p(x)\,y' + q(x)\,y = g(x) \quad \forall\ x \in I. \tag{3}$$

THEOREM #1. Let $S = \{\ y_1, y_2\ \}$ be a set of linearly independent solutions to the homogeneous equation (2). Now assume that we can find a (i.e one) particular solution $y_p(x)$ to the nonhomogeneous equation (2). Then $y(x) = y_p(x) + y_c(x)$ where $y_c(x)$ is the general solution of the associated homogeneous equation (also called the complementary equation) (2). That is, the general solution of (3) is given by:

$$y(x) = y_p(x) + c_1\, y_1(x) + c_2\, y_2(x). \tag{4}$$

This theorem reduces the problem of finding the general solution of the nonhomogeneous equation (3) to finding the three functions $y_p(x)$, $y_1(x)$, and $y_2(x)$. If the homogeneous equation has constant coefficients, we have developed a technique to for finding y_1 and y_2. For a large number of **forcing functions** g(x) in $A(I)$, we can find y_p by using either the method of undetermined coefficients or variation of parameters. However, if this is not the case, we do not have a technique for solution (unless for example we have an equation in which y is missing or x is missing). We wish a technique for the case of variable coefficients, that is when p and q are not constants.

MOTIVATION FOR USING POWER SERIES. We wish to develop a more general technique which works for the case when the solutions of (2) and (3) are analytic functions. Recall that a function $f \in C^\infty(I) = \{f{:}I \to R{:}\ f^{(n)}$ is continuous $\forall\ n \in \mathbf{N}\}$ is **analytic** on $I = (a,b)$ if for all $x_0 \in I$, there exists an open interval containing x_0 where f(x) equals its Taylor series. We first review power series and how a function (i.e. a real valued function of a real variable) can be defined by a power series. We then review how a function in $C^\infty(I)$ defines a power series at a point x_0 and hence how to compute Taylor series of an analytic function.

Handout No. 2 FUNCTIONS DEFINED BY POWER SERIES Professor Moseley

Let f(x) be a real (or complex) valued function of a real (or complex) variable. Recall how to find a power series expansion (or representation) of this function by finding its Taylor Series (or its Maclaurin series if the series is about zero). On the other hand, if we are given a power series with coefficients a_n, n = 0, 1, 2, 3, ... , then this power series defines an analytic function

$$y = f(x) \;=\; \sum_{n=0}^{\infty} a_n x^n \;=\; a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \ldots + a_n\, x^n + \ldots$$

on its (open) interval (or circle) of convergence. We may think of the sequence $\{a_n\}_{n=0}^{\infty}$ as the name of the function f since, given a value of x, we can compute a value for y = f(x), provided we can "sum the series". Practically, this may mean that we use a computer to obtain

$$y \;=\; f_A(x) = \sum_{n=0}^{N} a_n x^n \text{, where the remainder } R_N = \sum_{n=N+1}^{\infty} a_n x^n \text{ is small.}$$

The (open) **domain of the function** is the (open) **interval of convergence** of the power series. We review the following skills which you mastered in a previous course:

1. How to determine the interval of convergence of a power series using the Ratio test.
2. How to compute the (coefficients in the) power series for a given function (i.e. how do we compute the Taylor or Maclaurin series for a given function).

A third skill which you may have previously mastered:

3. How to change the index when using sigma notation.

will be covered when we learn how to use power series to solve a second order linear differential equation with (constant or) variable coefficients.

Consider the power series

$$y = \sum_{n=0}^{\infty} a_n x^n \quad = \quad a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots. \tag{1}$$

Let

$$R_n =_{df} \frac{|a_{n+1} x^{n+1}|}{|a_n x^n|} = \frac{|a_{n+1}|}{|a_n|} \; |x| \tag{2}$$

and

$$L =_{df} \lim_{n\to\infty} R_n = (\; \lim \; \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \;) \, |x| . \tag{3}$$

<u>THEOREM</u> (Ratio Test). Then the power series converges if $L < 1$. It diverges if $L > 1$. If $L = 1$, the test is inconclusive.

