A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 2

A COLLECTION OF HANDOUTS ON SCALAR LINEAR ORDINARY DIFFERENTIAL EQUATIONS (ODE"s)

CHAPTER 1

Theory of Second Order

Linear ODE's

1. Initial Value Problems (IVP's) and Boundary Value Problems (BVP's)

2. Application of Linear Theory to Second Order Linear ODE's

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5. Summary of Linear Theory for Second Order Linear ODE's

Recall that a "general" solution (or a second integral) for the ODE $y'' = f(x,y,y')$ will have two arbitrary constants. (Recall the special case $y'' = g(x)$. Why are there two arbitrary constants in the solution?) To determine these constants, <u>two</u> constraints or **side conditions** are specified. These can be at the same value of x or at different values.

The **Initial Value Problem** (IVP) for the general second order linear ODE is:

IVP

ODE $y'' + p(x)y' + q(x)y = g(x)$ (2)

IC's $y(x_0) = y_0 \quad y'(x_0) = v_0$

where we have specified **<u>Initial</u> <u>Condition's</u>** (IC's) for the values of y and y' at $x = x_0$. As an example, consider throwing a ball up. In order to uniquely determine the motion of the ball, one needs to specify values for the two state variables, position and velocity, at the same time. If $x_0 \in I=(a,b)$, and $p,g \in C(I)$, our theory will assure exactly one solution in $C^2(I)$.

A **Boundary Value Problem** (BVP) for the general second order linear ODE is:

IVP)

ODE $y'' + p(x)y' + q(x)y = g(x)$ (3)

BC's $y(x_0) = y_0 \quad y(x_1) = y_1$

where we have specified **<u>Boundary</u> <u>Conditions</u>** (BC's) for the values of y at both $x = x_0$ and at $x = x_1$. If we throw a ball up, we might wish to specify the initial position and the position at some later time. For example, If we throw the ball up at time t = 0, (so that y(0) = 0), we may wish to specify the time at which it comes down ($y(t_1) = 0$).

For elementary problems, the **solution technique** is essentially the same, first obtain the "general solution" for the ODE and then apply the IC's or BC's to obtain the arbitrary constants. However, the theoretical results for IVP's and BVP's are often different. Theorems for linear IVP's usually conclude that there exists a unique solution (i.e., the problem is set-theoretically well-posed.) Theorems for similar linear BVP's reflect the general linear theory in that they typically state that there exist no solution, one solution, or an infinite number of solutions. Special cases where BVP's are set-theoretically well-posed are of interest.

EXERCISES on Initial Value Problems and Boundary Value Problems

<u>EXERCISE #1.</u> The general solution of the second order linear ODE $y'' + y = 0$ is $y = c_1 \sin x + c_2 \cos x$ where c_1 and c_2 are arbitrary constants. Apply the boundary conditions to obtain all solutions to the following:

a) $y'' + y = 0$
$y(0) = 2,\ y'(0) = 0$

b) $y'' + y = 0$
$y(0) = 2,\ y(\pi/4) = 0$

c) $y'' + y = 0$
$y(0) = 2,\ y(\pi) = 0$

To solve second and higher order linear differential equations, instead of using a strategy of trying to isolate the dependent variable (i.e., the unknown function) on one side of the equation, we first formulate these equations as **mapping problems**. An **operator**, like a function, maps one set to another set, but we use the term operator when the mapping is from one **vector space** to another, instead of from one number to another. Since we can treat certain collections of functions as vector spaces (which we refer to as an **function spaces**), we use the term **operator** when mapping one function to another. Similar to the notation, $f: \mathbf{R} \to \mathbf{R}$, for real valued functions of a real variable, we use the notation, $T: V \to W$ to indicate that T is an operator where V and W may be unspecified vector spaces or specified function spaces. Recall the definition of a **linear operator** from one vector space to another:

DEFINITION #1. An operator $T:V \to W$ from a vector space V to another vector space W is said to be **linear** if for all vectors $\vec{x}$ and $\vec{y}$ in V and all scalars α and β, it is true that

$$T(\alpha \vec{x} + \beta \vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y}). \qquad (1)$$

The general **second order linear differential operator**, which we denote by L, is

$$L[y] = y'' + p(x)y' + q(x)y. \qquad (2)$$

We introduce three contexts (or settings). If $p,q \in C(I)$, $I = (a,b)$, we take $V = C^2(I)$, the set of functions which have two derivatives and whose second derivative is continuous on the open interval I, and $W = C(I)$, the space of continuous functions on I. If $p,q \in A(I)$, $I = (a,b)$, we take $V = W = A(I)$, the set of functions which are analytic on I. If $p,q \in H(\mathbf{C})$, we take $V = W = H(\mathbf{C})$, the set of functions which are analytic or holomorphic on **C**. For the first context, we prove directly using the above definition (DUD) that the general second order linear differential operator L given by (1) is a linear operator. That is, we write a clear explanation of why the operator L satisfies property (1) in this context. Proofs in the other contexts are similar.

THEOREM #1. The general second order linear differential operator $L: C^2(I) \to C(I)$ defined by (1) is a linear operator from the vector space $C^2(I)$ to the vector space C(I) where I= (a,b) is an open interval on which p and q are continuous.

Proof. By the above definition of a linear operator, to show that the operator L defined by (2) is a linear operator, we must show that if c_1 and c_2 are arbitrary scalars (constants in **R**) and $\varphi_1(x)$ and $\varphi_2(x)$ are "vectors" (functions in $C^2(I)$), then:

$$L(c_1 \varphi_1(x) + c_2 \varphi_2(x)) = c_1 L(\varphi_1(x)) + c_2 L(\varphi_2(x)) \quad \forall x \in I = (a,b)$$

Since this is an identity, we use the standard format for proving identities.

STATEMENT	REASONS
$L(c_1 \varphi_1(x) + c_2 \varphi_2(x)) = (c_1 \varphi_1(x) + c_2 \varphi_2(x))'' + p(x)(c_1 \varphi_1(x) + c_2 \varphi_2(x))' + q(x)(c_1 \varphi_1(x) + c_2 \varphi_2(x))$	Definition of L
$= c_1 \varphi_1''(x) + c_2 \varphi_2''(x) + c_1 p(x) \varphi_1'(x) + c_2 p(x) \varphi_2'(x) + c_1 q(x) \varphi_1(x) + c_2 q(x) \varphi_2(x)$	Properties of Derivatives and some algebra
$= c_1 (\varphi_1''(x) + p(x) \varphi_1'(x) + q(x) \varphi_1(x)) + c_2 (\varphi_2''(x) + p(x) \varphi_2'(x) + q(x) \varphi_2(x))$	Algebra
$= c_1 L(\varphi_1(x)) + c_2 L(\varphi_2(x))$	Definition of L

Since we have shown for arbitrary constants (scalars in **R**) and arbitrary functions ("vectors" in $C^2(I)$) that $L(c_1 \varphi_1(x) + c_2 \varphi_2(x)) = c_1 L(\varphi_1(x)) + c_2 L(\varphi_2(x))$, we have by the definition of a linear operator (as applied to a mapping from the vector space $C^2(I)$ to the vector space $C(I)$) that L is a linear operator.

QED.

<u>COROLLARY</u> <u>#2</u>. Let $L:C^2(I)\to C(I)$ where $I = (a,b)$ and L is given by (2). If y_1 and y_2 are solutions of $L[y] = 0$ (i.e. $L[y_1] = 0$, and $L[y_2] = 0$, then $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is also a solution to $L[y] = 0$.

<u>Proof.</u> Left to the exercises.

This corollary is sometimes called the **superposition principle** (for the <u>homogeneous</u> equation). Actually, it just states that the null set of L is a subspace. Hence, we call $N(L) = \{y\in C^2(I): L[y] = 0\}$ the **null space** of L. We sometimes denote N(L) by N_L.

