A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 2

A COLLECTION OF HANDOUTS ON SCALAR LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODE"s)

CHAPTER 5

Higher Order Linear ODE'S

1. Introduction and Theory for Higher Order ODE’s
2. Technique for Solving Higher Order Homogeneous ODE’s with Constant Coefficients
3. Technique for Solving Higher Order Nonhomogeneous ODE’s: Method of Undetermined Coefficients
4. Technique for Solving Higher Order Nonhomogeneous ODE’s: Method of Variation of Parameters

We first consider the general **n^{th} order linear differential operator** from the vector space $C^n(I)$ to the vector space $C(I)$ where $I = (a,b)$:

$$L[\,y\,] = y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \dots + p_1(x)y' + p_0(x)\, y. \tag{1}$$

where $p_i(x) \in C(I)$ for all i and $I = (a,b)$ is the interval of validity. If $p,q \in \mathcal{A}(I)$, we take L to map $\mathcal{A}(I)$ to $\mathcal{A}(I)$ and if $p,q \in H(\mathbf{C})$, we take L to map $H(\mathbf{C})$ to $H(\mathbf{C})$. Recall that an **operator** is like a function except that it maps a function to another function, instead of a number to another number. Algebraically, we treat collections of functions as vector spaces provided the Laws of Vector Algebra are satisfied. Hence if $p_i(x) \in C(I)$ for all i, we view L as mapping the vector space $C^n(I)$, the set of functions which have n derivatives and whose n^{th} derivative is continuous on the interval $I = (a,b)$, to the vector space $C(I)$ of continuous functions on I; if $p_i(x) \in \mathcal{A}(I)$ for all i, we view L as mapping the vector space $\mathcal{A}(I)$ to the vector space $\mathcal{A}(I)$; and if $p_i(x) \in H(\mathbf{C})$ for all i, we view L as mapping the vector space $H(\mathbf{C})$ to the vector space $H(\mathbf{C})$.

We also consider the linear **homogeneous** equation $L[y] = 0$:

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \dots + p_1(x)y' + p_0(x)\, y = 0 \qquad \forall\, x \in I = (a,b) \tag{2}$$

and the linear **nonhomogeneous** equation $L[y] = g$:

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \dots + p_1(x)y' + p_0(x)\, y = g(x) \qquad \forall\, x \in I = (a,b) \tag{3}$$

We review the **linear theory** and apply it to the n^{th} order linear operator L given in (1). We begin our review with the definition of a **linear operator**. To distinguish it from functions that map numbers to numbers, a "function" or mapping T from one vector space V to another vector space W is often call an **operator**. Similar to writing $f:\mathbf{R}\to \mathbf{R}$, we write $T:V \to W$. If we wish to think geometrically rather than algebraically we might call T a **transformation** or a **transform**.

<u>DEFINITION #1</u>. An operator $T:V \to W$ is said to be **<u>linear</u>** if for all $\vec{x}, \vec{y} \in V$ and for all scalars α,β we have

$$T(\alpha\, \vec{x} + \beta\, \vec{y}\,) = \alpha T(\vec{x}\,) + \beta T(\vec{y}\,). \tag{4}$$

<u>THEOREM #1</u>. The operator defined by L in (1) is linear and its null space has dimension n. Hence the general solution of the <u>homogeneous</u> equation (2) has the form

$$y(x) = c_1\, y_1(x) + \cdots + c_n\, y_n(x) = \sum_{i=1}^{n} c_i\, y_i(x) \tag{5}$$

where $B = \{y_1,\cdots,y_n\}$ is a basis for the null space N_L and c_i, i=1,...,n are arbitrary constants.

Since we know that the dimension of the null space N_L is n, if we have a set of n solutions to the homogeneous equation (1), to show that it is a basis of the null space N_L, it is sufficient to show that it is a linearly independent set.

It is important that you learn the definition of linear independence in an abstract vector space.

DEFINITION #1. Let V be a vector space. A finite set of vectors $S = \{\vec{x}_1,\ldots,\vec{x}_k\} \subseteq V$ is **linearly independent** (ℓ.i.) if the only set of scalars $c_1, c_2, \ldots, c_k$ which satisfy the (homogeneous) vector equation

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{0} \tag{1}$$

is $c_1 = c_2 = \cdots = c_n = 0$; that is, (1) has only the trivial solution. If there is a set of scalars not all zero satisfying (1) then S is **linearly dependent** (ℓ.d.).

DEFINITION #2. Let $f_1,\ldots,f_k \in \mathscr{F}(I,\mathbf{R})$ where $I = (a,b)$. Now let $J = (c,d) \subseteq (a,b)$ and for $i = 1,\ldots,k$, denote the restriction of f_i to J by the same symbol. Then we say that $S = \{f_1,\ldots,f_k\} \subseteq \mathscr{F}(J,\mathbf{R}) \subseteq \mathscr{F}(I,\mathbf{R})$ is **linearly independent** on J if S is linearly independent as a subset of $\mathscr{F}(J,\mathbf{R})$. Otherwise S is **linearly dependent** on J.

Applying Definitions #1 and 2 to a set of k functions in the function space $C^n(I)$ we obtain:

THEOREM #1. The set $S = \{f_1,\ldots,f_k\} \subseteq C^n(I)$ where $I = (a,b)$ is **linearly independent** on I if (and only if) the only solution to the equation

$$c_1 f_1(x) + \cdots + c_k f_k(x) = 0 \qquad \forall\, x \in I \tag{1}$$

is the trivial solution $c_1 = c_2 = \cdots = c_k = 0$ (i.e., S is a **linearly independent set** in the vector space $C^n(I)$). If there exists $c_1, c_2, \cdots, c_n \in \mathbf{R}$, not all zero, such that (1) holds, (i.e, there exists a nontrivial solution) then S is **linearly dependent** on I (i.e., S is a **linearly dependent set** in the vector space $C^n(I)$ which is a subspace of $\mathscr{F}(I,\mathbf{R})$).

