A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 2

A COLLECTION OF HANDOUTS ON SCALAR LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODE"s)

CHAPTER 3

Techniques for Solving

Second Order Linear

Nonhomogeneous ODE's

1. Nonhomogeneous Equations: Method of Undetermined Coefficients

2. Superposition for Nonhomogeneous Equations

3. Nonhomogeneous Equations: Method of Variation of Parameters

Handout # 1. NONHOMOGENEOUS EQUATIONS: METHOD OF UNDETERMINED COEFFICIENTS Professor Moseley

Let $L:C^2(I) \to C(I)$ where $I = (a,b)$ and $p,q \in C(I)$, $I = (a,b)$ be defined by

$$L[y] = y'' + p(x)\, y' + q(x)\, y. \tag{1}$$

Alternately consider $L:\mathcal{A}(I)\to\mathcal{A}(I)$ with $p,q \in \mathcal{A}(I)$ or $L:\mathcal{H}(\mathbf{C})\to\mathcal{H}(\mathbf{C})$ with $p,q \in\mathcal{H}(\mathbf{C})$. We now consider the second order linear <u>nonhomogeneous</u> equation $L[y] = g$:

$$y'' + p(x)\, y' + q(x)\, y = g(x) \qquad \forall\, x \in I \tag{2}$$

For all three contexts, the general solution of (2) is of the form

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a (i.e. any) particular solution of (1) and y_c is the general solution of the associated homogeneous (complementary) equation $L[g] = 0$:

$$y'' + p(x)\, y' + q(x)\, y = 0. \tag{3}$$

Since the null space $N(L)$ of the linear operator $L:D(L)\to CoD(L)$ given by

$$L[y] = y'' + p(x)\, y' + q(x)\, y \tag{3}$$

has dimension two, if $B = \{y_1, y_2\}$ is a set of linearly independent solutions to (2), then B is a basis of $N(L)$. (Note that in all three contexts, $D(L)\subseteq CoD(L)$ so that L always maps to the same or a large space.) Hence the general solution of the associated homogeneous (complementary) equation can be written as $y_c = c_1\, y_1 + c_2\, y_2$. In the language of linear algebra, we have that $D(L) = N(L)\oplus W$ is a **direct decomposition** of the domain $D(L)$ where W is a subspace of $D(L)$ that L maps onto the range $R(L)$ in a one-to-one fashion. (This is not easy to prove.)

There are two standard techniques for obtaining the particular solution y_p:

1. The **method of undetermined coefficients** (Also called the method of judicious guessing).
2. The method of **variation of parameters**.

We consider the **method of undetermined coefficients** first. To use this method, we must have:

1. The homogeneous equation must have constant coefficients .
2. The Right Hand Side (RHS) must be "nice". By this we mean that they must be analytic and such that the operator L maps them back into a finite dimensional subspace of $\mathcal{A}(\mathbf{R})$ (or $\mathcal{H}(\mathbf{C})$).

The ability to obtain a first guess for the form of the particular solution for certain elementary forcing functions g(x) stems from the fact that there are finite dimensional subspaces

$W_1 \subseteq W \subseteq N(L) \oplus W = A(\mathbf{R})$ (or $H(\mathbf{C})$) such that any linear operator with constant coefficients given by

$$L[y] = a\,y'' + b\,y' + c\,y, \qquad a \neq 0 \tag{4}$$

maps these functions back into W_1. Since we know a priori that L is one-to-one on W, it will be one-to-one on W_1 and hence given $g \in W_1$, there will be exactly one function $y_p \in W_1$ that L maps to g. Hence if a basis of W_1 is known, y_p must be a linear combination of a these known basis functions. If we can guess a basis of W_1, we can substitute a linear combination of these functions into the nonhomogeneous equation and solve for the correct coefficients. If L maps a finite dimensional subspace W_1 back into W_1 but $W_1 \cap N(L) \neq \varnothing$, we must modify our guess for the particular solution y_p. That is, we cannot just take it to be a basis of W_1. We need a larger subspace whose range is is all of W_1. (See Rule of Thumb #5 below.)

