A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR
    PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 1

A COLLECTION OF HANDOUTS ON
FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

CHAPTER 6

Introduction to Second Order ODE's and

Solution by First Order Techniques

1. Introduction to Second Order ODE'S: Initial and Boundary Value Problems

2. Solution to Second Order ODE'S by First Order Techniques

Read the introduction and Section 3.1 of Chapter 3 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.). Also read the introduction and Section 10.1 of Chapter 10. Pay particular attention to Examples 1-4 on pages 543-4.

Second order ODE's can be written in the general form

$$F(x,y,y',y'') = 0. \tag{1}$$

where F is a function of four variables and y'' must appear explicitly. However, we only consider second order ODE's of the form

$$y'' = f(x,y,y'), \tag{2}$$

that is, equations where we can explicitly solve for y''.

Recall that the "general" solution (or a first integral) for the first order ODE $y' = f(x,y)$ has one arbitrary constant. The "general" solution (or second integral) of the ODE (2) will have two arbitrary constants. To determine these constants, <u>two</u> constraints or **side conditions** are specified. These can be at the same value of x or at different values. Recall that when using Newton's second law, F = ma, to find the position of a point mass experiencing one-dimensional motion, we needed both the **initial position** and the **initial velocity**. However, we could instead specify the initial position and the final position at some time t_1. For example, if we throw a ball up, we could specify the time at which it is to come down.

The **<u>Initial</u> <u>Value</u> <u>Problem</u>** (IVP) for the general second order ODE is:

IVP
$$\begin{aligned} &\text{ODE} &&y'' = f(x,y) \\ &\text{IC} &&y(x_0) = y_0, \quad y'(x_0) = v_0 \end{aligned}$$

where we have specified **<u>Initial</u> <u>Condition's</u>** (IC) for the values of y and y' at $x = x_0$. As an example, consider throwing a ball up. In order to uniquely determine the motion of the ball, in addition to the force acting on the particle, one must also specify the initial position and the initial velocity. The interval of validity for the solution (i.e., where the ODE must be valid) must include $x = x_0$. (Finding the interval of validity can be thought of as part of the problem.)

A **<u>Boundary</u> <u>Value</u> <u>Problem</u>** (BVP) for the general second order ODE is:

BVP
$$\begin{aligned} &\text{ODE} &&y'' = f(x,y) \\ &\text{BC} &&y(x_0) = y_0, \quad y(x_1) = y_1 \end{aligned}$$

where we have specified **<u>Boundary</u> <u>Conditions</u>** (BC's) for the values of y at $x = x_0$ and at

$x = x_1$. If we throw a ball up, we might wish to specify the initial position and the position at some later time. For example, If we throw the ball up at time $t = 0$, (so that $y(0) = 0$), we may wish to specify the time at which it comes down ($y(t_1) = 0$). More often, BVP's involve spacial rather than temporal conditions so that $y = y(x)$ is appropriate.

For elementary problems, the solution technique for IVP's and BVP's is essentially the same. First obtain the "general solution" for the ODE. The apply the IC's or BC's to obtain the arbitrary constants. However, the theory of IVP's and the theory of BVP's are different. First, the concept of interval of validity changes. That is , our concept of solution changes. To be a solution, we now **require** that the function exist over the entire interval between the two points. For linear problems, we simply require the coefficient functions to be continuous. However, for nonlinear problems this is not sufficient to prevent "blow up"between the points. Hence we must require it. On the other hand, for Boundary conditions that only involve the values of the function at the endpoints, we are not as concerned with differentiablility at the end points. Hence we require only that solutions to the BVP be in $\Sigma = C(\bar{I}) \cap C^1(I)$ where I is the open interval I =

(x_0, x_1) and $\bar{I}$ is the close interval $\bar{I} = [x_0, x_1]$.

Second, the theorems for such problems are quite different in the following sense. Theorems involving linear IVP's usually assert that a unique solution exists, even if it is only a local solution and may "blow" up in finite time. However, theorems for linear BVP's with the same ODE (and even having analytic coefficient functions) but boundary conditions instead of initial conditions may only be able to conclude that there exist no solution, one solution, or an infinite number of solutions.

For elementary problems, the **solution technique** for IVP's and BVP's is essentially the same, first obtain the "general solution" for the ODE and then apply the IC's or BC's to obtain the arbitrary constants. However, since the theoretical results for IVP's and BVP's are different,. (theorems for linear IVP's usually conclude that there exists a unique solution i.e., the problem is set-theoretically well-posed whereas theorems for similar linear BVP's reflect the general linear theory in that they typically state that there exist no solution, one solution, or an infinite number of solutions), this solution technique may not yield a unique solution. Conditions where BVP's are set-theoretically well-posed are of interest.