Often people abuse the definition and say the functions in S are linearly independent or linearly dependent on I rather than the set S is linearly independent or dependent. Since it is in general use, this abuse is permissible, but not encouraged as it can be confusing. Note that Eq. (1) is really an infinite number of equations in the two unknowns c_1 and c_2, one for each value of x in the interval I. Four theorems are useful.

THEOREM #2. If a finite set $S \subseteq C^n(I)$ where $I = (a,b)$ contains the zero function, then S is linearly dependent on I.

THEOREM #3. If f is not the zero function, then $S = \{f\} \subseteq C^n(I)$ is linearly independent.

THEOREM #4. Let $S = \{f,g\} \subseteq C^n(I)$ where $I = (a,b)$. If either f or g is the zero function in $C^n(I)$ (i.e., is zero on I), then S is linearly dependent on I.

THEOREM #5. Let $S = \{f,g\} \subseteq C^n(I)$ where $I = (a,b)$ and suppose neither f or g is the zero function. Then S is linearly dependent if and only if one function is a scalar multiple of the other (on I).

PROCEDURE. To show that $S = \{f_1, f_2, f_3, \ldots f_n\}$ is linearly independent it is standard to assume (6) and try to show $c_1 = c_2 = c_3 = \ldots = c_n = 0$. If this can not be done, to show that S is linearly dependent, it is mandatory that a nontrivial solution to (6) be exhibited.

DEFINITION #3. If $y_1, \ldots, y_n \in C^1(I)$, where $I = (a,b)$, then

$$W(y_1,\ldots,y_2;x) =_{df} W(x) =_{df} \begin{vmatrix} y_1 & \cdot & \cdot & \cdot & y_n \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ y_1^{(n-1)} & \cdot & \cdot & \cdot & y_n^{(n-1)} \end{vmatrix} \tag{7}$$

is called the Wronski determinant or the Wronskian of $y_1, \ldots, y_n$ at the point x.

THEOREM #4. The null space N_L of the operator defined by L in (1) above has dimension n. Hence the solution of the homogeneous equation

$$L[y] = 0 \quad \forall\, x \in I = (a,b) = \text{interval of validity} \tag{8}$$

has the form

$$y(x) = c_1\, y_1(x) + \cdots + c_n\, y_n(x) = \sum_{i=1}^{n} c_i y_i(x) \tag{9}$$

where $\{y_1, \cdots, y_n\}$ is a basis for N_L and c_i, $i=1,\ldots,n$ are arbitrary constants. Given a set of n solutions to $L[y] = 0$, to show that they are linearly independent solutions, it is sufficient to compute the Wronskian

$$W(y_1,\ldots,y_2,x) =_{df} W(x) =_{df} \begin{vmatrix} y_1 & \cdot & \cdot & \cdot & y_n \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ y_1^{(n-1)} & \cdot & \cdot & \cdot & y_n^{(n-1)} \end{vmatrix} \tag{10}$$

and show that it is not equal to zero on the interval of validity.

<u>THEOREM #5</u>. The nonhomogeneous equation

$$L[y] = g(x) \qquad \forall\, x \in I = (a,b) = \text{interval of validity} \tag{11}$$

has at least one solution if the function g is contained in the range space of L, R(L). If this is the case then the general solution of (11) is of the form

$$y(x) = y_p(x) + y_h(x) \tag{12}$$

where y_p is a <u>particular</u> (i.e. any specific) solution to (11) and y_h is the general (e.g. a formula for all) solutions of (8). Since N(L) is finite dimensional with dimension n we have

$$y(x) = y_p(x) + \sum_{i=1}^{n} c_i y_i(x). \tag{13}$$

where $B = \{y_1, y_2, \ldots, y_n\}$ is a basis of the null space N(L).

EXERCISES on Introduction and Theory for Higher Order ODE's

EXERCISE #1 Compute $L[\phi]$ if the operator L is defined by $L[y] = y^{(4)} + y' + 3y$ and (a) $\phi(x) = \sin x$, (b) $\phi(x) = \cos x$, (c) $\phi(x) = e^x$.

EXERCISE #2 Compute $L[\phi]$ if the operator L is defined by $L[y] = y''' - y'' + 2y$ and (a) $\phi(x) = \sin x$, (b) $\phi(x) = e^x$, (c) $\phi(x) = e^{-x}$.

EXERCISE #3. Directly using the Definition (DUD) or by using Theorem 1, prove that the following operators L[y] which maps the vector space $C^n(I)$ (the set of function which have n derivatives and whose n^{th} derivative is continuous on the interval of validity I) into the space C(I) of continuous functions on I is linear.
(a) $L[y] = y^{(4)} + y' + 3y$, (b) $L[y] = y''' - y'' + 2y$

EXERCISE #4. Determine (and prove your answer directly using the definition (DUD), or by using Theorem A or Theorem B) if the following sets are ℓ.i. or ℓ.d. in $C^2(\mathbf{R})$.
(1) $\{e^x, e^{2x}, 3e^x, 2e^x\}$ (2) $\{\sin x, \cos x, 1 - \sin^2 x, \cos^2 x\}$ (3) $\{3e^x, 2e^{2x}, \sin x, \cos x\}$
(4) $\{1 - \sin^2 x, \cos^2 x, \sin 2x, \sin x \cos x\}$ Hint: Since (8) must hold $\forall\, x \in \mathbf{R}$, as your first try, pick several (distinct) values of x to show (if possible) that $c_1 = c_2 = c_3 = ... = c_n = 0$. Hint: If (8) can only hold $\forall\, x \in \mathbf{R}$ if $c_1 = c_2 = c_3 = ... = c_n = 0$, use Theorem B to show that the set is ℓ.i. If this is not possible find $c_1, c_2, c_3, ..., c_n$ not all zero s.t. (8) holds $\forall\, x \in \mathbf{R}$. Exhibiting (8) with these values provide conclusive evidence that $\{f_1, f_2, f_3, ... f_n\}$ is ℓ.d.