Thus the power series converges on the open interval where $L < 1$, diverges outside the closed interval where $L \leq 1$, and we must use some other test for the endpoints. However, since we wish to solve ODE's, we wish the domain (i.e. the interval of validity) of our solutions to be open intervals. Hence we are not particularly interested in the end points and will not review the tests needed to check for convergence at the endpoints (e.g., alternating series test). Finding the open interval $L < 1$ where the series converges (i.e. the open interval of convergence) is sufficient.

<u>EXAMPLE #1</u>. Find the open interval of convergence for the power series:
$y = \sum_{n=0}^{\infty} a_n x^n$, where $a_n = (-1)^{n+1} x^n/2^n$; that is, for the power series $y = \sum_{n=0}^{\infty} (-1)^{n+1} x^n \,/\, 2^n$

<u>Solution</u>. Let $R_n = \left| \dfrac{(-1)^{n+2} x^{n+1} \,/\, 2^{n+1}}{(-1)^{n+1} x^n \,/\, 2^n} \right| = \left| \dfrac{(-1)^{n+2} x^{n+1}}{2^{n+1}} \right| \left| \dfrac{2^n}{(-1)^{n+1} x^n} \right| = \dfrac{1}{2} \; |x| .$

Note that although R_n is usually a function of n, for this example, it is not. This makes computing the limit easy.

Let $\quad L = \lim_{n\to\infty} R_n = (\; \lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} \;) \, |x| = \lim_{n\to\infty} \dfrac{1}{2} \; |x| = \dfrac{1}{2} \; |x|$

Hence $L < 1 \Rightarrow \lim_{n\to\infty} \frac{1}{2} |x| < 1 \Rightarrow |x| < 2 \Rightarrow x \in I = (-2, 2)$.

The **radius of convergence**, usually denoted by ρ, is half of the length of the interval I. The reason that it is referred to as the radius of convergence is that a power series can be considered as a complex valued function of a complex variable. In this context, the Ratio Test applies to the series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n . \tag{4}$$

in which case $L < 1$ implies convergence within a circle of radius ρ about the origin, $L > 1$ implies divergence outside this circle, and the test is inconclusive on the circle where $L = 1$.

Recall from calculus

<u>THEOREM</u> (Taylor Series). For any function f(x) that is analytic at the point x_0, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$= f(x_0) + [f'(x_0)]\,(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots .$$

inside some interval (circle if x is replaced by the complex variable z) centered at x_0.

<u>EXAMPLE #1</u>. Find the Maclaurin series for $f(x) = e^x$

<u>Solution</u>. Find the Maclaurin series means find the Taylor series for $x_0 = 0$.

$$\begin{array}{ll}
f(x) = e^x & f(0) = e^0 = 1 \\
f'(x) = e^x & f'(0) = e^0 = 1 \\
f''(x) = e^x & f''(0) = e^0 = 1 \\
\vdots & \vdots \\
f^{(n)}(x) = e^x & f^{(n)}(0) = e^0 = 1
\end{array}$$

Hence

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + [f'(0)]\; x + \frac{f''(0)}{2!}\, x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\, x^n + \cdots$$
$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots .$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

<u>EXAMPLE #2</u>. Find the Maclaurin series for $f(x) = \sin(x)$

<u>Solution</u>. Find the Maclaurin series means find the Taylor series for $x_0 = 0$.

n	k	$f^{(n)}(x)$	$f^{(n)}(0)$
0	0	$f(x) = \sin(x)$	$f(0) = \sin(0) = 0$
1	0	$f'(x) = \cos(x)$	$f'(0) = \cos(0) = 1$
2	0	$f''(x) = -\sin(x)$	$f''(0) = -\sin(0) = 0$
3	0	$f'''(x) = -\cos(x)$	$f'''(0) = -\cos(0) = -1$
4	1	$f^{(4)}(x) = \sin(x)$	$f^{(4)}(0) = \sin(0) = 0$
.	.	.	.
.	.	.	.
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$$\begin{matrix} n = 4k+0 & k \\ n = 4k+1 & k \\ n = 4k+2 & k \\ n = 4k+3 & k \end{matrix} \qquad f^{(n)}(x) = \begin{cases} \sin x \\ \cos x \\ -\sin x \\ -\cos x \end{cases} \qquad f^{(n)}(0) = \begin{cases} \sin(0) = 0 \\ \cos(0) = 1 \\ -\sin(0) = 0 \\ -\cos(0) = -1 \end{cases}$$