It can be shown that $A(I)\subseteq C^2(I)\subseteq C^1(I)\subseteq C(I)$ where we recall that $A(I)$ is the set of analytic function on I. Initially we have assumed $p,q\in C(I)$ so that L maps $V=C^2(I)$ to $W=C(I)$. Hence $V\subseteq W$ so that the domain of L is a subset of its codomain. If we require that $p,q\in A(I)$, then L maps $A(I)$ into $A(I)$ so that $V=W=A(I)$. Much of our work will be where p and q are analytic, indeed, where they are constants and hence in $A(\mathbf{R})$. Recall that $A(\mathbf{R})$ can be embedded in $H(\mathbf{C})$, the set of functions which are analytic or holomorphic on **C** (see Exercise #5 in Handout #3 SUBSPACE OF A VECTOR SPACE in Chapter 2-3) . If we require that $p,q\in H(\mathbf{C})$, then L maps $H(\mathbf{C})$ into $H(\mathbf{C})$ so that $V=W= H(\mathbf{C})$. Whether p,q are in $A(I)$, $C(I)$, or $H(\mathbf{C})$, we have the following :

<u>THEOREM</u> <u>#3</u>. The dimension of the null space of L, N(L), is 2. If $L:C^2(I)\to C(I)$, then $N(L) \subseteq C^2(I)$; if $L:A(I)\to A(I)$, then $N(L) \subseteq A(I)\subseteq C^2(I)$; if $L:H(\mathbf{C})\to H(\mathbf{C})$, then $N(L) \subseteq H(\mathbf{C})$. We write $\dim N(L) = 2$.

Since we know that the dimension of the null space is two, if we have two linearly independent solutions, φ_1 and φ_2, of

$$y'' + p(x)y' + q(x)y = 0 \qquad (3)$$

we can prove that B = { φ_1, φ_2 } is a basis of N(L). Hence all solutions of the linear homogeneous equation (3) (i.e., L[y] = 0) can be written as a linear combination of the functions in B, i.e., as

$$y(x) = c_1\, \varphi_1(x) \quad + \quad c_2\ \varphi_2(x). \qquad (4)$$

Theorem#3 and other linear theory form the basis for the standard techniques for solving linear ODE's and PDE's (partial differential equations). Thus we abandon the strategy of isolating the unknown function and instead rely on linear theory. We view our problem as a **mapping problem** with a linear operator L mapping functions (vectors) from one function space to another. We solve the homogenous equation by finding a basis of the null space. That is, we wish to find all functions y in the domain that map to the zero function. Theorem #3 says that for the linear operator (2), the set of all such functions is just the set of all linear combinations of $\varphi_1(x)$ and $\varphi_2(x)$ where B = { φ_1, φ_2 } is a basis of N(L) where L. Unfortunately, a direct proof of Theorem #3 is beyond the scope of this course. However, an indirect proof can be obtained using the uniqueness of the solution to the associated IVP. We state this theorem later. However, a proof that this IVP is well-posed in a set-theoretic sense is also beyond the scope of this course. On the other hand (OTOH), it is easy to use Theorem #3 to prove:

THEOREM #4. Let $p,q\in C(I)$ and B = { φ_1, φ_2 } be a linearly independent set of solutions to (2) (so that it is a basis of N_L). Suppose $y_p\in C^2(I)$ and $L[y_p] = g$. Then the general solution of L[y] = g(x), i.e., to the linear nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = g(x) \qquad (5)$$

is

$$y(x) = y_p(x) + c_1\, \varphi_1(x) \quad + \quad c_2\ \varphi_2(x). \qquad (6)$$

Proof. Left as an exercise.

Theorem #4 says that the set of all $y\in C^2(I)$ that L[y] maps to g is the two parameter infinite set of functions given by (6). Similar theorems can be proved in the other two contexts for L.