EXERCISE #5. Compute the Wronskian $W(y_1, ..., y_n; x)$ of the following:
(a) $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = \sin x$ $y_4 = \cos x$, (b) $y_1 = \sin x$ $y_2 = \cos x$, $y_3 = 1$, $y_4 = x$,
(c) $y_1 = e^{ax}$, $y_2 = e^{bx}$, $y_3 = \sin wx$ $y_4 = \cos wx$

Handout #2 TECHNIQUE FOR SOLVING HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS Prof. Moseley

Recall the **homogeneous** equation:

$$L[y] = 0. \tag{1}$$

where L is a linear operator of the form

$$L[y] = y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \dots + p_1(x)' + p_0(x)\, y. \tag{2}$$

We consider the special case when the functions $p_i(x)$, i = 1,2,...,n, are constants. For notational convenience, we consider:

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + a_{n-2}\, y^{(n-2)} + \dots + a_1 y' + a_0\, y = 0 \qquad a_n \neq 0 \qquad \forall\, x \in R. \tag{3}$$

Since the coefficient functions are constant, they are continuous for all $x \in R$ and the interval of validity is the entire real line. By the linear theory we obtain:

<u>THEOREM</u>. Let $S = \{y_1,\cdots,y_n\}$ be a set of solutions to the homogeneous equation (3). Then the following are equivalent (i.e. they happen at the same time).

a. The set S is linearly independent. (This is sufficient for S to be a basis of the null space N(L) of the linear operator L[y] since the dimension of N(L) is n.)

b. $W(y_1,...,y_2;x) \neq 0 \qquad \forall\, x \in \mathbf{R}$.

c. All solutions of (3) can be written in the form

$$y(x) = c_1\, y_1(x) + \cdots + c_n\, y_n(x) = \sum_{i=1}^{n} c_i\, y_i(x) \tag{4}$$

where c_i, i=1,...,n are arbitrary constants. That is, since S is a basis of N_L it is a spanning set for N_L and hence every function ("vector") in N_L can be written as a linear combination of the functions ("vectors") in S.

The theorem reduces the problem of finding the general solution of the homogeneous equation (3) to finding the n linearly independent functions $y_1,...,y_2$. We can extend the technique for solving second order linear homogeneous equations with constant coefficients to finding the functions $y_1,...,y_2$ for the case of N^{th} order linear homogeneous equations with constant coefficients. We "guess" that there may be solutions to (3) of the form

$$y(x) = e^{rx} \tag{5}$$

where r is a constant to be determined later. We attempt to determine r by substituting y into the ODE and obtaining a condition on r in order that (3) have a solution of the form (5). Using our standard technique for substituting into a linear equation we obtain:

$$
\begin{array}{ll}
a_0) & y = e^{rx} \\
a_1) & y' = r\, e^{rx} \\
\vdots & \\
a_{n-1}) & y^{(n-1)} = r^{n-1} e^{rx} \\
a_n) & y^{(n)} = r^n \; e^{rx} \\
\hline
\end{array}
$$

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)\prime} + \ldots + a_1\, y' + a_0\, y = (a_n\, r^n + a_{n-1}\, r^{n-1} + \ldots + a_0\,)\, e^{rx} = 0.$$

Hence we obtain the Characteristic or Auxiliary equation:

$$a_n\, r^n + a_{n-1}\, r^{n-1} + \ldots + a_1 r + a_0 = 0. \qquad a_n \neq 0 \qquad \underline{\text{Auxiliary}}\text{ equation} \qquad (6)$$

Hence we have changed the "calculus" problem of solving an n^{th} order linear differential equation with constant coefficients to a high school algebra problem of solving a n^{th} degree polynomial equation. (Hence no antiderivatives need be computed.) In accordance with the fundamental theorem of algebra, (6) has "n solutions" i.e. the left hand side (LHS) can be factored into

$$a_n\,(r - r_1)(r - r_2)\,\ldots\,(r - r_n) = 0. \qquad a_n \neq 0 \qquad (7)$$

Hence your **factoring skills** for higher degree polynomial equations are a must. Two or more of the factors in Equation (7) may be the same. This makes the semantics of the standard verbiage sometimes confusing, but once you understand what is meant, there should be no problem. As in second order equations, we speak of the special case of "repeated roots".)

EXAMPLE #1. Solve (i.e. find the general solution of) $y^{iv} - y = 0$ subject to the initial conditions $y(0) = 1,\ y'(0) = 0,\ y''(0) = 3,\ y'''(0) = 0.$

Solution. Let $y(x) = e^{rx}$. Hence substituting into the ODE we obtain:

$$
\begin{array}{ll}
-1) & y = e^{rx} \\
 & y' = r\, e^{rx} \\
 & y'' = r^2\, e^{rx} \\
 & y''' = r^3\, e^{rx} \\
1) & y^{iv} = r^4\, e^{rx} \\
\hline
\end{array}
$$

$$y^{iv} - y = (\, r^4 - 1\,)\, e^{rx} = 0.$$

Hence we obtain: $r^4 - 1 = 0$. Auxiliary equation

This factors so we obtain: $(r^2 + 1)(r^2 - 1) = (r^2 + 1)(r - 1)(r + 1) = 0$

so that $(r^2 + 1) = 0$ or $(r - 1) = 0$ or $(r + 1) = 0$.