We substitute into Taylor's formula and look for a pattern:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + [f'(0)]\ x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

$$= 0 + x + 0 - \frac{1}{3!} x^3 + 0 + \frac{1}{5!} x^5 + 0 - \frac{1}{7!} x^7 \cdots .$$

We see that if we let $k = 0,1,2,3,4, \cdots$, we obtain

$$f(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 \cdots . + \frac{(-1)^k}{(2m+1)!} x^{2m+1} + \cdots .$$

Hence, $\quad \sin(x) = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$

Alternately, since we have four cases for the formula for the n^{th} derivative of sin(x), we break up the formula for the Maclaurin series into four separate series (absolute convergence for each $x \in \mathbf{R}$ can be shown).

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{k=0}^{\infty} \frac{f^{(4k)}(0)}{(4k)!} x^{4k} + \sum_{k=0}^{\infty} \frac{f^{(4k+1)}(0)}{(4k+1)!} x^{4k+1} + \sum_{k=0}^{\infty} \frac{f^{(4k+2)}(0)}{(4k+2)!} x^{4k+2} + \sum_{k=0}^{\infty} \frac{f^{(4k+3)}(0)}{(4k+3)!} x^{4k+3}$$

We may now substitute in the values for the four cases:

$$f(x) = \sum_{k=0}^{\infty} \frac{0}{(4k)!}\, x^{4k} + \sum_{k=0}^{\infty} \frac{1}{(4k+1)!}\, x^{4k+1} + \sum_{k=0}^{\infty} \frac{0}{(4k+2)!}\, x^{4k+2} + \sum_{k=0}^{\infty} \frac{-1}{(4k+3)!}\, x^{4k+3}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(4k+1)!}\, x^{4k+1} - \sum_{k=0}^{\infty} \frac{1}{(4k+3)!}\, x^{4k+3}$$

$$= (\, x + \frac{1}{5!}\, x^5 + \cdots + \frac{1}{(4k+1)!}\, x^{4k+1}\, x + \cdots) - (\, \frac{1}{3!}\, x^3 + \frac{1}{7!}\, x^7 + \cdots + \frac{1}{(4k+1)!}\, x^{4k+1} + \cdots).$$

Taking a term from one series and then the other, these two series may then be combined as

$$f(x) = x - \frac{1}{3!}\, x^3 + \frac{1}{5!}\, x^5 - \frac{1}{7!}\, x^7 \cdots .+ \frac{(-1)^m}{(2m+1)!}\, x^{2m+1} + \cdots .$$

and hence we again get $\quad \sin(x) = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\, x^{2m+1}$

Thus there is not necessarily a unique way to write the series. You need to check your answer to see if it is just the answer in the back of the book in a different form.

HOW TO USE POWER SERIES TO SOLVE SECOND ORDER ODE's WITH VARIABLE COEFFICIENTS.

Recall the general second order linear differential operator

$$L[y] = y'' + p(x)y + q(x)y \tag{1}$$

where $p,q \in C(I)$, $I = (a,b)$. We consider the homogeneous equation:

$$L[y] = y'' + p(x)\, y' + q(x)\, y = 0 \qquad \forall\, x \in I \tag{2}$$

and the nonhomogeneous equation:

$$L[y] = y'' + p(x)\, y' + q(x)\, y = g(x) \qquad \forall\, x \in I. \tag{3}$$

THEOREM #1. Let $S = \{ y_1, y_2 \}$ be a set of linearly independent solutions to the homogeneous equation (2). Now assume that we can find a (i.e one) particular solution $y_p(x)$ to the nonhomogeneous equation (3). Then $y(x) = y_p(x) + y_c(x)$ where $y_c(x)$ is the general solution of the associated homogeneous equation (also called the complementary equation) (2). Thus:

$$y(x) = y_p(x) + c_1\, y_1(x) + c_2\, y_2(x).$$

Theorem #1 reduces the problem of finding the general solution of the nonhomogeneous equation (3) to the finding of the three functions $y_p(x)$, $y_1(x)$, and $y_2(x)$. We wish to see how we can use the concept of representing a function by a power series to find (power series) representations of these three functions. On the other hand, the solution to an initial value problem is typically unique. If explicit initial conditions are given, then we wish to find (a power series representation of) this function. Thus we assume that the solutions of (2) or (3) are analytic n y so that they can be written in the form:

$$y = a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \dots + a_n\, x^n + \dots \quad = \sum_{n=0}^{\infty} a_n x^n$$

PROCEDURE. We know the function y if we know the coefficients a_n, n = 1, 2, 3, ... Hence we substitute the power series into the ODE and attempt to solve for the coefficients a_n, n = 1, 2, 3,... . However, we first make two comments.

COMMENT #1. Since there are an infinite number of coefficients to solve for we expect to have to solve for them recursively.

COMMENT #2. Since the general solution of the homogeneous equation contains two arbitrary constants, we expect that there will be two constants we can not solve for. Since the usual IVP requires that to solve for these constants we usually have $y(0) = a_0$ and $y'(0) = a_1$. Hence we might expect to have to solve for the rest of the coefficients in terms of these two.

EXAMPLE. Using power series, solve y" + y = 0

Solution. This ODE is easily solved by previous techniques to obtain:

$y = c_1 \sin(x) + c_2 \cos(x)$

Knowing the answer before we start should help you to understand the method. We then consider an example that you cannot solve by previous methods. Let

$$1) \quad y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

$$y' = \sum_{n=0 \text{ (or } n=1)}^{\infty} a_n n\, x^{n-1} = a_1 + a_2 2x + a_3 3x^2 + \cdots + a_n n x^{n-1} + \cdots$$

$$1) \quad y'' = \sum_{n=0 \text{ (or } n=1 \text{ or } n=2)}^{\infty} a_n n(n-1)\, x^{n-2} = a_2 2(1) + a_3 3(2) x + \cdots + a_n n(n-1) x^{n-2} + \cdots$$

$$y'' + y = \sum_{n=2}^{\infty} a_n n(n-1)\, x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = a_2 2(1) + a_1 + (a_3 3(2) + a_2 2)\, x + \cdots = 0$$

Hence

$x^0 \quad a_2 2(1) + a_0 = 0 \quad \Rightarrow \quad a_2 = -(½)\, a_0$

$x^1 \quad a_3 3(2) + a_2 x = 0 \quad \Rightarrow \quad a_3 = -(1/[(3)(2)])\, a_1$

$\ddots$

and in general $\displaystyle\sum_{n=2}^{\infty} a_n n(n-1)\, x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$

Note that we have obtained a_2 in terms of a_0 and a_3 in terms of a_1. We could write out more terms and then solve for a_4, a_5, a_6, ... , but it is more expedient to work with the general case. To do this we must review ("learn") how to work with sigma notation, specifically how to change the index of summation so that all the summations can be put together (i.e. they are all written as (coefficient) times x^n. In the first series we let k = n-2 (so that n = k+2):

$$S = \sum_{n=2}^{\infty} a_n n(n-1)\, x^{n-2} = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$$

$$= a_2\, 2(1) + a_3\, 3(2)\, x + \cdots + a_{k+2}\, (k+2)(k+1)\, x^k + \cdots.$$

After the change of index is complete, we may use any index we choose and can go back to using n:

$$S = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)\, x^n$$

This is completely analogous to changing the variable used in a definite integral. Consider:

$$I = \int_{x=0}^{x=4} x \sin(x^2)\,dx = \int_{u=0}^{u=2} \sin(u)\,du = \int_{x=0}^{x=2} \sin(x)\,dx$$