EXERCISES on Second Order Linear Differential Operators

EXERCISE #1
(a) Let $L[y] = y'' + y$. Compute $L[\varphi]$ if (i) $\varphi(x) = \sin x$, (ii) $\varphi(x) = \cos x$, (iii) $\varphi(x) = e^x$.
(b) Let $L[y] = y'' - y$ Compute $L[\varphi] =$ if (i) $\varphi(x) = \sin x$, (ii) $\varphi(x) = e^x$ (iii) $\varphi(x) = e^{-x}$.

EXERCISE #2. Using DUD, prove that each of the following is a linear operator.
(a) $L[y] = y'$ (b) $L[y] = 3y' + y$ (c) $L[y] = y'' + 3y' + 4y$ (d) $L[y] = y'' + ay' + by$
(e) $L[y] = y'' + x^2y' + e^xy$ (f) $L[y] = y'' + p(x)y' + q(x)y$

EXERCISE #3. Prove Corollary #2.

EXERCISE #4. Use Theorem #3 to prove Theorem #4.

Read Section 3.3 of Chapter 3 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.) again. Pay particular attention to informal definitions of linear dependence and independence for two functions (on page 147) and Theorems 3.3.1 (on page 148) and 3.3.3 (on page 150).

It is important that you know and understand the definition of linear independence in an abstract vector space and how to apply it directly to various specific vector spaces including $\mathbf{R}^n$ and the function spaces (and not that you just learn a process that only works in one vector space).

DEFINITION #1. Let V be a vector space. A finite set of vectors $S = \{\vec{x}_1, \ldots, \vec{x}_k\} \subseteq V$ is **linearly independent** (ℓ.i.) if the only set of scalars c_1, c_2, ..., c_k which satisfy the (homogeneous) vector equation

$$c_1\vec{x}_1 +.c_2\vec{x}_2 +..+c_k\vec{x}_k = \vec{0} \qquad (1)$$

is $c_1 = c_2 = \cdots = c_n = 0$; that is, (1) has only the trivial solution. If there is a set of scalars not all zero satisfying (1) then S is **linearly dependent** (ℓ.d.).

DEFINITION #2. Let $f_1,...,f_k \in \mathscr{F}(I,\mathbf{R})$ where I = (a,b). Now let J = (c,d) $\subseteq$ (a,b) and for i = 1,...,k, denote the restriction of f_i to J by the same symbol. Then we say that $S = \{f_1,...,f_k\} \subseteq \mathscr{F}(J,\mathbf{R}) \subseteq \mathscr{F}(I,\mathbf{R})$ is **linearly independent** on J if S is linearly independent as a subset of $\mathscr{F}(J,\mathbf{R})$. Otherwise S is **linearly dependent** on J.

Applying Definitions #1 and 2 to a set of two functions in the function space $\mathscr{F}(I,\mathbf{R})$ we obtain:

THEOREM #1. (Definition of linear independence of two functions in $\mathscr{F}(I,\mathbf{R})$.) The set $S = \{f,g\} \subseteq \mathscr{F}(I,\mathbf{R})$, I = (a,b) is **linearly independent** on I if (and only if) the only solution to the equation

$$c_1\, f(x) + c_2\, g(x) = 0 \qquad \forall\, x \in I \qquad (1)$$

is the trivial solution $c_1 = c_2 = 0$ (i.e., S is a **linearly independent set** in the vector space $\mathscr{F}(I,\mathbf{R})$). If there exists c_1, $c_2 \in \mathbf{R}$, not both zero, such that (1) holds, (i.e, there exists a nontrivial solution) then S is **linearly dependent** on I (i.e., S is a **linearly dependent set** in the vector space $\mathscr{F}(I,\mathbf{R})$).

Often people abuse the definition and say the functions f and g are linearly independent or linearly dependent on I rather than the set {f,g}is linearly independent or dependent. Since it is in general use, this abuse is permissible, but not encouraged as it can be confusing. Note that Eq.

