A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 2

A COLLECTION OF HANDOUTS ON SCALAR LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODE"s)

CHAPTER 2

Techniques for Solving

Second Order Linear

Homogeneous ODE’s

Let $L:C^2(I) \to C(I)$ where I = (a,b) and p,q ∈ C(I), I = (a,b) be defined by

$$L[y] = y'' + p(x)\, y' + q(x)\, y. \qquad (1)$$

Alternately consider $L:A(I)\to A(I)$ with p,q ∈ A(I) or $L:H(\mathbf{C})\to H(\mathbf{C})$ with p,q ∈H(**C**). We consider the second order linear homogeneous equation L[y] = 0:

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad \forall\, x \in I \qquad (2)$$

We consider the special case where p and q are real constants. Since the coefficient functions are constant, they are not only continuous for all x ∈ **R**, but also analytic on **R**. Hence the interval of validity for all solutions is the entire real line, that is, I = (−∞, ∞) = **R** and L maps A(**R**) back into A(**R**) with N(L)⊆A(**R**). Recall that A(**R**) can be embedded in H(**C**), the set of functions which are analytic or holomorphic on **C** (see Exercise #5 in Handout #3 SUBSPACE OF A VECTOR SPACE in Chapter 2-3) . For notational convenience, we consider:

$$L[y] = a\, y'' + b\, y' + c\, y \qquad a,b,c\in\mathbf{C} \text{ with } a\neq 0. \qquad (2)$$

Since as constant functions, a,b,c∈H(**C**), L maps H(**C**) into H(**C**) and we can extend L to map H(**C**) to H(**C**). We first solve L[y] = 0 in this setting. When x is time (which we model as a real variable), a, b, and c, are real-valued physical parameters, and y is a real physical variable. For applications where a, b, c ∈**R**, we consider A(**R**) as a subset of H(**C**) and use **Euler's formula** to find solutions in A(**R**).

For the **homogeneous** equation L[y] = 0:

$$a\, y'' + b\, y' + c\, y = 0 \qquad \forall\, x \in \mathbf{C} \qquad a,b,c\in\mathbf{C} \text{ with } a \neq 0 \qquad (3)$$

the linear theory implies:

THEOREM #1. Let S = { y_1, y_2 } be a set of solutions to the homogeneous equation (3) for x∈**C**. Then the following are equivalent (i.e. they happen at the same time).

a. The set S is linearly independent on I (i.e., y_1 and y_2 are linearly independent in the vector space H(**C**)). Since the dimension of the null space N_L is two, S is then a basis of N_L .

b. $W[\, y_1, y_2\, ; x] \neq 0 \quad \forall\, x \in \mathbf{C}$.

c. The general solution of (3) is

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x) \quad \forall\, x \in \mathbf{C} \tag{4}$$

That is; all solutions of (3) can be written in the form (4) where c_1 and c_2 are arbitrary constants. Since S is a basis of N_L, it is a spanning set for N_L and hence every function ("vector") in N_L can be written as a linear combination of the functions ("vectors") in S.

Equation (4) gives the parametric representation of a two dimensional linear manifold in an infinite dimensional function space. If y_1 and y_2 were vectors in $\mathbf{R}^3$, (4) would be the parametric equations of a plane through the origin. (Recall that planes through the origin in $\mathbf{R}^3$ are subspaces.)

Theorem #1 reduces the problem of finding the general solution of the homogeneous equation (2) to the finding of the two linearly independent solutions $y_1(x)$ and $y_2(x)$. We now develop a technique for finding the functions $y_1(x)$ and $y_2(x)$. We "guess" (actually, we repeat a guess that was made a long time ago) that there may be solutions to (2) of the form

$$y(x) = e^{rx} \qquad \forall\, x \in \mathbf{C} \tag{5}$$

where r is a constant to be determined later. (Euler's formula can be used to extend the exponential function to the complex plane. It can then be shown that $de^x/dx = e^x$.) We determine the required condition on r by substituting $y = e^{rx}$ into the ODE. Using our standard technique for substituting into a linear ODE we obtain:

a) $y = e^{rx}$
b) $y' = r\, e^{rx}$
c) $y'' = r^2\, e^{rx}$

$$a\,y'' + b\,y' + c\,y = (a\,r^2 + b\,r + c)\, e^{rx} = 0.$$

Hence we obtain the **<u>Characteristic</u>** or **<u>Auxiliary</u> equation**:

$$a\,r^2 + b\,r + c = 0. \qquad \underline{\text{Auxiliary}}\ \text{equation} \tag{6}$$

Thus if r satisfies (6), then (5) gives a solution of (3). Using the theory of abstract linear operators and our knowledge of the dimension of N(L), we have changed the "calculus" problem of solving a second order linear differential equation with constant coefficients to a high school algebra problem of solving a quadratic equation. Hence computation of antiderivatives is not required.