Hence $r^2 = -1$ or $r = 1$ or $r = -1$

We let $r_{1,2} = \pm i$ and $r_3 = 1$ and $r_4 = -1$. Hence we let

$y_1 = \sin(x)$, $y_2 = \cos(x)$, $y_3 = e^x$, and $y_4 = e^{-x}$

so that the general solution to the ODE $y^{iv} - y = 0$ is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + c_4 y_4(x) = c_1 \sin(x) + c_2 \cos(x) + c_3 e^x + c_4 e^{-x}.$$

We now apply the initial conditions to obtain c_1, c_2,c_3 and c_4. First we compute

$$\begin{aligned}
y(x) &= c_1 \sin(x) + c_2 \cos(x) + c_3 e^x + c_4 e^{-x}.\\
y'(x) &= c_1 \cos(x) - c_2 \sin(x) + c_3 e^x - c_4 e^{-x}.\\
y''(x) &= - c_1 \sin(x) - c_2 \cos(x) + c_3 e^x + c_4 e^{-x}.\\
y'''(x) &= - c_1 \cos(x) + c_2 \sin(x) + c_3 e^x - c_4 e^{-x}.
\end{aligned}$$

Applying the initial conditions: $y(0) = 1$, $y'(0) = 0$, $y''(0) = 3$, $y'''(0) = 0$

we get the 4 linear algebraic equations:

$$\begin{aligned}
1 &= c_1 (0) + c_2 (1)) + c_3 e^0 + c_4 e^0.\\
0 &= c_1 (1) - c_2 (0) + c_3 e^0 - c_4 e^0.\\
3 &= - c_1 (0) - c_2 (1) + c_3 e^0 + c_4 e^0.\\
0 &= - c_1 (1) + c_2 (0) + c_3 e^0 - c_4 e^0.
\end{aligned}
\qquad \text{or} \qquad
\begin{array}{rrrrl}
 & c_2 & + c_3 & + c_4 & = 1\\
c_1 & & + c_3 & - c_4 & = 0\\
 & - c_2 & + c_3 & + c_4 & = 2\\
- c_1 & & + c_3 & - c_4 & = 0
\end{array}$$

We could use Gauss elimination (matrix reduction) to solve this system. However, for these (textbook) problems an ad hoc procedure is often faster. Adding the equations we get $4 c_3 = 4$ so that $c_3 = 1$. Now adding the second and fourth equations we get $2 - 2 c_4 = 0$. Hence $c_4 = 1$. From the first equation we get $c_2 = -1$ and from the second equation we get $c_1 = 0$. Hence

$$y(x) = - \cos(x) + e^x + e^{-x}.$$

<u>EXAMPLE #2</u>. Solve $y''' + 4 y'' + 5 y' + 2 y = 0$.

<u>Solution</u> Let $y(x) = e^{rx}$. Hence substituting into the ODE we obtain the auxiliary equation:

$r^3 + 4\,r^3 + 5\,r^2 + 4\,r + 2 = 0.$ Auxiliary equation

To factor the polynomial: $p(r) = r^3 + 4\,r^3 + 5\,r^2 + 4\,r + 2$

we recall the **RATIONAL ROOT THEOREM**:

THEOREM #2 (Rational Root). Consider the polynomial:

$$p(r) = a_n\, r^n + a_{n-1}\, r^{n-1} + \dots + a_1 r + a_0 \tag{8}$$

where $a_i \in \mathbf{Z}$ for $i = 1, 2, \dots, n$. Then if the equation $p(r) = 0$ has any rational roots, (if the polynomial $p(r)$has any zeros) they are of the form: $r = \pm\, k/m$ where k is a factor of a_0 and m is a factor of a_n. If $a_n = 1$ as is often the case, then the possible zeros of p are simply plus or minus the factors of the constant a_0 in the polynomial.

In the example, the possible rational zeros of $p(r) = r^3 + 4\,r^3 + 5\,r^2 + 4\,r + 2$ are therefore $r = \pm 1$ and ± 2. However, note that all of the coefficients of p are positive. Hence there can be no positive zeros. Hence we try $r = -1$.

$p(-1) = (-1)^3 + 4\,(-1)^2 + 5\,(-1) + 2 = -1 + 4 - 5 + 2 = 0$

Hence $r = -1$ is a zero of p and $r + 1$ is a factor. We wish to divide $r + 1$ into $p(r)$. You can use synthetic division if you wish, but the propensity for error indicates that long division is better.

$$\begin{array}{r l}
 & \;r^2 + 3\,r + 2 \\
r + 1 \,\big) & \overline{\;r^3 + 4\,r^2 + 5\,r + 2\;} \\
 & \overset{-}{\oplus}\, r^3 \;\overset{-}{\oplus}\, r^2 \\ \cline{2-2}
 & \quad 3\,r^2 + 5\,r \\
 & \quad \overset{-}{\oplus}\, 3\,r^2 \;\overset{-}{\oplus}\, 3\,r \\ \cline{2-2}
 & \qquad\quad 2\,r + 2 \\
 & \qquad\quad \overset{-}{\oplus}\, 2\,r \,\overset{-}{\oplus}\, 2 \\ \cline{2-2}
 & \qquad\qquad\quad 0
\end{array}$$

Hence $p(r) = r^3 + 4\,r^2 + 5\,r + 2 = (r + 1)(r^2 + 3\,r + 2)$.
But now the last term is a quadratic and can be factored.