(1) is really an infinite number of equations in the two unknowns c_1 and c_2, one for each value of x in the interval I. Four theorems are useful.

THEOREM #2. If a finite set $S \subseteq \mathscr{F}(I,\mathbf{R})$, I = (a,b) contains the zero function, then S is linearly dependent on I.

THEOREM #3. If f is not the zero function, then $S = \{f\} \subseteq \mathscr{F}(I,\mathbf{R})$ is linearly independent on I.

THEOREM #4. Let $S = \{f,g\} \subseteq \mathscr{F}(I,\mathbf{R})$ where I= (a,b). If either f or g is the zero function in $\mathscr{F}(I,\mathbf{R})$ (i.e., is zero on I), then S is linearly dependent on I.

THEOREM #5. Let $S = \{f,g\} \subseteq \mathscr{F}(I,\mathbf{R})$ where I= (a,b) and suppose neither f or g is the zero function. Then S is linearly dependent if and only if one function is a scalar multiple of the other (on I).

Since a subspace of a vector space is in fact a vector space, we can apply the definition of linear independence and linear dependence to each of the subspaces

$$A(I) \subseteq C^2(I) \subseteq C^1(I) \subseteq C(I) \subseteq \mathscr{F}(I,\mathbf{R}).$$

PROCEDURE. To show that {f,g} is linearly independent directly using the definition (i.e., using DUD), assume (1) and try to show $c_1 = c_2 = 0$. If this appears not to be the case, then try to find explicit nontrivial values of c_1 and c_2 so that (1) is true ($\forall$ x $\in$ I). If there is one such nontrivial solution, then there will in fact be an infinite number of such solutions. However, it is not necessary to find all solutions of (1) to show that S is linearly dependent. The exhibition of a single nontrivial solution to (1) is sufficient to prove that S is linearly dependent. (It is the linear theory that says that there can not be exactly two solutions. Since the trivial solution is always a solution, if we exhibit a nontrivial solution, then there are at least two solutions and hence an infinite number of solutions.) To help you better understand the definition of linear dependence in a vector space, **to show that S is linearly dependent, an exhibition of a nontrivial solution is required**. On the other hand, to show that a set is linearly independent, you must do the complete computation that shows that the only solution to (1) is $c_1 = c_2 = 0$

EXERCISES on Linear Independence of Functions

<u>EXERCISES</u> <u>#1</u>. Determine (and prove your answer directly using the definition, i.e. using DUD) if the following sets are linearly independent or linearly dependent on I = **R** (i.e., in the vector space $C^2(\mathbf{R})$).

a) $\{e^x, e^{2x}\}$ b) $\{\sin x, \cos x\}$ c) $\{3e^x, 2e^x\}$ d) $\{1 - \sin^2 x, \cos^2 x\}$ e) $\{\sin 2x, \sin x \cos x\}$.

Hint: For the set to be linearly independent, (1) must hold $\forall\, x \in \mathbf{R}$. Pick two (distinct) values of x to obtain two equations in c_1 and c_2. Now show (if possible) that $c_1 = c_2 = 0$. Hence use Theorem 2 to show that the set is linearly independent. If this is not possible, see if one of the functions is a scalar multiple of the other. You may finish the proof by citing Theorem 2 or by finding explicit values of c_1 and c_2, not both zero such that (1) holds $\forall\, x \in \mathbf{R}$. Exhibiting (1) with these values provide conclusive evidence that {f,g} is linearly dependent.

Handout No. 4 THE WRONSKIAN Professor Moseley

DEFINITION #1. If $y_1, y_2 \in C^1(I)$, where I = (a,b), then

$$W(y_1,y_2,x) \;=_{df}\; W(x) \;=_{df}\; \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \;=_{df}\; y_1\, y'_2 - y'_1\, y_2 \qquad (1)$$

is called the Wronski determinant or the Wronskian of y_1 and y_2 at the point x.