In accordance with the fundamental theorem of algebra, (5) has "two solutions" as given by the quadratic equation. However, the two solutions may be the same. (The semantics of the standard verbiage is sometimes confusing, but once you understand what is meant, you should have no problem. To handle the case when there is really only one solution, we speak of the special case of "repeated roots".) The two roots are given by the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = (-|b|/(2|a|))e^{i(\beta-\alpha)} \pm (d/(2|a|))e^{i(\delta-\alpha)/2}. \tag{7}$$

where, as complex numbers, $a = |a|e^{i\alpha}$, $b = |b|e^{i\beta}$, and $b^2-4ac = de^{i\delta}$ with $d>0$, $0\le\alpha<2\pi$, $0\le\beta<2\pi$, and $0\le\delta<2\pi$. Now let

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = (-|b|/(2|a|))e^{i(\beta-\alpha)} + (d/(2|a|))e^{i(\delta-\alpha)/2}, \tag{8}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = (-|b|/(2|a|))e^{i(\beta-\alpha)} - (d/(2|a|))e^{i(\delta-\alpha)/2}. \tag{9}$$

If the discriminant $b^2 - 4ac = 0$, then $r_1 = r_2$ and we obtain only one real linearly independent solution. We must then come up with a method for finding a second solution. But if $b^2 - 4ac \ne 0$, then $r_1 \ne r_2$ and we obtain the two solutions

$$y_1 = e^{r_1 x} \quad\text{and}\quad y_2 = e^{r_2 x}. \tag{10}$$

It can be shown that in this case ($r_1 \ne r_2$) that the set B = { y_1, y_2 } is linearly independent (compute the Wronskian) and we have

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}. \tag{11}$$

as the general solution of (2) in $H(\mathbf{C})$.

If a, b, c $\in\mathbf{R}$, we wish solutions in $A(\mathbf{R})$. If $b^2 - 4ac > 0$, then r_1 and r_2 are real so that when x is real so that $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ can be considered to be in $A(\mathbf{R})$ as as a subset of $H(\mathbf{C})$. However, if $b^2 - 4ac < 0$, then the solutions involve exponentials of nonreal complex numbers. We will see that we can still find two linearly independent solutions in $A(\mathbf{R})$ by using Euler's formula. We will also see how to use the process of "reduction of order to obtain a second solution in $A(\mathbf{R})$ when $b^2 - 4ac = 0$.

EXERCISES on Homogeneous Equations with Constant Coefficients: Introduction

<u>EXERCISE #1</u>. Show that $y_1 = e^{-3x}$ and $y_2 = e^{-2x}$ are linearly independent solutions to $y'' + 5y' + 6y = 0$. Hint: First show that they are solutions. Then compute the wronskian.

Recall that we are considering the operator

$$L[y] = a y'' + b y' + c y \tag{1}$$

and the **homogeneous** equation L[y] = 0:

$$a y'' + b y' + c y = 0 \qquad \forall x \in \mathbf{C} \qquad a,b,c \in \mathbf{C} \text{ with } \quad a \neq 0 \tag{2}$$

Since a, b, c ∈ $H(\mathbf{C})$, the domain of validity for all solutions is $H(\mathbf{C})$. The linear theory implies that the general solution is :

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{3}$$

where $S = \{ y_1, y_2 \}$ is a linearly independent set of solutions. Thus the problem of finding the general solution of the homogeneous equation (2) is reduced to finding of the two linearly independent solutions $y_1(x)$ and $y_2(x)$. Once we have two solutions, we can check to see if they are linearly independent by computing the Wronskian. We "guessed" that there may be solutions to (2) of the form

$$y(x) = e^{rx} \tag{5}$$

where r is a constant to be determined later. We determined the required condition on r by substituting $y = e^{rx}$ into the ODE to be that r satisfies the Characteristic or Auxiliary equation:

$$a r^2 + b r + c = 0. \qquad \underline{\text{Auxiliary}} \text{ equation} \tag{6}$$

