

A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND
NONLINEAR
PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 2

A COLLECTION OF HANDOUTS ON SCALAR
LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODE"s)

CHAPTER 4

Mathematical Modeling

Using Second Order ODE's

1. Mechanical Vibrations: Spring Problems
2. Electrical Vibrations: Electric Circuits

Applied mathematics really begins with a desire to answer specific questions about “real world” problems. Hence in investigating applications, we will begin with specific questions that drive us to find answers. For our applications, answers require that we first develop and solve a **mathematical model** (i.e., a “find” problem) that is an **initial value problem**.

QUESTION: A (physical object of) mass m (we let m denote the physical object as well as the numerical value of the mass of the physical object in appropriate mass units, e.g. slugs) stretches a spring (which is ℓ_s units in length) $\Delta\ell$ units of length (e.g. feet or inches which can be converted to feet). Assume that the mass or the weight $W = mg$ is known. Also assume that the spring constant is k in appropriate units (e.g. lbs force/ft) and that the air resistance is proportional to the velocity with known proportionality constant c in appropriate units (e.g. lbs force sec/ft). Now assume the mass is lowered u_0 units (e.g. feet) below its equilibrium position and given an initial velocity of v_0 units (e.g. ft/sec) downward. Find the position of the mass three seconds later.

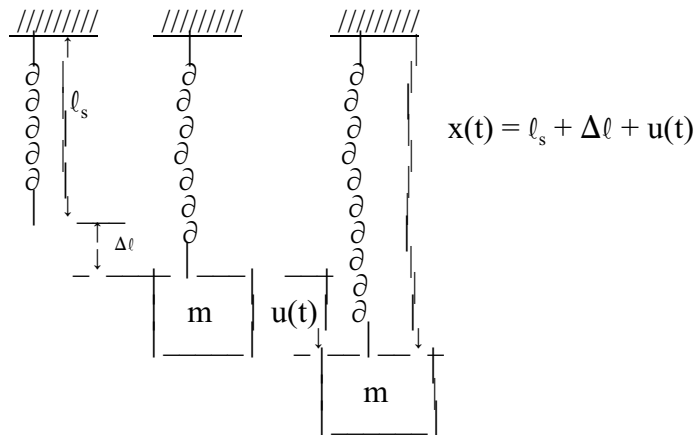
To answer this question we use our **five step procedure**.

Step 1: Understand the Concepts in the Application Area Where the Questions are Asked. We first describe the phenomenon to be modeled, including the **laws** it must follow (e.g., that are imposed by nature, by an entrepreneurial environment or by the modeler). To understand vibrating springs, we consider again the following **theoretical physical law**.

THEORETICAL (AND EMPIRICAL) PHYSICAL LAW. (Newton’s Second Law of Motion, Conservation of Momentum.) This law states that at any given time, the net **force** on a particle (point mass) is equal to its **mass** times its **acceleration** ($F = MA$).

Step 2: Understand the Needed Concepts in Mathematics. 1. High School Algebra, 2. Calculus, 3. Linear algebra, 4. Solution Techniques covered in this Part of the Notes.

Step 3. DEVELOP THE MATHEMATICAL MODEL. If the problem is not complicated, a **general model** may be developed and solved first. This **general model** may then be used for any specific data where the modeling assumptions used to obtain the **general model** are satisfied. If they are not, a new model must be formulated and solved. Thus we first develop a general mathematical model (i.e. an IVP) that models the mass/spring system; that is, the IVP that can be solved to determine the position of the mass with respect to the equilibrium position. We begin by drawing an appropriate sketch and listing our variables and parameters. We then give any additional equations that can be used to solve for parameters in terms of quantities that are known or can be measured. First we **sketch** the picture.



List of variables and parameters.

$x(t)$ = distance of (point) mass from the inertial point of reference where the spring is attached.

$u(t)$ = distance of (point) mass from equilibrium.

$\dot{x}(t) = \dot{u}(t)$ = velocity of (point) mass

$\ddot{x}(t) = \ddot{u}(t)$ = acceleration of (point) mass

m = mass of the physical object

W = weight of the mass (physical object)

l_s = length of unstretched spring

Δl = initial stretching of the spring to get to the equilibrium position.

k = spring constant.

c = constant of proportionality for air resistance.

g = acceleration due to gravity.

u_0 = initial displacement below equilibrium position. If the mass is raised, then the numerical value of u_0 will be negative.

v_0 = initial velocity in the downward direction. If the mass is given an initial velocity in the upward direction, then the numerical value of v_0 will be negative.