We develop the **method of undetermined coefficients** using examples and **rules of thumb** to learn how to judiciously guess the form of the solution for certain forcing functions.

EXAMPLE #1. $y'' + y = e^x$

Solution: Homogeneous Equation $\quad y'' + y = 0$

Auxiliary Equation $\quad r^2 + 1 = 0 \quad \Rightarrow \quad r = \pm i \quad \Rightarrow \quad y_c = C_1 \sin(x) + C_2 \cos(x)$

Particular Solution of Nonhomogeneous: $\quad y'' + y = e^x$

RULE OF THUMB #1. Choose y_p of the same form as the Right Hand Side (RHS). (To find a finite dimensional subspace W_1 containing g(x) so that L maps W_1 back to W_1, we must have that all scalar multiples of g are in W_1.)

1) $\quad y_p = A\,e^x \quad$ Note that A is the coefficient that remains to be determined.

$\quad y_p' = A\,e^x$

1) $\quad y_p'' = A\,e^x$

$$y_p'' + y_p = A\,e^x + A\,e^x = e^x \;\Rightarrow\; 2A = 1 \Rightarrow A = \tfrac{1}{2} \Rightarrow y_p = \tfrac{1}{2}\,e^x$$

Since $L[y] = y'' + y$ maps the one-dimensional subspace $W_1 = \{Ae^x : A \in \mathbf{R}\} \subseteq A(\mathbf{R})$ (or $H(\mathbf{C})$) back into itself in a one-to-one fashion, the only question is which Ae^x gets mapped to e^x by the linear operator $L\{y\} = y'' + y$. Our computation shows that $1/2\ e^x$ gets mapped to e^x. Hence $y_p = \tfrac{1}{2} e^x$ and

$$y(x) = y_p(x) + y_c(x) = \tfrac{1}{2} e^x + C_1 \sin(x) + C_2 \cos(x)$$

EXAMPLE #2. $y'' - y' = \sin(x)$

Solution: Homogeneous Equation: $\quad y'' - y' = 0$

Auxiliary Equation: $\quad r^2 - r = 0 \quad \Rightarrow \quad r(r-1)=0 \quad \Rightarrow \quad r_1=0, r_2=1 \quad \Rightarrow \quad y_c = C_1 + C_2 e^x$

Particular Solution of the Nonhomogeneous Equation. $\quad y'' - y' = \sin(x)$

RULE OF THUMB #2. Choose y_p as a linear combination of RHS and all derivatives of RHS. (Given that scalar multiples of g are to be in W_1, to find a finite dimensional subspace W_1 containing g(x) so that L maps W_1 back to W_1, we must have that at least two derivatives of g are in W_1. To simplify, we require all derivatives of g to be in W_1.)

y_p = A sin(x) + B cos(x)   A and B are the undetermined coefficients. We do not need
-1) y_p' = A cos(x) - B sin(x)   a term with - cos(x) or - sin(x) since this can be
1) y_p'' = -A sin(x) - B cos(x)   incorporated into the coefficients A and B.

$y_p'' - y_p = (-A-B)\sin(x) + (-B-A)\cos(x) = \sin(x)$

Hence

$$\begin{aligned} -A + B &= 1 \\ -A - B &= 0 \\ \hline -2A \quad &= 1 \Rightarrow A = -1/2 \Rightarrow B = -A = \tfrac{1}{2} \end{aligned}$$

Since $L[y] = y'' - y'$ maps the two-dimensional subspace W = {A sin x + B cos x: A,B$\in$**R**} back into itself, the only question is which A sin x + B cos x gets mapped to sin x. Our computation shows that $y_p = -1/2\sin(x) + \tfrac{1}{2}\cos(x)$ gets mapped to sin x. Hence