Hence

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

Now let

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{8}$$

Case 1. REAL AND UNEQUAL ROOTS. ($b^2 - 4ac > 0$)

If a, b, c ∈ **R** and the discriminant $b^2 - 4ac > 0$, then $r_1 \neq r_2$, where $r_1, r_2 \in \mathbf{R}$ and we obtain the two solutions

$$y_1 = e^{r_1 x} \quad \text{and} \quad y_2 = e^{r_2 x}. \tag{9}$$

in $H(\mathbf{C})$. It can be shown that in this case ($r_1 \neq r_2$) the set B = { y_1, y_2 } is linearly independent (compute the Wronskian) so that the general solution of (2) is

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x) = c_1\, e^{r_1 x} + c_2\, e^{r_2 x}. \qquad (10)$$

If we restrict x to R, then (9) gives two linearly independent solution in $A(\mathbf{R})$ and (10) gives the general solution of (2) in $A(\mathbf{R})$.

<u>EXAMPLE #1.</u> Solve (i.e. find the general solution of) y'' + 5 y' + 6 y = 0.
<u>Solution.</u> Let $y(x) = e^{rx}$. Hence substituting into the ODE we obtain:

6) $y = e^{rx}$
5) $y' = r\, e^{rx}$
1) $y'' = r^2\, e^{rx}$

$y'' + 5\,y' + 6\,y = (\, r^2 + 5\,r + 6 \,)\, e^{rx} = 0.$

Hence we obtain: $r^2 + 5\,r + 6 = 0.$ <u>Auxiliary</u> equation

This quadratic factors so we can solve to obtain:

<u>STATEMENT</u>	<u>REASON</u>
(r + 3) (r + 2) = 0	Factoring
(r + 3) = 0 or (r + 2) = 0	A fundamental theorem of real numbers
r = - 3 or r = - 2	Algebra

We let $r_1 = -3$ and $r_2 = -2$ and $y_1 = e^{-3x}$ and $y_2 = e^{-2x}$.

Using the theory, we see that the general solution to the ODE y'' + 5 y' + 6 y = 0 is

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x) = c_1\, e^{-3x} + c_2\, e^{-2x}.$$

<u>EXAMPLE #2.</u> Solve the Initial Value Problem (IVP): ODE y'' + 5 y' + 6 y = 0.
IC's $y(0) = 1$, $y'(0) = 2$

<u>Solution</u>. From the previous problem the general solution of the ODE y'' + 5 y' + 6 y = 0 is $y(x) = c_1\, e^{-3x} + c_2\, e^{-2x}$. Applying the IC y(0) = 1 we obtain $c_1\, e^{-3(0)} + c_2\, e^{-2(0)} = c_1 + c_2 = 1$. Now $y'(x) = (-3)\, c_1\, e^{-3x} + (-2)c_2\, e^{-2x}$ so that applying the IC $y'(0) = 2$ we obtain $(-3)\, c_1\, e^{-3(0)} + (-2)c_2\, e^{-2(0)} = (-3)\, c_1 + (-2)c_2 = 2$. Hence we obtain the two linear algebraic equations:

$$c_1 + c_2 = 1$$
$$(-3)\, c_1 + (-2)c_2 = 2$$

We could use Gauss elimination or Cramer's Rule, but, as is often the case for these problems we can solve by multiplying the first equation by 2 and adding the second equation. We obtain: $-c_1 = 4$ so that $c_1 = -4$ and hence $c_2 = 1 - c_1 = 1 - (-4) = 5$. Hence $y = -4\,e^{-3x} + 5\,e^{-2x}$.

EXERCISES on Homogeneous Equations with Constant Coefficients: Real and Unequal Roots

<u>EXERCISE #1</u>. Show that $y_1 = e^{-3x}$ and $y_2 = e^{-2x}$ are linearly independent solutions to $y'' + 5\,y' + 6\,y = 0$..