EXERCISES on Introduction to Second Order ODE'sInitial Value Problems and Boundary Value Problems

EXERCISE #1. The general solution of the second order linear ODE $y'' + y = 0$ in $A(\mathbf{R})$ is
$y = c_1 \sin x + c_2 \cos x$ where c_1 and c_2 are arbitrary constants. Apply the boundary conditions to obtain all solutions to the following:

a) $y'' + y = 0$
   $y(0) = 2, y'(0) = 0$

b) $y'' + y = 0$
   $y(0) = 2, y(\pi/4) = 0$

c) $y'' + y = 0$
   $y(0) = 2, y(\pi) = 0$

Read the introduction and Section 3.1 of Chapter 3 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.). Also read the introduction and Section 10.1 of Chapter 10. Pay particular attention to Examples 1-4 on pages 543-4.

Recall that we only consider second order ODE's of the form

$$y'' = f(x,y,y'),\tag{1}$$

that is, equations where we can explicitly solve for y".

SPECIAL CASES   Several special cases of interest are listed below. The first three give examples of second order equations which can be **solved using the first order techniques** you already know. Thus you already know how to **solve** some **second order nonlinear equations**. The fourth case is the general form for a **second order linear ODE**. We will spend the rest of the semester on linear theory and its implications for solving linear problems.

1. $y'' = f(x)$ (y,y' missing)
2. $y'' = f(x,y')$ (y missing)
3. $y'' = f(y,y')$ (x missing)
4. $y'' = -p(x) y' - q(x)y + g(x)$ (linear)

Case 1. $y'' = f(x)$ (y,y' missing, f is only a function of x)

Since f is only a function of x, this equation can always be solved by **quadrature** (i.e. by integrating twice). Recall the example using Newton's second law with only the force of gravity.

Case 2. $y'' = f(x,y')$ (y missing, f is only a function of x and y')

Since $y'' = (y')'$ we can let $v = y'$ and convert the second order equation to the first order equation

$$v' = f(x,v).\tag{2}$$

We can now use any first order techniques that are applicable (i.e., first order linear, separable, exact and substitutions). Once v is obtained, y can be obtained by **quadrature** (i.e. by integration). You have already worked application problems like this using Newton's second law. First you found the velocity and then integrated the velocity to find the position.

<u>EXAMPLE</u>. Solve the Boundary Value Problem (BVP) $y'' = y'$, $y(0)=0$, $y(1)=1$.

Solution. Let $v=y'$ so that $y''=v'$. Substituting into the ODE we obtain the first order problem $v'=v$ which is linear and separable. Its solution is $v = C_1 e^x$. Hence $y' = C_1 e^x$ and we can integrate to obtain $y = C_1 e^x + C_2$. Now applying the boundary conditions we obtain the two equations:

$$\left. \begin{array}{l} 0 = C_1 e^0 + C_2 \quad \Rightarrow C_2 = -C_1 \\ 1 = C_1 e^1 + C_2 \end{array} \right\} \Rightarrow \quad C_1 (e+1) = 1 \Rightarrow C_1 = 1/(e+1) \Rightarrow C_2 = -1/(e+1)$$

Hence $\quad y = 1/(e+1)(e^x - 1)$.

Case 3. $y'' = f(y,y')$ \qquad\qquad (x missing, f is only a function of y and y′)

Let $v = y'$. Now using the chain rule for derivatives, we obtain

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}.$$

Hence $y'' = f(y,y')$ can be written as the first order equation

$$v\frac{dv}{dy} = f(y,v) \tag{3}$$

Note that if we can now solve for v, we get v in terms of y instead of x. More likely we have an equation involving v and y. Since $v = y'$, this is a first order equation, which we may (or may not) be able to solve. In the special case where, in addition to x, y' is also missing,

$$y'' = f(y) \tag{4}$$

we call the equation an **autonomous** equation. If (5) is the model of a point mass whose mass is equal to one (m = 1), and x is time, the **total energy** of the point mass can be written in terms of the velocity (**kinetic energy**) and position (**potential energy**). We sometimes refer to this method of solution as an **energy method** since we can derive **conservation of energy** from **conservation of momentum** (F = ma).

$$v\frac{dv}{dy} = f(y) \Rightarrow \int v\, dv = \int f(y)\, dy = -P(y) + c \quad \text{where} \ -P'(y) = f(y).$$

That is P(y) is an antiderivative of $-f(y)$ and is referred to as a **potential** function. Hence, $K(v) =_{df} \frac{1}{2} v^2 = -P(y) + c$ so that

$$K(v) + P(y) = \tfrac{1}{2}\, v^2 + P(y) = c = \text{constant}.$$

that is; the kinetic energy plus the potential energy equals the total energy which is constant. There is no energy transfer from the system.

Case 4. $y'' = -p(x)\, y' - q(x)y + g(x)$ (linear)

This case will be covered in detail later. Note that the example given in Case 2 is in fact linear.