Development of IVP using Newton's second law.

First we note that $W = mg$ so that if either the mass or the weight is known, then the other can be computed. To develop the mathematical model for the mass/spring system, we use Newton's second law: The (net) force, F , (on the physical object) equals the (numerical value of the) mass, m , times the acceleration, a , (of the physical object). i.e. $F = ma$ or $ma = m \ddot{x} = F$. The forces on the spring are:

Force of gravity on the physical object. $F_g = mg = W$ = weight of physical object. There is no minus sign since we take positive displacement as down.

Force of the spring on the physical object. $F_s = -k(\text{elongation of the spring}) = -k(\Delta l + u(t))$. The minus sign indicates that when the spring is elongated (i.e. stretched and the displacement $x(t) - l_s = \Delta l + u(t)$ is positive) that the force on the physical object is up.

Force of air resistance on the physical object. $F_r = -c(\text{velocity}) = -c(\dot{x}(t) x(t))$

Applying Newton's second law we obtain:

$$m \ddot{x} = F_g + F_s + F_r = mg - k \dot{x} - cx$$

Now since $x(t) = \ell_s + \Delta\ell + u(t)$ we have $\dot{x}(t) = \frac{d(\ell_s + \Delta\ell + u(t))}{dt} = \dot{u}(t)$ and $\ddot{x}(t) = \ddot{u}(t)$,

we obtain $m \ddot{u} = mg - k(\Delta\ell + u) - c \dot{u}$

$$m \ddot{u} + c \dot{u} + k u = mg - k \Delta\ell.$$

Since all forces are independent of time the system is **autonomous** and we may consider the **equilibrium** (static) equation. At equilibrium, the velocity and the acceleration are zero. Also, the displacement from equilibrium is zero. Hence $\ddot{x}(t) = \ddot{u}(t) = 0$, $\dot{x}(t) = \dot{u}(t) = 0$, and $u = 0$. Hence we have

$$mg - k \Delta\ell = 0 \text{ or } k = mg/\Delta\ell$$

which can be used to solve for k if m and $\Delta\ell$ are measured. This simplifies our ODE to:

$$m \ddot{u} + c \dot{u} + k u = 0.$$

The initial value problem (IVP) is complete if we add the initial conditions.

GENERAL MATHEMATICAL MODEL OF A MASS/SPRING SYSTEM:

$$\text{ODE } m \ddot{u} + c \dot{u} + k u = 0.$$

IVP

$$\text{IC } u(0) = u_0 \quad \dot{u}(0) = v_0$$

Step 4. Solve the Mathematical Model. We are to solve a mathematical problem which is an IVP. The ODE is a second order linear ODE with constant coefficients. The standard solution process says find the general solution of the ODE first. But there are three cases. To determine which case, we first solve for system parameters and write down the specific model to be solved. We then solve the model to obtain the position of the mass with respect to the equilibrium position. Finally we answer the questions asked.

APPLICATION OF SPECIFIC DATA Once a general model has been formulated and solved, it can be applied using specific data. Alternately, the model can be written in terms of the specific data and resolved. Although redundant, this resolving of the model provides much needed practice in the process of formulating and solving models. This is useful in preparation for exams since solutions of general models are not normally given on exams and are usually not

memorized. Also specific data may simplify the process and the formulas obtained. Suppose that the following specific information is given:

SPECIFIC DATA. A mass weighing 3 lbs. stretches a spring (which is 4 ft. long) 3 inches. If the mass is raised 1 inch above its equilibrium position and given an initial velocity of 2 ft./sec. upward, determine the subsequent motion (i.e. find the distance from the equilibrium position as a function of time). Assume that the air resistance is negligible.