$p(r) = r^3 + 4r^2 + 5r + 2 = (r+1)(r^2 + 3r + 2)$
$= (r+1)(r+2)(r+1)$

Hence the zeros of p are $r_1 = r_2 = -1$ and $r_2 = -2$. Hence the three linearly independent solutions of
$y''' + 4y'' + 5y' + 2y = 0$ are $y_1 = e^{-x}$, $y_2 = x e^{-x}$, and $y_3 = e^{-2x}$.

Note that to find a second solution associated with the zero $r = -1$, we multiplied the first solution $y_1 = e^{-x}$ by x. We learned that this works for second order equations by using reduction of order. It works in a similar manner for repeated roots which have algebraic multiplicity greater than two. Hence the general solution of $y''' + 4y'' + 5y' + 2y = 0$ is:

$y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x)$
$y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{-2x}$.

If initial conditions are given, these can be used to obtain c_1, c_2, and c_3.

EXAMPLE #3. Solve $y^{(n)} - y = 0$ in $H(\mathbf{C})$. Also find a solution in $A(\mathbf{R})$.

Solution: Let $y(x) = e^{rx}$. Hence substituting into the ODE we obtain the auxiliary equation:

$r^n - 1 = 0$. Auxiliary equation

To factor the polynomial: $p(r) = r^n - 1$ we recall **DE MOIVRE'S THEOREM**: for finding the n^{th} roots of unity. But first we note that since the coefficients are all real, the unreal roots come in complex conjugate pairs.

THEOREM #3 (DE MOIVRE) Let $\theta = \frac{2\pi}{n}$. The n^{th} roots of unity are $e^{i\theta}, e^{i2\theta}, \ldots, e^{i(n-1)\theta}, 1$ or $r_k = \cos(k\theta) + i\sin(k\theta) = \mu_k + i\,\nu_k$ for $k = 1, 2, \ldots, n$.

Hence the general solution of $y^{(n)} - y = 0$ in $H(\mathbf{C})$ may be written as

$$y = c_1\, e^{e^{i\theta}x} + \cdots + c_{n-1}\, e^{e^{i(n-1)\theta}x} + c_n e^x = c_1\, e^{(\mu_1 + i\nu_1)x} + \cdots + c_{n-1}\, e^{(\mu_{n-1} + i\nu_{n-1})x} + c_n e^x$$

$$= c_1\, e^{\mu_1 x}\cos(\nu_1 x) + c_1\, i\, e^{\mu_1 x}\sin(\nu_1 x) \cdots + c_{n-1}\, e^{\mu_{n-1}x}\cos(\nu_{n-1}x) + c_{n-1}\, i\, e^{\mu_{n-1}x}\sin(\nu_{n-1}x) + c_n e^x$$

$$= c_1 e^{\mu_1 x}\cos(\nu_1 x) + \cdots + c_{n-1} e^{\mu_{n-1}x}\cos(\nu_{n-1}x) + i\,(c_1\, e^{\mu_1 x}\sin(\nu_1 x) \cdots + c_{n-1} e^{\mu_{n-1}x}\sin(\nu_{n-1}x)) + c_n e^x$$

where $c_1, \ldots, c_n \in \mathbf{C}$.

If $n = 2p+1$ is odd then $\theta = \frac{2\pi}{2p+1}$ and we may take the roots to be $r_k = \cos(k\theta) + i\sin(k\theta)$ for

$k = 0, \pm 1, \pm 2, ..., \pm p$ or $r_0 = 1$, $r_k = \cos(k\theta) \pm i \sin(k\theta) = \mu_k \pm \nu_k$ for $k = 0, 1, 2, ..., p$. Hence the solution of $y^{(n)} - y = 0$ in $A(\mathbf{R})$ is

$$y = c_0 e^x + c_{1c} e^{\mu_1 x} \cos(\nu_k x) + c_{1s} e^{\mu_1 x} \sin(\nu_k x) + ... + c_{pc} e^{\mu_p x} \cos(\nu_p x) + c_{ps} e^{\mu_p x} \sin(\nu_p x)$$

where $c_0, c_{1c}, c_{1s}, ..., c_{pc}, c_{ps} \in \mathbf{R}$.

If $n = 2p$ is even then $\theta = \dfrac{2\pi}{2p} = \dfrac{\pi}{p}$ and we may take the roots to be $r_k = \cos(k\theta) + i \sin(k\theta)$ for $k = 0, \pm 1, \pm 2, ..., \pm(p-1), p$ or $r_0 = \pm 1$, $r_k = \cos(k\theta) \pm i \sin(k\theta) = \mu_k \pm i\nu_k$ for $k = 0, 1, 2, ..., p-1$. Hence the solution of $y^{(n)} - y = 0$ in $A(\mathbf{R})$ is

$$y=c_0 e^x + c_{1c} e^{\mu_1 x} \cos(\nu_1 x) + c_{1s} e^{\mu_1 x} \sin(\nu_1 x) + ... + c_{(p-1)c} e^{\mu_{(p-1)} x} \cos(\nu_{(p-1)} x) + c_{(p-1)s} e^{\mu_{(p-1)} x} \sin(\nu_{(p-1)} x) + c_p e^{-x}$$

where $c_0, c_{1c}, c_{1s}, ..., c_{(p-1)c}, c_{(p-1)s}, c_p \in \mathbf{R}$.