<u>EXERCISE #2</u>. Find the general solution (i.e. a formula for all solutions) to
a) $y'' + 5\,y' + 6\,y = 0$ b) $y'' + 6\,y' + 8\,y = 0$ c) y'') y') 6 y = 0 d) $y'' + 2\,y' + \; y = 0$

Handout # 3 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS: NONREAL COMPLEX ROOTS Professor Moseley

Recall the second order linear differential operator from the vector space $C^2(\mathbf{R})$ to the vector space $C(\mathbf{R})$ given by:

$$L[y] = a\,y'' + b\,y' + c\,y \tag{1}$$

We consider the associated second order linear **homogeneous** equation L[y] = 0:

$$a\,y'' + b\,y' + c\,y = 0 \qquad \forall\, x \in \mathbf{R} \qquad a \neq 0 \tag{2}$$

Note that the interval of validity for all solutions is the entire real line. The linear theory implies that the general solution is :

$$y(x) = c_1\,y_1(x) + c_2\,y_2(x) \tag{3}$$

where $S = \{\, y_1, y_2 \,\}$ is a linearly independent set of solutions. Thus the problem of finding the general solution of the homogeneous equation (2) is reduced to finding of the two linearly independent solutions $y_1(x)$ and $y_2(x)$. Once we have two solutions, we can check to see if they are linearly independent by computing the Wronskian. We "guessed" that there may be solutions to (2) of the form

$$y(x) = e^{rx} \tag{4}$$

where r is a constant to be determined later. We determined the required condition on r by substituting $y = e^{rx}$ into the ODE to be the Characteristic or Auxiliary equation:

$$a\,r^2 + b\,r + c = 0. \qquad \text{Auxiliary equation} \tag{5}$$

Thus if r satisfies (5), then (4) gives a solution of (2) in $H(\mathbf{C})$. Using the theory of abstract linear operators and our knowledge of the dimension of N(L), we have changed the "calculus" problem of solving a second order linear differential equation with constant coefficients to a high school algebra problem of solving a quadratic equation. Hence no antiderivatives are required.

In accordance with the fundamental theorem of algebra, (4) has "two solutions" as given by the quadratic equation. However, the two solutions may be the same. (The semantics of the standard verbiage is sometimes confusing, but once you understand what is meant, you should have no problem. To handle the case when there is really only one solution, we speak of the special case of "repeated roots".) The two roots are given by the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{6}$$

Let

$$r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}. \tag{7}$$

Case 2. NONREAL COMPLEX ROOTS. ($b^2 - 4ac < 0$)

If a, b, c $\in$ **R** and the discriminant $b^2 - 4ac < 0$, then the two roots to the auxiliary equation may be written as

$$r = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \mu \pm i\nu \tag{10}$$

where $\mu = -\frac{b}{2a}$ and $\nu = \frac{\sqrt{4ac-b^2}}{2a}$ are real numbers so that

$$r_1 = \frac{-b+i\sqrt{4ac-b^2}}{2a} = \mu + i\nu, \qquad r_2 = \frac{-b-i\sqrt{4ac-b^2}}{2a} = \mu - i\nu. \tag{11}$$

Rather then continue with the general case, we consider an example. This will make the problem less messy. You should be able to see how the procedure works for the general case.

EXAMPLE. Solve (i.e. find the general solution of) $y'' + y' + y = 0$.

Solution The Auxiliary equation is: $r^2 + r + 1 = 0$

so that $r = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2}$.

Let $r_1 = \frac{-1+\sqrt{-3}}{2}$, $r_2 = \frac{-1-\sqrt{-3}}{2}$.

Using **Euler's formula** we obtain:

$$y_1 = e^{r_1 x} = e^{(-1/2 + i\sqrt{3}/2)x}$$

$$= e^{(-1/2)x} e^{i(\sqrt{3}/2)x} = e^{(-1/2)x}\left[\cos\left(\frac{\sqrt{3}}{2}x\right) + i\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$y_2 = e^{r_2 x} = e^{(-1/2 - i\sqrt{3}/2)x}$$

$$= e^{(-1/2)x} e^{i(-\sqrt{3}/2)x} = e^{(-1/2)x}\left[\cos\left(-\frac{\sqrt{3}}{2}x\right) + i\sin\left(-\frac{\sqrt{3}}{2}x\right)\right]$$

$$= e^{(-1/2)x}\left[\cos\left(\frac{\sqrt{3}}{2}x\right) - i\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

as two linearly independent solutions to $y'' + y' + y = 0$ in $H(\mathbf{C})$. The general solution to $y'' + y' + y = 0$ in $H(\mathbf{C})$ may then be written as

$$y(x) = c_1\ y_1 + c_2\ y_2$$

$$= c_1\ e^{(-1/2)\,x}\ [\cos(\frac{\sqrt{3}}{2}x) + i\sin(\frac{\sqrt{3}}{2}x)] + c_2\ e^{(-1/2)\,x}\ [\cos(\frac{\sqrt{3}}{2}x) - i\sin(\frac{\sqrt{3}}{2}x)].$$

