A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 1

A COLLECTION OF HANDOUTS ON
FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

CHAPTER 3

More Techniques for Solving First Order ODE'S and a Classification Scheme for Techniques

1. Technique for Solving Exact Differential Equations
2. Substitutions as a Technique for Solving First Order ODE's
3. Technique for Solving Bernoulli's Differential Equation
4. Technique for Solving Homogeneous Differential Equations
5. A Classification of Techniques for Solving First Order ODE'S

ODE's-I-3 TECHNIQUE FOR SOLVING
Handout #1 EXACT ODE's Professor Moseley

Read Section 2.6. of Chapter 2 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.). Pay particular attention to the test for exactness and how to solve exact equations. Be aware that there exist techniques to find integrating factors that can change non-exact equations to exact equations.

Let $\psi(x,y) = c$ be a family of curves. By **differentiating implicitly** we obtain

$$\frac{\partial\psi}{\partial x} \quad \frac{\partial\psi}{\partial y}\frac{dy}{dx} \qquad = 0 \tag{1}$$

Compare (1) with the alternate general form of a first order ODE

$$M(x,y) + N(x,y)\,\frac{dy}{dx} \qquad = 0. \tag{2}$$

Given the ODE (2), suppose that we can find a **potential function** $\psi(x,y)$ such that

$$\frac{\partial\psi}{\partial x} \quad = M(x,y), \quad \frac{\partial\psi}{\partial y} \qquad = N(x,y), \tag{3}$$

then the family of curves $\psi(x,y) = c$ is the "general" solution of (2). But we have two problems:

1. Given the ODE (2), how do we know that a "potential " function $\psi(x,y)$ exists?
2. Even if we know that it exists, how do we find it?

The following theorem answers the first question.

<u>THEOREM #1</u> Let $M(x,y), N(x,y) \in C^2(\mathbf{R}^2, \mathbf{R})$. There exists a $\psi(x,y)$ satisfying (3) if and only if

$$\frac{\partial M}{\partial y} \quad \frac{\partial N}{\partial x} \qquad \textbf{(check for exactness)} \tag{4}$$

<u>Proof idea</u>. Using (3), (4) becomes

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) = \frac{\partial N}{\partial y}$$

Recall that if such a ψ exists and all second partials are continuous (i.e., $\psi \in C^2(\mathbf{R}^2,\mathbf{R})$), then it should not matter which way you compute the derivatives.

EXAMPLE #1: $(y \cos x + 2xe^y) + (\sin x + x^2e^y + 2)y' = 0$ (6)

Solution: We rewrite this equation in differential form

$$(y \cos x + 2xe^y)dx + (\sin x + x^2e^y + 2)dy = 0. \quad (7)$$

If the check for exactness is satisfied, we call both the general form (2) (e.g., (6))and the differential form (e.g., (7)) of the ODE **exact**.

$M = y \cos x + 2x e^y$ $N = \sin x + x^2 e^y + 2$

$\frac{\partial M}{\partial y} = \cos x + 2x e^y$ $\frac{\partial M}{\partial x} = \cos x + 2x e^y$ implies exactness.

To find $\psi(x,y)$, we recall that it must satisfy $\partial\psi/\partial x = M$.

$\frac{\partial \psi}{\partial x} = y \cos x + 2x e^y$ which implies $\psi(x,y) = \sin x + x^2 e^y + h(y)$ where the integration

"constant" h(y) may be a function of y. (Check that $\partial \psi/\partial x = M$ by computing the partial derivative. What is the partial of h(y) with respect to x?) To find $\psi(x,y)$, it remains to find h(y). Recall that we have not yet involved N in our solution process and that $\partial\psi/\partial y = N$.

$$\frac{\partial \psi}{\partial y} = \sin x + x^2 e^y + h'(y) = N = \sin x + x^2 e^y + 2$$

which implies $h'(y) = 2$. (All x's must cancel, Why? If they do not, go back and find your error.) Hence $h(y) = 2y + c$. We may let $c = 0$ so that $h(y) = 2y$. We have that a potential function is

$$\psi(x,y) = y \sin x + x^2e^y + 2y$$

and the family of solution curves $\psi(x,y) = c$ is

$$y \sin x + x^2e^y + 2y = c.$$

EXAMPLE 2 $(3x^2 + 2xy)dx + (x+y^2)y' = 0$
Solution:

$M = 3x^2 + 2xy$ $N = x + y^2$

$$\frac{\partial M}{\partial y} = 2x, \quad \frac{\partial N}{\partial x} = 1 \quad \Rightarrow \quad \text{Not exact}$$

We are stuck! You can try the other techniques you know, but unfortunately none will work. This problem is not solvable by the techniques you have learned. Recall that the antiderivatives of all elementary functions are not elementary functions. Hence we should not be surprised to find that there are first order ODE's that are not solvable by analytic techniques.