$u = x^2 \Rightarrow du = 2x\, dx$

Note that although the u in the second integral is related to the x in the first integral, the x used in the last integral is not the same as the x used in the first integral. Returning to our power series, we obtain:

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)\, x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

or

$$\sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1)a_n]x^n = 0.$$

so that $a_{n+2}\,(n+2)(n+1) + a_n = 0$ for n = 0, 1, 2, 3, ... Hence we obtain the **recursion formula**:

$a_{n+2} = - a_n / [(n+2)(n+1)]$ for n = 0, 1, 2, 3, ...

$n=0 \Rightarrow$ (as before) $a_2\, 2(1) + a_0 = 0 \quad \Rightarrow \quad a_2 = -(½)\, a_0$
$n=1 \Rightarrow$ (as before) $a_3\, 3(2) + a1 = 0 \quad \Rightarrow \quad a_3 = -(1/[(3)(2)])\, a_1$
$n=2 \Rightarrow \qquad a_4\, 4(3) + a_2 = 0 \quad \Rightarrow \quad a_4 = -\ (1/[(4)(3)])\, a_2 =(1/[(4)(3)2])\, a_0$
$n=3 \Rightarrow \qquad a_5\, 5(4) + a_1 = 0 \quad \Rightarrow \quad a_5 = -(1/[(5)(4)])\, a_3 = (1/(5!))\, a_1$

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arbitrary n $\Rightarrow$

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} = \begin{cases} [(-1)^{k+1} / (2k+2)!]a_0 \text{ if } & n = 2k, \quad k = 0,1,2,3,\ldots \\ [(-1)^{k+1} / (2k+3)!]a_1 \text{ if } & n = 2k+1, \ k = 0,1,2,3,\ldots \end{cases}$$

Thus:

$$a_n = = \begin{cases} [(-1)^{k} / (2k)!]a_0 \text{ if } & n = 2k, \ k = 0,1,2,3,\ldots \\ [(-1)^{k} / (2k+1)!]a_1 \text{ if } & n = 2k+1, k = 0,1,2,3,\ldots \end{cases}$$

which may be proved by mathematical induction. For this example, we were able to use the recursion formula to obtain a general formula for a_n in terms of a_0 and a_1. Since we know the solution of this problem in terms of elementary functions, we can continue. For many problems, the recursion formula is as far as we can go. However, if you are given initial conditions $y(0) = y_0 = a_0$ and $y'(0) = v_0 = a_1$, you can use the recursion formula (and a computer) to compute as many values a_n as you wish. For this example, we have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\ &= \sum_{k=0}^{\infty} [(-1)^k / (2k)!]a_0\, x^{2k} + \sum_{k=0}^{\infty} [(-1)^k / (2k+1)!]a_1\, x^{2k+1} \\ &= a_0 - \frac{1}{2!}\, a_0 x^2 + \cdots + (-1)^k \frac{1}{(2k)!} a_0 x^{2k} \cdots \\ &\qquad + a_1 x - \frac{1}{3!}\, a_1 x^3 + \cdots + (-1)^k \frac{1}{(2k+1)!} a_1 x^{2k+1} \cdots \\ &= a_0 \left(1 - \frac{1}{2!}\, x^2 + \cdots + (-1)^k \frac{1}{(2k)!} x^{2k} \cdots\right) + a_1 \left(x - \frac{1}{3!}\, x^3 + \cdots + (-1)^k \frac{1}{(2k+1)!} x^{2k+1} \cdots\right) \end{aligned}$$

We recognize both of these series. The second we computed as the Maclaurin series for sin(x).

The first is cos(x). Hence we obtain: $\quad y = a_0 \cos(x) + a_1 \sin(x)$

Letting $y_1 = \cos(x)$ and $y_2 = \sin(x)$ we obtain y in the standard form: $\quad y = a_0\, y_1 + a_1\, y_2$.