Since the coefficients $a = b = c = 1$ are real numbers, we wish solutions which are real valued functions of a real variable in $A(\mathbf{R})$ when c_1 and c_2 are real numbers. However, we have come up with complex valued functions in $H(\mathbf{C})$ even when c_1 and c_2 are real numbers. On the other hand (OTOH), we recall that since $L[y] = a\,y'' + b\,y' + c\,y$ is a linear operator, linear combinations of solutions in $H(\mathbf{C})$ are also solutions. (The null set is a subspace.) Hence we consider

$$y_3 = (½)\,(y_1 + y_2) = e^{(-1/2)\,x}\ \cos(\frac{\sqrt{3}}{2}x)$$

$$y_4 = (-i/2)\,(y_1 - y_2) = e^{(-1/2)\,x}\ \sin(\frac{\sqrt{3}}{2}x).$$

It can be shown that $\{ y_3, y_4 \}$ is a linearly independent set. Check the Wronskian! Hence we can write the general solution as

$$y(x) = c_1\ y_3(x) + c_2\ y_4(x) = c_1\ e^{(-1/2)\,x}\cos(\frac{\sqrt{3}}{2}x) + c_2\ e^{(-1/2)\,x}\sin(\frac{\sqrt{3}}{2}x).$$

Hence when c_1 and c_2 are real numbers we have the general solution to $y'' + y' + y = 0$ in $A(\mathbf{R})$.

<u>GENERAL CASE FOR NONREAL COMPLEX ROOTS</u>. Suppose the roots of (5) are $r = \mu \pm i\,\nu$ where μ and ν are real numbers and let $r_1 = \mu + i\,\nu$ and $r_2 = \mu - i\,\nu$. Repeating the process for the general case it can be shown that the general solution of (2) is given by

$$y(x) = c_1\ e^{\mu x}\cos(\nu x) + c_2\ e^{\mu x}\sin(\nu x).$$

Since for applications problems we wish real solutions to real problems, writing the **solutions in this form is mandatory.**

EXERCISES on Homogeneous Equations with Constant Coefficients: Nonreal Complex Roots

EXERCISE #1. Show that $y_1 = e^{(-1/2)\,x}\cos(-\frac{\sqrt{3}}{2}x)$ and $y_2 = e^{(-1/2)\,x}\sin(-\frac{\sqrt{3}}{2}x)$ are linearly independent solutions to y'' + y' + y = 0..

EXERCISE #2. Find the general solution (i.e. a formula for all solutions) to
a) y'' + y' + 6 y = 0 b) y'' + y' + 8 y = 0 c) y'') y' + 6 y = 0 d) y'' + 2 y' + 5 y = 0

Handout #4 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS: REAL AND EQUAL ROOTS Professor Moseley

Recall the second order linear differential operator from the vector space $C^2(\mathbf{R})$ to the vector space $C(\mathbf{R})$ given by:

$$L[y] = a\,y'' + b\,y' + c\,y \qquad (1)$$

We consider the associated second order linear **homogeneous** equation L[y] = 0:

$$a\,y'' + b\,y' + c\,y = 0 \qquad \forall\, x \in \mathbf{R} \qquad a \neq 0 \qquad (2)$$

Note that the interval of validity for all solutions is the entire real line. The linear theory implies that the general solution is :

$$y(x) = c_1\,y_1(x) + c_2\,y_2(x) \qquad (3)$$

where $S = \{ y_1, y_2 \}$ is a linearly independent set of solutions. Thus the problem of finding the general solution of the homogeneous equation (2) is reduced to finding of the two linearly independent solutions $y_1(x)$ and $y_2(x)$. Once we have two solutions, we can check to see if they are linearly independent by computing the Wronskian. We "guessed" that there may be solutions to (2) of the form

$$y(x) = e^{rx} \qquad (4)$$

where r is a constant to be determined later. We determined the required condition on r by substituting $y = e^{rx}$ into the ODE to be the Characteristic or Auxiliary equation:

$$a\,r^2 + b\,r + c = 0. \qquad \text{Auxiliary equation} \qquad (5)$$

Thus if r satisfies (5), then (4) gives a solution of (2). Using the theory of abstract linear operators and our knowledge of the dimension of N(L), we have changed the "calculus" problem of solving a second order linear differential equation with constant coefficients to a high school algebra problem of solving a quadratic equation. Hence no antiderivatives are required.