$y(x) = y_p(x) + y_c(x) = -1/2\sin(x) + \tfrac{1}{2}\cos(x) + C_1 + C_2 e^x.$

FUNCTIONS THAT L MAPS BACK INTO A FINITE DIMENSIONAL SUBSPACE:
Now consider the more general case of

$$a\, y'' + b\, y' + c\, y = g(x) \qquad (4)$$

where

1) $g(x) = a\, e^{\alpha x}$. $(a \neq 0)$
2) $g(x) = a\sin(\omega x) + b\cos(\omega x)$.   (a or b may be zero, but not both)
3) $g(x) = P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$.   ($a_n \neq 0$, but others may be)

Similar to $A e^{\alpha x}$ and A sin(x) + B cos(x), since the derivative of a polynomial of degree less than n is a polynomial of degree less than n, L maps any polynomial of degree n into the subspace spanned by the polynomials of degree n.

RULE OF THUMB #3. For the cases listed above, y_p is assumed to be of the same form; that is,

1) $y_p(x) = A e^{\alpha x}$.
2) $y_p(x) = A\sin(\omega x) + B\cos(\omega x)$.
3) $y_p(x) = \overline{P}_n(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0$.

RULE OF THUMB #4. For products of the above functions, assume corresponding products for y_p eliminating duplicate coefficients. For example, if $g(x) = P_n(x)\, e^{\alpha x}$, then
$y_p(x) = \overline{P}_n(x)\, e^{\alpha x} = A_n x^n e^{\alpha x} + A_{n-1} x^{n-1} e^{\alpha x} + \ldots + A_1 x e^{\alpha x} + A_0 e^{\alpha x}$ and if $g(x) = P_n(x) \sin(x)$, then $y_p(x) = A_n x^n \sin(x) + A_{n-1} x^{n-1} \sin(x) + \ldots + A_1 x \sin(x) + A_0 \sin(x)$
$+ B_n x^n \cos(x) + B_{n-1} x^{n-1} \cos(x) + \ldots + B_1 x \cos(x) + A_0 \cos(x)$.

Our last Rule of Thumb is the trickiest. It is needed when $W_1 \cap N(L) \neq \varnothing$. Thus it requires that you compare the FIRST GUESS obtained by applying Rules of Thumb #'s 1,2 3, and 4 with the solution to the homogeneous (complementary) equation. This may (or may not) require a second and a third guess. If a second (or third) guess is not necessary, do not give one.

RULE OF THUMB #5. If any of the terms in the first guess at y_p (as determined by Rules of Thumb #'s 1,2,3, and 4) is a solution to the homogeneous equation, then multiply the first guess at y_p by x to obtain a second (judicious) guess for y_p. If any of the terms in this second guess at y_p is a solution to the homogeneous equation, then multiply the second guess at y_p by x to obtain a third (judicious) guess for y_p.

For a second order equation, you will never have to multiply by anything except x or x^2, (i.e. multiply by x twice) to obtain the appropriate (i.e. most efficient) judicious guess. However, higher order equations, additional "guesses" may be requires.

EXERCISES on Nonhomogeneous Equations: Method of Undetermined Coefficients

EXERCISE #1. Given that

$$y_h = c_1 + c_2\, e^{-3x}$$

is the general solution of

$$y'' + 3y' = 0,$$

use the method discussed in class to determine the proper (most efficient) form of the judicious guess for the particular solution y_p of the following ode's. Do not give a second or third guess if these are not needed.