Recall the **homogeneous** equation L[y] = 0 where L is a linear operator of the form

$$L[y] = y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \dots + p_1(x)' + p_0(x)\, y \tag{1}$$

and the theorem:

<u>THEOREM #1</u>. The **nonhomogeneous** equation

$$L[y] = g(x) \qquad \forall\, x \in I = (a,b) = \text{interval of validity} \tag{2}$$

has at least one solution if the function g is contained in the range space of L, R(L). If this is the case then the general solution of (1) is of the form

$$y(x) = y_p(x) + y_c(x) \tag{3}$$

where y_p is a <u>particular</u> (i.e. any specific) solution to (3) and y_c is the general (e.g. a formula for all) solutions of (1). Since N_L is finite dimensional with dimension n we have

$$y(x) = y_p(x) + \sum_{i=1}^{n} c_i\, y_i(x). \tag{4}$$

There are two standard methods for obtaining the particular solution y_p:

1. The METHOD OF UNDETERMINED COEFFICIENTS (Also called the method of judicious guessing).
2. The METHOD OF VARIATION OF PARAMETERS.

To use the Method of Undetermined Coefficients we must have:
1. The Homogeneous equation must have constant coefficients.
2. The Right Hand Side (RHS) (i.e. the forcing function g(x)) must be "nice.

As with second order equations, we explore the method of undetermined coefficients using examples and the RULES OF THUMB previously developed.

RULE OF THUMB #1. Choose y_p of the same form as the Right Hand Side (RHS).
(To find a finite dimensional subspace W_1 containing g(x) so that L maps W_1 back to W_1, we must have that all scalar multiples of g are in W_1.)

RULE OF THUMB #2. Choose y_p as a linear combination of RHS and all derivatives of RHS.
(Given that scalar multiples of g are to be in W_1, to find a finite dimensional subspace W_1 containing g(x) so that L maps W_1 back to W_1, we must have that at least two derivatives of g are in W_1. To simplify, we require all derivatives of g to be in W_1.)

FUNCTIONS THAT L MAPS BACK INTO A FINITE DIMENSIONAL SUBSPACE:

Now consider the more general case of

$$a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0\, y = g(x) \tag{5}$$

where

1) $g(x) = a\, e^{\alpha x}$. $(a \neq 0)$
2) $g(x) = a \sin(\omega x) + b \cos(\omega x)$. (a or b may be zero, but not both)
3) $g(x) = P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$. ($a_n \neq 0$, but others may be)

Similar to $A\, e^{\alpha x}$ and $A \sin(x) + B \cos(x)$, since the derivative of a polynomial of degree less than n is a polynomial of degree less than n, L maps any polynomial of degree n into the subspace spanned by the polynomials of degree n.

RULE OF THUMB #3. For the cases listed above, y_p is assumed to be of the same form; that is,

1) $y_p(x) = A\, e^{\alpha x}$.
2) $y_p(x) = A \sin(\omega x) + B \cos(\omega x)$.
3) $y_p(x) = \overline{P}_n(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0$.

RULE OF THUMB #4. For products of the above functions, assume corresponding products for y_p eliminating duplicate coefficients. For example, if $g(x) = P_n(x)\, e^{\alpha x}$, then $y_p(x) = \overline{P}_n(x)\, e^{\alpha x} = A_n x^n e^{\alpha x} + A_{n-1} x^{n-1} e^{\alpha x} + \ldots + A_1 x e^{\alpha x} + A_0 e^{\alpha x}$ and if $g(x) = P_n(x) \sin(x)$, then

$$y_p(x) = A_n x^n \sin(x) + A_{n-1} x^{n-1} \sin(x) + \ldots + A_1 x \sin(x) + A_0 \sin(x) + B_n x^n \cos(x) + B_{n-1} x^{n-1} \cos(x) + \ldots + B_1 x \cos(x) + A_0 \cos(x).$$

Our last Rule of Thumb is the trickiest. It requires that you compare the FIRST GUESS obtained by applying Rules of Thumb #'s 1,2 3, and 4 with the solution to the homogeneous (complementary) equation. This may require a second guess, a third guess and maybe more. But it may not. Give no more guesses than necessary to obtain the correct solution.

RULE OF THUMB #5. If any of the terms in the first guess at y_p (as determined by Rules of Thumb #'s 1,2,3, and 4) is a solution to the homogeneous equation, then multiply the first guess at y_p by x to obtain a second (judicious) guess for y_p. If any of the terms in this second guess at y_p is a solution to the homogeneous equation, then multiply the second guess at y_p by x to obtain a third (judicious) guess for y_p. Continue this process until no term in the judicious guess matches a term in the homogeneous solution.

For a second order equation, you will never had to multiply by anything except x or x^2, (i.e. twice) to obtain the appropriate (i.e. most efficient) judicious guess. For an nth order equation, you will never have to multiply by more than x^n.

We also extend the use of superposition. Recall that we developed a divide and conquer strategy in case g(x) is complicated. Consider the equations:

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = 0 \qquad \forall\, x \in I, \tag{6}$$

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = g_1(x) \qquad \forall\, x \in I, \tag{7}$$

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = g_2(x) \qquad \forall\, x \in I, \tag{8}$$

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = g_1(x) + g_2(x) \qquad \forall\, x \in I, \tag{9}$$

where $p_i(x) \in C(I)$ for all i, and I = (a,b) is the interval of validity.

THEOREM. (Superposition Principle for Nonhomogeneous Equations) Suppose

(i) y_c is the general solution to the homogeneous equation (6),
(ii) y_{p1} is a particular solution to the nonhomogeneous equation (7),
(iii) y_{p2} is a particular solution to the nonhomogeneous equation(8).

Then $y_p = y_{p1} + y_{p2}$ is a particular solution to the nonhomogeneous equation (9) and $y = y_p\, y_c = y_c + y_{p1} + y_{p2}$ is the general solution to (9).