In accordance with the fundamental theorem of algebra, (4) has "two solutions" as given by the quadratic equation. However, the two solutions may be the same. (The semantics of the standard verbiage is sometimes confusing, but once you understand what is meant, you should have no problem. To handle the case when there is really only one solution, we speak of the special case of "repeated roots".) The two roots are given by the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (6)$$

Let

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

<u>Case 3.</u> <u>REAL REPEATED ROOTS.</u> ($b^2 - 4ac = 0$)

If a, b, c $\in$ **R** and the discriminant $b^2 - 4ac = 0$, we have repeated roots so that

$$r_1 = r_2 = r = -b/(2a) \in \mathbf{R}. \tag{8}$$

Since there is only one root to the auxiliary equation, we only get one (linearly independent) solution, namely,

$$y_1 = e^{rx}. \tag{9}$$

This results in the one dimensional subspace of solutions
$S = \{y \in C^2(\mathbf{R}): y = c_1 e^{rt} \text{ where } c_1 \in \mathbf{R}\}$. Recall that to obtain the general solution to (1), we need a set of two linearly independent solutions since the null space of the linear operator $L[y] = ay'' + by' + cy$ has dimension two.

To find a second solution y_2 we use the method of **<u>Reduction of Order</u>** (also known as variation of parameter or finding a second solution from a given one). To apply this method, we assume that y_2 has the gereral form

$$y_2 = v(x)\, y_1(x) = v\, e^{rx} \tag{4}$$

Since y_1 is a solution, we know that $cy_1(x)$ is also (by linearity). The name variation of parameter (a.k.a. variation of constant) comes from allowing the constant (or parameter) c to be a function of x. To indicate that it is a function, we renamed it $v(x)$. To find y_2, it remains to find v so that $y_2 = ve^{rx}$ is a solution of (1). Substituting $y_2 = ve^{rx}$ into (1) gives a condition on v that must be satisfied. Substituting (4) into (1) we obtain

c) $\quad y_2 = v\, e^{rx}$

b) $\quad y_2' = v' e^{rx} + v\, re^{rx}$

a) $\quad y_2'' = v'' e^{rx} + v' re^{rx} + v' re^{rx} + vr^2 e^{rx}$

$$ay_2'' + by_2' + cy_2 = [av'' + (2ra + b)v' + (ar^2 + br + c)v]e^{rx} = 0$$

But r is always chosen so that $ar^2 + br + c = 0$ and for this case ($b^2 - 4ac = 0$). Hence $r = -b/(2a)$ so that $2ra + b = 0$. Since $a \neq 0$ we obtain $v'' = 0$ so that $v = c_1 x + c_2$. Since we wish $\{y_1, y_2\}$ to be linearly independent, we choose $c_1 = 1$ and $c_2 = 0$ so that $y_2 = xe^{rx}$. It can be shown (how?) that $\{y_1, y_2\}$ is linearly independent. Hence

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

is the general solution of (1). Since this procedure always works for (1) and is not dependent on the specific values of the constants a, b, and c, we have the following:

THEOREM #2. If the Auxiliary equation (2) has repeated roots, then a second linearly independent solution y_2 of (1) can be obtained by multiplying the first solution y_1 by the independent variable x.

EXERCISES on Homogeneous Equations with Constant Coefficients: Real and Equal Roots

EXERCISE #1. Show that $y_1 = e^{-x}$ and $y_2 = x\,e^{-x}$ are linearly independent solutions to $y'' + 2y' + y = 0$..

EXERCISE #2. Find the general solution (i.e. a formula for all solutions) to
a) $y'' + 4y' + 4y = 0$ b) $y'' + 6y' + 9y = 0$ c) y'') $4y' + 4y = 0$ d) y'') $2y' + y = 0$

Even if the equation does not have constant coefficients, **reduction of order** can be used to obtain a second linearly independent solution provides one solution is already known. However, the first order equations obtained will require antiderivatives.

<u>EXAMPLE.</u> Given that $y_1(x) = x$ is one solution, solve (i.e. find the general solution of) $(1-x^2)y'' - 2xy' + 2y = 0$.

<u>Solution</u>. First verify that $y_1 = x$ is a solution.