EXAMPLE #1. Compute the Wronskian of $y_1 = x$ and $y_2 = x^2$.

$$W(y_1,y_2;x) \;=_{df}\; W(x) \;=_{df}\; \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 \;)\; x^2 = x^2.$$

THEOREM #1. Let $p,g \in C(I)$, where I = (a,b), Then the set S = {f,g} $\subseteq C^1(I,\mathbf{R})$ is linearly independent on I (i.e., in the function $C^1(I,\mathbf{R})$) if there exists $x_0 \in I$ such that $W(y_1,y_2;x_0) \neq 0$. On the other hand, if S = {f,g} $\subseteq C^1(I,\mathbf{R})$, I = (a,b) is linearly dependent on I, then $W(y_1,y_2;x) = 0$ for all $x \in I$.

THEOREM #2. (Abel) If φ_1, φ_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0 \qquad (2)$$

then there exists $c \in \mathbf{R}$ (the value of c will depend on φ_1 and φ_2) such that

$$W(\varphi_1,\varphi_2;x) = c \exp(\;)\; \textstyle\int^x p(t)\, dt\,). \qquad (3)$$

THEOREM #3. Suppose $f,g \in C^2(I,\mathbf{R})$ where I = (a,b) are solutions to the homogeneous equation (2) where $p,q \in C(I,\mathbf{R})$. Then S={f,g} is linearly dependent on I if and only if $W(y_1,y_2;x) = 0$ for all $x \in I$. Also, S is linearly independent on I if and only if $W(y_1,y_2;x) \neq 0$ for all $x \in I$.

EXAMPLE #1. Show that $y_1 = \sin(x)$ and $y_2(x) = \cos(x)$ are solutions of $y'' + y = 0$. Using Theorem #3, show that the set S = {y_1, y_2} is linearly independent.

Solution. To use Theorem #3, we must first show that y_1 and y_2 are solutions to $y'' + y = 0$. Here I = $\mathbf{R}$.

1) $y_1 = \sin(x)$
$y_1' = \cos(x)$
1) $y_1'' = -\sin(x)$

$y_1'' + y_1 = (-1 + 1)\sin(x) = 0$ so that y_1 is a solution to $y'' + y = 0$.

1) $y_2 = \cos(x)$

$y_2' = -\sin(x)$

1) $y_2'' = -\cos(x)$

$y_2'' + y_2 = (-1 + 1)\cos(x) = 0$ so that y_2 is a solution to $y'' + y = 0$.

Instead of showing that the two functions are linearly independent using DUD, we use Theorem #3 and compute the Wronskian.

$$W[y_1, y_2; x] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -\sin^2(x) - \cos^2 x = -1 \neq 0, \quad \forall x \in \mathbf{R}.$$

Since we have shown that $y_1 = \sin(x)$ and $y_2(x) = \cos(x)$ are solutions of $y'' + y = 0$ and that $W[y_1, y_2; x] = -1 \neq 0$, $\forall x \in R$, we have by Theorem #3 that $S = \{ y_1, y_2 \}$ is a set of linearly independent solutions to $y'' + y = 0$.

EXERCISES on The Wroskian

<u>EXERCISE</u> <u>#1</u>. Compute the Wronskian $W(y_1,y_2;x)$ of the following: (a) $y_1 = e^x$, $y_2 = e^{-x}$, (b) $y_1 = \sin x$ $y_2 = \cos x$, (c) $y_1 = e^{ax}$, $y_2 = e^{bx}$. (d) $y_1 = \sin wx$ $y_2 = \cos wx$

<u>EXERCISE</u> <u>#2</u>. Compute the Wronskian $W(x)$ for two solutions φ_1 and φ_2 of (2) if the initial value of $W(x)$ is as given: (a) $y'' + q(x)y = 0$, $W(0) = 3$, (b) $y'' + q(x)y = 0$, $W(x_0) = 0$, (c) $y'' + y' + q(x)y = 0$, $W(0) = 5$, (d) $y'' + xy' + q(x)y = 0$, $W(0) = 3$. Hint: Use (3).