1. $y'' + 3y' = e^x$ First guess: y_p = ________________________

   Second guess (if needed): y_p = ________________________

   Third guess (if needed): y_p = ________________________

2. $y'' + 3y' = 3x$ First guess: y_p = ________________________

   Second guess (if needed): y_p = ________________________

   Third guess (if needed): y_p = ________________________

3. $y'' + 3y' = 5e^{-3x}$ First guess: y_p = ________________________

   Second guess (if needed): y_p = ________________________

   Third guess (if needed): y_p = ________________________

4. $y'' + 3y' = \sin(x)$ First guess: y_p = ________________________

   Second guess (if needed): y_p = ________________________

   Third guess (if needed): y_p = ________________________

Recall the general second order linear operator

$$L[y] = y'' + p(x)\, y' + q(x)\, y \tag{1}$$

from the vector space $C^2(I)$ where $I = (a,b)$, the general second order linear <u>homogeneous</u> equation with variable coefficients:

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad \forall\, x \in I, \tag{2}$$

and the general second order <u>nonhomogeneous</u> equation with variable coefficients:

$$y'' + p(x)\, y' + q(x)\, y = g(x) \qquad \forall\, x \in I. \tag{3}$$

<u>THEOREM</u> <u>#1</u>. Since the dimension of the null space N(L) is two, if $S = \{ y_1, y_2 \}$ is a set of linearly independent solutions to the homogeneous equation (2). Now assume that we can find a (i.e one) particular solution $y_p(x)$ to the nonhomogeneous equation (3). Then $y(x) = y_p(x) + y_c(x)$ where $y_c(x)$ is the general solution of the associated homogeneous equation (also called the complementary equation) (1). Thus the general solution to (3) is:

$$y(x) = y_p(x) + c_1\, y_1(x) + c_2\, y_2(x). \tag{4}$$

We note that the notation y_h is also used to denote the solution to the complementary (homogeneous) equation.

This theorem reduces the problem of finding the general solution of the nonhomogeneous equation (2) to the finding of the three functions $y_p(x)$, $y_1(x)$, and $y_2(x)$. We continue to consider techniques for finding the particular solution y_p. Next we develop a divide and conquer strategy in case the forcing function g(x) is complicated. Consider the three equations:

$$y'' + p(x)\, y' + q(x)\, y = g_1(x) \qquad \forall\, x \in I. \tag{5}$$

$$y'' + p(x)\, y' + q(x)\, y = g_2(x) \qquad \forall\, x \in I. \tag{6}$$

$$y'' + p(x)\, y' + q(x)\, y = g_1(x) + g_2(x) \qquad \forall\, x \in I. \tag{7}$$

<u>THEOREM</u> <u>#2</u>. (Superposition Principle for Nonhomogeneous Equations) Suppose

(i) y_c is the general solution to the homogeneous equation (2),
(ii) y_{p1} is a particular solution to the nonhomogeneous equation (5),
(iii) y_{p2} is a particular solution to the nonhomogeneous equation (6).

Then $y_p = y_{p1} + y_{p2}$ is a particular solution to the nonhomogeneous equation (7) and .

$$y(x) = y_p(x) + c_1\, y_1(x) + c_2\, y_2(x) = y_{p1}(x) + y_{p2}(x) + c_1\, y_1(x) + c_2\, y_2(x)$$

is the general solution of (7).

<u>EXAMPLE</u> <u>#1</u>. Solve $y'' + y = x^2 + \sin(x)$

<u>Solution:</u> $y'' + y = x^2 + \sin(x)$

General Solution of the Homogeneous Equation: $y'' + y = 0$

Auxiliary Equation: $r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i \Rightarrow y_c = c_1 \sin(x) + c_2 \cos(x)$

Particular Solution of the Nonhomogeneous Equation #1: $y'' + y = x^2$

$$\begin{array}{rl}
1)\quad y_{p1} &= A x^2 + B x + C \\
y'_{p1} &= 2 A x + B \\
1)\quad y''_{p1} &= 2 A \\
\hline
y''_{p1} + y_{p1} &= (A)\, x^2 + B x + (2A + C) = x^2
\end{array}$$

$$\begin{array}{lllll}
x^2) & A = 1 & & & A = 1 \\
x^1) & B = 0 & \Rightarrow & & C = -2A = -2 \\
x^0) & 2A + C = 0 & & &
\end{array}$$