$2\,)$	$y_1 = x$
$-2x\,)$	$y_1' = 1$
$(1-x^2)\,)$	$y_1'' = 0$

$$(1-x^2)\,y_1'' - 2x\,y_1' + 2\,y_1 = (1-x^2)(0) - 2x(1) + 2(x) = 0$$

Hence y_1 is a solution to $(1-x^2)y'' - 2xy' + 2y = 0$.
Now let $y_2(x) = v\,y_1(x) = v\,x$.

$2\,)$	$y_2 = v\,x$
$-2x\,)$	$y_2' = v'x + v$
	$y_2'' = v''x + v' + v'$
$(1-x^2)\,)$	$y_2'' = v''x + 2v'$

$$(1-x^2)\,y_2'' - 2x\,y_2' + 2\,y_2 = (1-x^2)(x)\,v'' + \left((1-x^2)(2) - 2x(x)\right)v' + (-2x + 2x)\,v = 0$$

Simplifying we obtain: $x(1 - x^2)v'' + (2 - 4x^2)v' = 0$.

Since v is missing, let $u = v'$ to obtain the separable equation: $\dfrac{du}{u} = \dfrac{-(2-4x^2)dx}{x(1-x^2)}$.

We do the integrals separately.

$\int (1/u)\,du = \ln *u* + c$

Let

$$I = \int \frac{-(2-4x^2)}{x(1-x^2)}dx = \int \frac{2-4x^2}{x(x^2-1)}dx = \int \frac{2-4x^2}{x(x-1)(x+1)}dx \Rightarrow \text{Partial Fractions}$$

$$\frac{2-4x^2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

$$2 - 4x^2 = A(x-1)(x+1) + B\,x(x+1) + C\,x(x-1)$$
$$2 - 4x^2 = A(x^2-1) + B(x^2+x) + C(x^2-x)$$

so that

x^2)   A + B + C = - 4   A = - 2
x^1)   B - C = 0   ⇒   B = C
x^0) - A   = 2   2 B = - 4 - A = - 4 + 2 = - 2

Hence A = - 2, B = -1, and C = -1 so that

$$I = \int \frac{2-4x^2}{x(x^2-1)}dx = \int \frac{-2}{x}dx + \int \frac{-1}{x-1}dx + \int \frac{-1}{x+1}dx .$$

= - 2 ln * x * - ln * x - 1 * - ln * x + 1 * + c

= - ln * x^2 (x^2 - 1)* + c.

Hence the solution of the ODE is

ln *u* = - ln * x^2 (x^2 - 1)* + c   or   ln *u (x^2)(x^2 - 1)* = c.

Letting A = ± e^c we obtain: u (x^2)(x^2 - 1) = A.

Recalling that we are looking for a second linearly independent solution (and not the general solution) we may take A = 1 to obtain:

$$u = v' = \frac{1}{x^2(x^2-1)} \quad \Rightarrow \text{ Partial Fractions}$$

$$\frac{1}{x^2\ (x-1)(x+1)} = \frac{A}{x^2} + \frac{B}{x}.+ \frac{C}{x-1} + \frac{D}{x+1}$$

1 = A (x - 1)(x + 1) + B x (x^2 - 1) + C x^2 (x + 1) + D x^2 (x - 1)
1 = A (x^2 - 1) + B (x^3 - x) + C (x^3 + x^2) + D (x^3 - x^2)

so that

x^3)   B + C + D = 0   A = - 1
x^2) A   + C - D = 0   ⇒   B = 0
x^1)   - B   = 0   C + D = 0
x^0) - A   = 1   C - D = - A = 1

Hence A = - 1, B = 0, 2 C = 1, D = - C so that C = ½ and D = -1/2.

Hence

$$v = \int \frac{1}{x^2 (x^2 - 1)} dx = \int \frac{-1}{x^2} dx \ .+ \int \frac{1/2}{x-1} dx + \int \frac{-1/2}{x+1} dx$$

= 1/x + ½ ln * x - 1 * - ½ ln * x + 1 * + c
= 1/x - (½) ln * (x + 1)/(x - 1) * + c

Letting c = 0, we obtain

y_2 = v x
= (1/x - (½) ln * (x + 1)/(x - 1) *) x
= 1 - (x/2) ln *(x+1)/(x-1)*

so that

$y = c_1x + c_2(1 - (x/2) \ln *(x+1)/(x-1)*)$.

EXERCISES on Another Use for Reduction of Order