<u>EXERCISE</u> <u>#3</u>. (a) Assume that φ_1 and φ_2 are solutions of (2). Show that $W(\varphi_1,\varphi_2;x) = W(x)$ satisfies the ODE

$$W' + p(x)W = 0 \qquad (3)$$

Hint: From $\varphi''_1 + p\varphi'_1 + q\varphi_1 = 0$ and $\varphi''_2 + p\varphi'_2 + q\varphi_2 = 0$ obtain $(\varphi,\varphi''_2 - \varphi_2\varphi''_1) + p(\varphi,\varphi_2' - \varphi'_1\,\varphi_2) = 0$.

(b) Prove Theorem #1. Hint: What is the solution of (3).

Handout #5 SUMMARY OF LINEAR THEORY FOR SECOND ORDER ODE'S Professor Moseley

Let $L:C^2(I) \rightarrow C(I)$ where $I = (a,b)$ and $p,q \in C(I)$, $I = (a,b)$ be defined by

$$L[y] = y'' + p(x)\, y' + q(x)\, y. \qquad (1)$$

Alternately consider $L:A(I)\rightarrow A(I)$ with $p,q \in A(I)$ or $L:H(\mathbf{C})\rightarrow H(\mathbf{C})$ with $p,q \in H(\mathbf{C})$. We consider the second order linear homogeneous equation $L[y] = 0$:

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad \forall\, x \in I \qquad (2)$$

and the nonhomogeneous equation $L[y] = g$:

$$y'' + p(x)\, y' + q(x)\, y = g(x) \qquad \forall\, x \in I. \qquad (3)$$

where $g \in C(I)$ (or $A(I)$ or $H(\mathbf{C})$).

THEOREM #1. (Homogeneous Equation). Let $S = \{ y_1, y_2 \}$ be a set of solutions to (2) in $C^2(I)$ (or $A(I)$ or $H(\mathbf{C})$) to the homogeneous equation (2). Then the following are equivalent (i.e. for a particular equation (2) with solution set S, if one is true, they all are true).

a. The set S is linearly independent (or y_1 and y_2 are linearly independent). Since the dimension of N(L) is two, if S is a linearly independent set, then it is a basis of the null space N(L).

b. $W[\, y_1, y_2\, ; x] \neq 0 \quad \forall\, x \in I$ (or $\mathbf{C}$).

c. The general solution of (2) is

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x) \qquad (4)$$

where c_1 and c_2 are arbitrary constants; that is, all solutions of (1) can be written in this form. Since S is a basis of N(L), it is a spanning set for N(L) and hence every function ("vector") in N(L) can be written as a linear combination of the functions ("vectors") in S.

Theorem #1 reduces the problem of finding the general solution of the homogeneous equation (2) to that of finding of two linearly independent solutions.

EXAMPLE #1. Using Theorem #1, show that the set $S = \{ y_1, y_2 \}$ where $y_1 = \sin(x)$ and $y_2(x) = \cos(x)$ are linearly independent solutions of $y'' + y = 0$. Then give the general solution of $y'' + y = 0$.

Solution. To use Theorem #1, we must first show that y_1 and y_2 are solutions to $y'' + y = 0$.

1) $y_1 = \sin(x)$
   $y_1' = \cos(x)$
   $y_1'' = -\sin(x)$
   $y_1'' + y_1 = (-1 + 1)\sin(x) = 0$ so that y_1 is a solution to $y'' + y = 0$.

1) $y_2 = \cos(x)$
   $y_2' = -\sin(x)$
   $y_2'' = -\cos(x)$
   $y_2'' + y_2 = (-1 + 1)\cos(x) = 0$ so that y_2 is a solution to $y'' + y = 0$.