EXAMPLE #1. Solve $y''' + 3\,y'' + 3\,y' + y = x^2 + \cos(x) + e^{-x}$

Solution:
Homogeneous Equation: $y''' + 3\,y'' + 3\,y' + y = \; = 0$
Auxiliary Equation: $r^3 + 3\,r^2 + 3\,r + 1 = 0$

By the rational root theorem we see that the possible rational roots are $r = \pm 1$. Since the coefficients are all positive, there are no positive zeros. Hence r = -1 is the only possible rational root. We check that it is indeed a root. Let

$p(r) = r^3 + 3\,r^2 + 3\,r + 1$

Then $p(-1) = (-1)^3 + 3\,(-1)^2 + 3\,(-1) + 1 = -1 + 3 - 3 + 1 = 0$ so that r = -1 is a zero of p. We could divide to obtain the factors of p. However, we recall **Pascal's triangle**:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
\qquad
\begin{array}{l}
(x + y)^0 = 1 \\
(x + y)^1 = x + y \\
(x + y)^2 = x^2 + 2\,xy + y^2 \\
(x + y)^3 = x^3 + 3\,x^2y + 3\,xy^2 + y^3 \\
(x + y)^4 = x^4 + 4\,x^3y + 6\,x^2y^2 + 4\,xy^3 + y^4 \\
(x + y)^5 = x^5 + 5\,x^4y + 10\,x^3y^2 + 10\,x^3y^2 + 5\,xy^4 + y^5
\end{array}
$$

Hence we see that $(x + 1)^3 = x^3 + 3\,x^2 + 3\,x + 1$ so that $p(r) = (r + 1)^3$
Hence $r_1 = r_2 = r_3 = -1$. Hence three linearly independent solutions are:

$y_1 = e^{-x}$, $y_2 = x\,e^{-x}$, and $y_3 = x^2\,e^{-x}$.

and the general solution of the homogeneous equation is:

$$y_c(x) = c_1\,y_1(x) + c_2\,y_2(x) + c_3\,y_3(x)$$
$$y_c(x) = c_1\,e^{-x} + c_2\,x\,e^{-x} + c_3\,x^2\,e^{-x}.$$

Particular Solution to Nonhomogeneous #1: $y''' + 3\,y'' + 3\,y' + y = x^2$

1) $y_{p1} = A\,x^2 + B\,x + C$ 1st guess (Only guess needed)
3) $y'_{p1} = 2\,A\,x + B$
3) $y''_{p1} = 2\,A$
1) $y'''_{p1} = 0$

$$y'''_{p1} + 3\,y''_{p1} + 3\,y'_{p1} + y_{p1} = (\,A\,)\,x^2 + (6A + B)\,x + (6A + 3B + C) = x^2$$

x^2)	A	$= 1$	$A = 1$
x^1)	$6A + B$	$= 0$	$B = -\,6A = -\,6$
x^0)	$6A + 3B + C$	$= 0$	$C = -\,6A - 3\,B = -\,6 - 3(-\,6) = 12$

$$y_{p1} = x^2 - 6x + 12$$

Particular Solution to Nonhomogeneous #2: $y''' + 3\,y'' + 3\,y' + y = \cos(x)$

$y_{p2} = A\sin(x) + B\cos(x)$ 1st guess (Only guess needed)

1) $y_{p2} = A\sin(x) + B\cos(x)$
3) $y'_{p2} = A\cos(x) - B\sin(x)$
3) $y''_{p2} = -\,A\sin(x) - B\cos(x)$
1) $y'''_{p2} = -\,A\cos(x) + B\sin(x)$

$$y'''_{p2} + 3y''_{p2} + 3y'_{p2} + y_{p2} = (\,B - 3A - 3B + A\,)\sin(x) + (\,-A - 3B + 3A + B\,)\cos(x) = \cos(x)$$

sin(x)) $-2A \; - 2B = 0 \;\Rightarrow\;$ $A = -B$
cos(x)) $2A \; -2B = 1 \;\Rightarrow\;$ $-4B = 1 \;\Rightarrow B= -1/4 \Rightarrow A= 1/4$

$y_{p2} = (1/4) \sin(x) + (- 1/4) \cos(x)$

Particular Solution of Nonhomogeneous #3: $y''' + 3 y'' + 3 y' + y = e^{-x}$

$y_{p3} = A e^{-x}$ 1st guess
$y_{p3} = A x e^{-x}$ 2nd guess
$y_{p3} = A x^2 e^{-x}$ 3rd guess
$y_{p3} = A x^3 e^{-x}$ 4th guess

1) $y_{p3} = A x^3 e^{-x}$
3) $y'_{p3} = - A x^3 e^{-x} + 3 A x^2 e^{-x}$
$y''_{p3} = A x^3 e^{-x} - 3 A x^2 e^{-x} - A\, 3 x^2 e^{-x} + 6 A x e^{-x}$
3) $y''_{p3} = A x^3 e^{-x} - 6 A x^2 e^{-x} + 6 A x e^{-x}$
$y'''_{p3} = - A x^3 e^{-x} + 3 A x^2 e^{-x} + 6 A x^2 e^{-x} - 12 A x e^{-x} \; - 6 A x e^{-x} + 6 A e^{-x}$
1) $y'''_{p3} = - A x^3 e^{-x} + 9 A x^2 e^{-x} - 18 A x e^{-x} + 6 A e^{-x}$

$y'''+3y''+3 y'+y = [(-1+3-3+1)Ax^3+(9-18+9)Ax^2+(-18+18)A x+(6)A] e^{-x} = e^{-x}$

$\Rightarrow 6 A = 1 \Rightarrow A = 1/6 \Rightarrow \; y_{p3} = (1/6) x^3 e^{-x}$

$y_p = y_{p1} + y_{p2} + y_{p3} = x^2 - 6x + 12 + (1/4) \sin(x) - (1/4) \cos(x) + (1/6) x^3 e^{-x}$

$y = y_c + y_p = y_c + y_{p1} + y_{p2}+ y_{p3}$

$= c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + x^2 - 6x + 12 + (1/4) \sin(x) - (1/4) \cos(x) + (1/6) x^3 e^{-x}$

EXERCISES on Introduction and Theory for Higher Order ODE's

EXERCISE #1.. Given that

$$y_h = c_1 + c_2 \sin(2x) + c_3 \cos(2x)$$

is the general solution of

$$y''' + 4 y' = 0,$$

use the method discussed in class to determine the proper (most efficient) form of the judicious guess for the particular solution y_p of the following ode's. Do not give a second or third guess if these are not needed.