$$y_{p1} = x^2 - 2$$

Particular Solution of the Nonhomogeneous Equation #2: $y'' + y = \sin(x)$

$$\begin{array}{lll}
y_{p2} = & A \sin(x) + B \cos(x) & 1^{st} \text{ guess} \\
y_{p2} = & A\, x \sin(x) + B\, x \cos(x) & 2^{nd} \text{ guess}
\end{array}$$

$$\begin{array}{rl}
1)\quad y_{p2} &= A\, x \sin(x) + B\, x \cos(x) \\
y'_{p2} &= A \sin(x) + B \cos(x) + A\, x \cos(x) - B\, x \sin(x) \\
y''_{p2} &= A \cos(x) - B \sin(x) + A \cos(x) - B \sin(x) - A\, x \sin(x) - B\, x \cos(x) \\
1)\quad y''_{p2} &= 2 A \cos(x) - 2 B \sin(x) - A\, x \sin(x) - B\, x \cos(x) \\
\hline
y''_{p2} + y_{p2} &= (-A + A)\, x \sin(x) + (-B + B)\, x \cos(x) - (2B) \sin(x) + (2A) \cos(x) = \sin(x)
\end{array}$$

$$\begin{array}{llll}
x \sin(x)) & A - A = 0 & & \\
x \cos(x)) & B - B = 0 & & \\
\sin(x)) & -2B = 1 & \Rightarrow & B = -\tfrac{1}{2} \\
\cos(x)) & 2A = 0 & & A = 0
\end{array}$$

$$y_{p2} = (-\tfrac{1}{2})\, x \cos(x)$$

Hence $y = y_c + y_{p1} + y_{p2}$
$= c_1 \sin(x) + c_2 \cos(x) + x^2 - 2 + (-½)\, x \cos(x)$

EXERCISES on Nonhomogeneous Equations: Superposition

EXERCISE #1. Solve the following. Be sure to give the correct "first guess" for the form of yp and then give the second or third guess if these are necessary.

1. $y'' + 4y = 3 \sin x$ $(y = c \sin 2x + c_2 \cos 2x + \sin x)$
2. $y'' - y' - 6y = 2\ 0\ e^{-2x}$ $y(0) = 0$ $y'/0) = 6$
3. $y'' - 3y' + 2y = 6e^{3x}$
4. $y'' + y' = 3x^2$ $y(0) = 4$ $y'/0) = 0$
5. $y'' - 2y' + y = -4e^x$
6. $y'' - 4y' + 4y = 6xe^{2x}$ $y(0) = 0$ $y'/0) = 3$
7. $y'' - 7y' + 10y = 100x$ $y(0) = 0$ $y'(0) = 5$
8. $y'' + y = 1 + x + x^2$
9. $y'' + y' = x^3 - x^2$ $\quad (y = c_1 + c_2e^{-x} + \quad - \quad x^3 + 4x^2 - 8x)$
10. $y'' + 4y = 16\, x \sin 2x$

<u>EXERCISE</u> <u>#2.</u> Use the principle of superposition to find the general solution of the following.

1. $y'' + y = 1 + 2 \sin x$
2. $y'' - 2y' - 3y = x - x^2 + e^x$
3. $y'' + 4y = 3 \cos 2x - 7x^2$
4. $y'' + 4y' + 4y = xe^x + \sin x$

Handout # 3 NONHOMOGENEOUS EQUATIONS: Professor Moseley

METHOD OF VARIATION OF PARAMETERS

Recall the general second order **<u>homogeneous</u> equation** with **variable coefficients**:

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad \forall\, x \in I \qquad (1)$$

and the general second order **<u>nonhomogeneous</u> equation** with **variable coefficients**:

$$y'' + p(x)\, y' + q(x)\, y = g(x) \qquad \forall\, x \in I. \qquad (2)$$

<u>THEOREM #1.</u> Let $S = \{ y_1, y_2 \}$ be a set of linearly independent solutions to the homogeneous equation (1). Now assume that we can find a (i.e one) particular solution $y_p(x)$ to the nonhomogeneous equation (2). Then $y(x) = y_p(x) + y_c(x)$ where $y_c(x)$ is the general solution of the associated homogeneous equation (also called the complementary equation) (1). Thus:

$$y(x) = y_p(x) + c_1\, y_1(x) + c_2\, y_2(x).$$

We note that the notation y_h is also used to denote the solution to the complementary (homogeneous) equation.