Solution: The general model is given above as

$$\begin{aligned} \text{ODE} \quad & m \ddot{u} + c \dot{u} + k u = 0 . \\ \text{IVP} \quad & \\ \text{IC} \quad & u(0) = u_0 \quad \dot{u}(0) = v_0 \end{aligned}$$

There are five numbers to be determined before solving the model: m , c , k , u_0 and v_0 . First, $c = 0$ since we are told that the air resistance is negligible. The initial conditions are given: $u_0 = -1/12$ ft (length units are feet, it is negative since it was moved up) $v_0 = -2$ ft/sec (it is negative since the initial velocity is upwards)

To obtain the mass m we use the equation $W = mg$. Hence $3 = m(32)$ so that $m = 3/32$. ($g = 32$ ft/sec²).

To obtain the spring constant k we use the equilibrium equation $k \Delta\ell = W = mg$. First note that $\Delta\ell = 3/12 = 1/4$ ft. Hence $k = mg/\Delta\ell = W/\Delta\ell = 3/(1/4) = 12$.

Hence the general model becomes for this particular case

$$\begin{aligned} \text{ODE} \quad & (3/32) \ddot{u} + 12 u = 0 . \\ \text{IVP} \quad & \\ \text{IC} \quad & u(0) = -1/12 \quad \dot{u}(0) = -2 \end{aligned}$$

We begin by solving the ODE

$$\begin{aligned} 3/32 \ddot{u} + 12 u = 0 & \Rightarrow 3/32 r^2 + 12 = 0 \Rightarrow r^2 = -4(32) \Rightarrow r = 8\sqrt{2} i \\ \Rightarrow u & = c_1 \sin(8\sqrt{2} t) + c_2 \cos(8\sqrt{2} t) \end{aligned}$$

Now apply IC:

$$\begin{aligned} u & = c_1 \sin(8\sqrt{2} t) + c_2 \cos(8\sqrt{2} t) \\ \dot{u} & = c_1 8\sqrt{2} \cos(4\sqrt{2}t) - c_2 8\sqrt{2} \sin(4\sqrt{2}t) \end{aligned}$$

$$\text{Hence } u(0) = 1/12 \Rightarrow 1/12 = c_1(0) + c_2(1) \Rightarrow c_2 = 1/12$$

$$\begin{aligned} \ddot{u}(0) = 2 &\Rightarrow 2 = 8\sqrt{2} c_1(1) - 8\sqrt{2} c_2(0) \Rightarrow c_1 = 2/(8\sqrt{2}) = \sqrt{2}/8 \\ \Rightarrow u &= \sqrt{2}/8 \sin(8\sqrt{2} t) + 1/12 \cos(8\sqrt{2} t) \\ \Rightarrow \dot{u} &= (\sqrt{2}/8)(8\sqrt{2}) \cos(8\sqrt{2} t) - (1/12)(8\sqrt{2}) \sin(8\sqrt{2} t) \\ &= 2 \cos(8\sqrt{2} t) - 2\sqrt{2}/3 \sin(8\sqrt{2} t) \end{aligned}$$

Step 5. Interpret the Results of the Solution to the Mathematical Model. Having solved the model developed above under the conditions given, we interpret the results of the solution to the model (i.e. answer the questions asked). Begin by writing down the solution obtained. End by reviewing the answers to the questions asked.

A mass weighing 3 lbs. stretches a spring (which is 4 ft. long) 3 inches. If the mass is raised 1 inch above its equilibrium position and given an initial velocity of 2 ft./sec. upward, determine the subsequent motion (i.e. find the distance from the equilibrium position as a function of time). Assume that the air resistance is negligible.

Express the solution in the form: $u(t) = A \cos(\omega t - \varphi)$. Determine the amplitude of the vibration. Determine the angular frequency of the motion. Determine the frequency of the motion. Determine the phase angle of the motion. Determine the exact time t_1 when the mass first reaches its equilibrium position. Determine the velocity of the mass at t_1 .

Solution: The solution to the IVP is:

$$\begin{aligned} u &= \sqrt{2}/8 \sin(8\sqrt{2} t) + 1/12 \cos(8\sqrt{2} t) \\ \dot{u} &= 2 \cos(8\sqrt{2} t) - 2\sqrt{2}/3 \sin(8\sqrt{2} t) \end{aligned}$$

Using standard trig identities we obtain:

$$u(t) = A \cos(\omega t - \varphi) = A \cos(\omega t) \cos(\varphi) + A \sin(\omega t) \sin(\varphi)$$

Hence $\omega = 8\sqrt{2} \approx 11.3137085$, $A \cos(\varphi) = 1/12$, and $A \sin(\varphi) = \sqrt{2}/8$. Hence

$$A^2 \cos^2(\varphi) + A^2 \sin^2(\varphi) = (1/12)^2 + (\sqrt{2}/8)^2 = 1/((12)(12)) + 2/64$$

so that $A^2 (\cos^2(\varphi) + \sin^2(\varphi))^2 = A^2 = (16 + 9)/((64)(9)) = 25/((64)(9))$.