1. $y''' + 4y' = e^x$ First guess: y_p = ________________________

   Second guess (if needed): y_p = ________________________

   Third guess (if needed): y_p = ________________________

2. $y''' + 4y' = 3x$ First guess: y_p = ________________________

   Second guess (if needed): y_p = ________________________

   Third guess (if needed): y_p = ________________________

3. $y''' + 4y' = 5 \sin(2x)$ First guess: y_p = ________________________

   Second guess (if needed): y_p = ________________________

   Third guess (if needed): y_p = ________________________

4. $y''' + 4y' = 3x \sin(2x)$ First guess: y_p = ________________________

   Second guess (if needed): y_p = ________________________

   Third guess (if needed): y_p = ________________________

<u>EXERCISE #2</u>. Solve. Be sure to give the correct "first guess" for the form of y_p and then give the second or third guess only if these are necessary.

1. $y''' + 4y' = 3 \sin x$
2. $y''' - y'' - 6y = 2e^{-2x}$, $y(0) = 0$, $y'(0) = 6$, $y''(0) = 0$
3. $y''' - 3y'' + 2y' = 6e^{3x}$
4. $y''' + y'' = 3x^2$, $y(0) = 4$, $y'(0) = 0$, $y''(0) = 0$
5. $y''' - 2y'' + y' = -4e^x$
6. $y''' - 4y'' + 4y' = 6xe^{2x}$, $y(0) = 0$, $y'(0) = 3$, $y''(0) = 0$
7. $y''' - 7y'' + 10y' = 100x$, $y(0) = 0$, $y'(0) = 5$, $y''(0) = 0$
8. $y^{iv} + y'' = 1 + x + x^2$
9. $y^{iv} + y''' = x^3 - x^2$
10. $y^{iv} + 4y'' = 16x \sin 2x$

EXERCISE #3. Use the principle of superposition to find the general solution of the following.

1. $y''' + y' = 1 + 2 \sin x$
2. $y''' - 2y'' - 3y' = x - x^2 + e^x$
3. $y''' + 4y' = 3 \cos 2x - 7x^2$
4. $y''' + 4y'' + 4y' = xe^x + \sin x$

METHOD OF VARIATION OF PARAMETERS

Recall the general second order <u>homogeneous</u> equation with variable coefficients:

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \dots + p_1(x)' + p_0(x)\, y = 0 \qquad \forall\, x \in I \tag{1}$$

and the general second order <u>nonhomogeneous</u> equation with variable coefficients:

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \dots + p_1(x)' + p_0(x)\, y = g(x) \qquad \forall\, x \in I \tag{2}$$

<u>THEOREM</u>. Let $S = \{ y_1, y_2, \dots, y_n \}$ be a set of linearly independent solutions to the homogeneous equation (1). Now assume that we can find a (i.e one) particular solution $y_p(x)$ to the nonhomogeneous equation (2). Then $y(x) = y_p(x) + y_c(x)$ where $y_c(x)$ is the general solution of the associated homogeneous equation (also called the complementary equation) (1). Thus:

$$y(x) = y_p(x) + c_1\, y_1(x) + c_2\, y_2(x) + \dots + c_n\, y_n(x).$$

The notation y_h is also used to denote the solution to the complementary (homogeneous) equation.

The theorem reduces the problem of finding the general solution of the nonhomogeneous equation (2) to the finding of the n+1 functions $y_p(x), y_1(x), y_2(x), \dots, y_n(x)$. We have considered the technique for finding the particular solution y_p which we called <u>undetermined coefficients</u> or <u>judicious guessing</u>. We now wish to extend the <u>method of variation of parameters</u> to find a particular solution $y_p(x)$. Recall that the methods strong points are that it works for the general (variable coefficient) case and that it works even when the forcing function (right hand side) is not "nice". One draw back is that it requires that we know the general solution of the complementary (associated homogeneous) equation. Another is that we must be able to find antiderivatives. Similar to second order, we assume a solution of the form:

$$y_p(x) = v_1(x)\, y_1(x) + v_2(x)\, y_2(x) + \dots + v_n(x)\, y_n(x) \tag{3}$$

By calculating derivatives and making the appropriate set of assumptions, we obtain the set of equations:

$$\begin{aligned}
&v_1'\, y_1 + v_2'\, y_2 + \dots + v_n'\, y_n = 0\\
&v_1'\, y_1' + v_2'\, y_2' + \dots + v_n'\, y_n' = 0\\
&v_1'\, y_1'' + v_2'\, y_2'' + \dots + v_n'\, y_n'' = 0\\
&\quad\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots\\
&v_1'\, y_1^{(n-1)} + v_2'\, y_2^{(n-1)} + \dots + v_n'\, y_n^{(n-1)} = g(x)
\end{aligned}$$

These can be solved to obtain $v_1', v_2', \dots, v_n'$. Integration yields $v_1, v_2, \dots, v_n$, and hence y_p.