Instead of showing that the two functions are linearly independent using DUD, we use Theorem #1 and compute the Wronskian.

$$W[y_1, y_2; x] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -\sin^2(x) - \cos^2 x = -1 \neq 0, \quad \forall x \in \mathbf{R}.$$

Since we have shown that $y_1 = \sin(x)$ and $y_2(x) = \cos(x)$ are solutions of $y'' + y = 0$ and that $W[y_1, y_2; x] = -1 \neq 0$, $\forall x \in R$, we have by Theorem #1 that $S = \{ y_1, y_2 \}$ is a set of linearly independent solutions to $y'' + y = 0$. Now by part c Theorem #1, we can assert that the general solution to $y'' + y = 0$ on $I = \mathbf{R}$ is $y = c_1 \sin x + c_2 \cos x$ (This two-parameter formula gives all solutions as we change the value of the real numbers c_1 and c_2.) In fact, since $p(x) = 0$ and $q(x) = 1$ can be considered to be in $H(\mathbf{C})$, $y = c_1 \sin x + c_2 \cos x$, $x \in \mathbf{C}$ is the general solution of $y'' + y = 0$ in $H(\mathbf{C})$.

THEOREM #2. (Nonhomogeous Equation). Let $S = \{ y_1, y_2 \}$ be a set of linearly independent solutions to the homogeneous equation (2). Now assume that we can find a (i.e one) particular solution $y_p(x)$ to the nonhomogeneous equation (3). Then

$$y(x) = y_p(x) + y_c(x) \qquad (5)$$

where $y_c(x)$ is the general solution of the associated homogeneous equation (also called the **complementary equation**) (2). The notation y_h is also used to denote the solution to the complementary (homogeneous) equation. The general solution of (3) is then:

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x). \qquad (6)$$

Recall that for a first order linear ODE's one can use an integrating factor to isolate the unknown variable on the left side of the equation so that the right hand side of the equation gives all of the family (because of the constant of integration) of unknown functions that satisfy the ODE. There is no extension of this technique (and no other technique) for isolating the unknown variable for a second order linear ODE. However, Theorem #2 reduces the problem of finding the general solution of the nonhomogeneous equation (3) to finding the three functions $y_p(x)$, $y_1(x)$, and $y_2(x)$. Again, we use the linear theory to develop a solution technique. We first solve the associated homogeneous (complementary) equation using techniques for finding a basis of N(L) (i.e., the functions $y_1(x)$ and $y_2(x)$) for the special case where p and q are constant functions. We then consider two standard techniques for finding a particular solution y_p: 1) The Method of Undetermined Coefficients and 2) The Method of Variation of Parameters.

Finally, we compare the linear theory as it applies to the solution of (2) and (3) with the solution process for $A\vec{x} = \vec{0}$ and $A\vec{x} = \vec{b}$ where the vector spaces are finite dimensional. We learned how to write solutions of $A\vec{x} = \vec{0}$ as $\vec{x} = \vec{x}_1 + \cdots + \vec{x}_k$ and solutions to $A\vec{x} = \vec{b}$ as $\vec{x} = \vec{x}_p + \vec{x}_1 + \cdots + \vec{x}_k$ where $B = \{\vec{x}_1, \cdots, \vec{x}_k\}$ is a basis of N(A) and $\vec{x}_p$ is a particular solution to $A\vec{x} = \vec{b}$. However, we do not know a priori the dimension of N(A). The strategy for solving is to eliminate variables until we have oone scalar equation in one variable. This strategy sometimes works for a system of nonlinear scalar equations. However, for a system of m linear algebraic equations in n unknown scalars, it always works. Gaussian elimination can be learned with no reference to the linear theory. The thing that may be new is that there are three possibilities: no solution, one solution, and an infinite number of solutions. For second order linear ODE's apply the theory to obtain a solution process. We (usually) do not have the possibility of no solution since our codomain is the range so that there is always a solution. Since dim N(L)=2, we are always in the case of an infinite number of solutions. However, we can add two initial conditions to obtain a unique solution. We develop this technique in the next chapter.

EXERCISES on Summary of Linear Theory for Second Order Linear ODE's