Hence $A = 5/((8)(3)) = 5/24 \approx 0.208333$ feet ≈ 2.5 inches (actually, this is exact, why?).

Also $(A \sin(\varphi))/(A \cos(\varphi)) = \tan(\varphi) = (\sqrt{2}/8)/(1/12) = 12\sqrt{2}/8 = 3\sqrt{2}/2$. Hence

$\varphi = \text{Arctan}(3\sqrt{2}/2) \approx \text{Arctan}(2.121320344) \approx 64.76059818^\circ \approx 1.130285664$ radians.

The mass reaches the equilibrium position when $\cos(\omega t - \varphi) = 0$ or when

$$\omega t - \varphi = \pi/2. \text{ Thus } t_1 = (\varphi + \pi/2)/\omega \approx (1.130285664 + 1.5780)/(11.3137085) \\ \approx 0.122836284 \text{ seconds}$$

$$\ddot{u}(t_1) = -A \omega \sin(\omega t_1 - \varphi) = -A \omega \sin(\pi/2) = -A \omega = - (5/24)(8\sqrt{2}) = -5\sqrt{2}/3 \\ \approx -2.357022604 \text{ ft/sec} \approx -28.28427125 \text{ inches/sec.}$$

Summary.

Amplitude = $A = 5/24 \approx 0.208333$ feet ≈ 2.5 inches (actually, this is exact, why?).

Angular Frequency = $\omega = 8\sqrt{2} \approx 11.3137085$ (radians/sec)

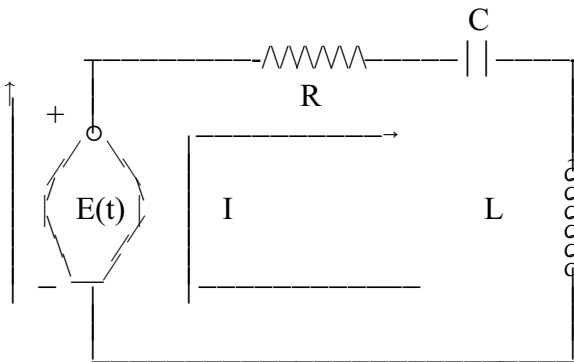
Frequency = $f = \omega/(2\pi) = 8\sqrt{2}/(2\pi) = 4\sqrt{2}/\pi \\ \approx 1.800632632$ Hertz (cycles/sec)

Phase angle = $\varphi = \text{Arctan}(3\sqrt{2}/2) \approx \text{Arctan}(2.121320344) \\ \approx 64.76059818^\circ \approx 1.130285664$ radians.

$t_1 = (\varphi + \pi/2)/\omega \approx (1.130285664 + 1.5780)/(11.3137085) \\ \approx 0.122836284$ seconds

$\ddot{u}(t_1) = -5\sqrt{2}/3 \approx -2.357022604$ ft/sec ≈ -28.28427125 inches/sec.

Electrical Network



I = current
 E = electromotive
 force

Kirchhoff's (Voltage) Law: In a closed circuit the impressed voltage ($E(t)$) is equal to the sum of the voltage drops in the rest of the circuit.

The voltage drop across the Resistor = IR (Ohm's Law)

The voltage drop across the Capacitor = Q/C where Q = Charge on the Capacitor

The voltage drop across the Inductor = $L \frac{dI}{dt}$

Hence we obtain the model: $L \frac{dI}{dt} + IR + \frac{Q}{C} = E(t)$

If we let $Q(0) = Q_0$ = initial charge on the capacitor and $I(0) = I_0$ = initial current in the circuit and recall that $I = \frac{dQ}{dt}$ we obtain the model:

MATH MODEL FOR A SIMPLE SERIES CIRCUIT

$$\text{ODE } L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = E(t)$$

IVP

$$\text{IC } Q(0) = Q_0, \quad \dot{Q}(0) = I_0$$