This theorem reduces the problem of finding the general solution of the nonhomogeneous equation (2) to finding the three functions $y_p(x)$, $y_1(x)$, and $y_2(x)$. We continue to consider techniques for finding the particular solution y_p and develop a more general method. Its strong points are that it works for the general (variable coefficient) case and when the forcing function (right hand side) is not "nice". The big draw back is that it requires that we know the general solution of the **complementary (associated homogeneous) equation**. Another drawback is that we must be able to find antiderivatives. We illustrate the procedure with an example.

<u>EXAMPLE #1.</u> Solve $y'' + y = \sec(x)$ $I = (0, \pi/2)$ (i.e. $0<x< \pi/2$)

<u>Solution:</u> First find the general solution of the **<u>homogeneous</u>** equation: $y'' + y = 0$.

Auxiliary Equation: $r^2 + 1 = 0 \Rightarrow r = \pm\, i$

$$\boxed{y_h = c_1 \sin(x) + c_2 \cos(x)}$$

<u>Particular Solution of Nonhomogeneous</u>: $y'' + y = \sec(x)$

Since the RHS is not "nice", we <u>must</u> use variation of parameters.

1) $y_p = u_1 \sin(x) + u_2 \cos(x)$

$y'_p = u'_1 \sin(x) + u'_2 \cos(x) + u_1 \cos(x) - u_2 \sin(x)$

Let $u'_1 \sin(x) + u'_2 \cos(x) = 0$. (This assumption is justified by the results it gives.)

Hence substituting into the **nonhomogeneous equation**.

$y'_p = u_1 \cos(x) - u_2 \sin(x)$

1) $y''_p = u'_1 \cos(x) - u'_2 \sin(x) - u_1 \sin(x) - u_2 \cos(x)$

$y''_p + y_p = u'_1 \cos(x) - u'_2 \sin(x) + [- u_1 + u_1]\sin(x) + [- u_2 + u_2]\cos(x) = \sec(x)$

Summarizing, we obtain two linear equations in u'_1 and- u'_2 which can be solved by elimination.

$$
\begin{array}{ll}
\sin(x) & u'_1 \sin(x) + u'_2 \cos(x) = 0 \\
\cos(x) & \underline{u'_1 \cos(x) - u'_2 \sin(x) = \sec(x)} \\
 & u'_1 [\sin^2(x) + \cos^2(x)] \quad = 1
\end{array}
$$

$\Rightarrow \quad u'_1 = 1 \quad \Rightarrow \quad u_1 = x + c_1$ (let c_1=0)

$\Rightarrow \quad u'_2 = -u'_1 \sin(x) / \cos(x) = -\sin(x) / \cos(x)$

$\Rightarrow \quad u_2 = \int - \sin(x) / \cos(x)\, dx = \int 1/v\, dv = \ln|v| + c_2 = \ln|\cos(x)| + c_2$ (Let $c_2 = 0$)

$v = \cos(x) \qquad dv = -\sin(x)\, dx$

$$\Rightarrow \quad \boxed{y_p = x \sin(x) + [\ln|\cos(x)|] \cos(x)}$$

$$\boxed{y = y_h + y_p = c_1 \sin(x) + c_2 \cos(x) + x \sin(x) + [\ln|\cos(x)|] \cos(x)}$$

EXERCISES on Nonhomogeneous Equations: Method of Variation of Parameters

EXERCISE #1. Use the method of variation of parameters to find a particular solution to the following nonhomogeneous equations.

a) $y'' + y = \tan(x) \quad I = (0, \pi/2)$