EXERCISES on Technique for Solving Exact Differential Equations

EXERCISE #1. Show that the following ODE's are exact.

EXERCISE #2. Determine which of the following ODE's are exact.

EXERCISE #3. Solve the following exact ODE's. Begin by showing that thaey are exact.

EXERCISE #4. Solve the following ODE's by any appropriate method.

Sometimes we can make a substitution (change of variable) for the unknown function (dependent variable) that we are looking for and change an unsolvable problem into a solvable (e.g., linear or separable) one. Consider the general equation

$$\frac{dy}{dx} = f(x,y) \quad . \tag{1}$$

Let $v = g(y;x)$ where $g:D_1 \to R$ is a bijection for each $x \in D_2$ so that $y = g^{-1}(v;x)$ where $g^{-1}:R \to D_1$. Then

$$\frac{dy}{dx} = \frac{\partial g^{-1}(v;x)}{\partial v}\frac{dv}{dx} + \frac{\partial g^{-1}(v;x)}{\partial x}$$

Hence (1) becomes

$$\frac{\partial g^{-1}(v;x)}{\partial v}\frac{dv}{dx} + \frac{\partial g^{-1}(v;x)}{\partial x} \qquad f(x,g^{-1}(v)) \qquad =$$

$$\frac{dv}{dx} \quad f(x,g^{-1}(v))\left[\frac{\partial g^{-1}(v)}{\partial v}\right]^{-1} \quad \frac{\partial g^{-1}(v;x)}{\partial x}\left[\frac{\partial g^{-1}(v)}{\partial v}\right]^{-1} \quad -$$

If this problem is solvable and we obtain $v = h(x)$ with domain D_3. We may then compute y as

$$y = g^{-1}(h(x);x). \tag{5}$$

with domain $D = D_2 \cap D_3$.

Read the discussion on Bernoulli equations on page 73 and Section 2.4 of Chapter 2 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.). Pay particular attention to the idea of how a change of variable can change an unsolvable (nonlinear) problem to a solvable (linear or separable) one.

First order ODE's of the form

$$y' + p(x)\, y = q(x)\, y^n, \quad n \neq 0,\ n \neq 1 \qquad (1)$$

are called **Bernoulli equations**. There is a substitution (change of variable) that will convert (1) into a linear equation. But if n = 0 then (1) is already linear and if n = 1 then (1) is both linear and separable. Thus for the purpose of **classification of solution techniques**, we do not consider (1) to be a Bernoulli equation if either n = 0 or n = 1.

THEOREM #1. The substitution $v = y^{1-n}$ will convert a Bernoulli equation ($n \neq 0,1$) into a first order linear ODE.

Proof. As $v = g(y) = y^{1-n}$ ($n \neq 0,1$), we have $g:D_1 \to R$ where D_1 depends on n, but $[0,\infty) \subseteq D_1$. Hence $y = g^{-1}(v) = v^{1/(1-n)}$ where $g^{-1}:R \to D_1$. Hence $dy/dx = (1/(1-n))\, v^{(1/(1-n))-1}\, dv/dx = (1/(1-n))\, v^{n/(1-n)}\, dv/dx$. Hence (1) becomes

$$(1/(1-n))\, v^{n/(1-n)}\, dv/dx + p(x)\, v^{1/(1-n)} = q(x)\, v^{n/(1-n)}$$
$$dv/dx + (1-n)\, p(x)\, v^{1/(1-n)-n/(1-n)} = q(x)$$
$$dv/dx + (1-n)\, p(x)\, v = (1-n)\, q(x), \quad n \neq 0,\ n \neq 1 \qquad (2)$$

which is first order linear. Q.E.D.

EXAMPLE. Solve (i.e., find the "general solution" of) $y' - (y/x) = [y^2 \sin(x)]/x$.
Solution. Since this equation has the form given in (1) with n = 2, $p(x) = -1/x$ and $q(x) = (\sin x)/x$, we classify it as a Bernoulli equation. Let $v = g(y) = y^{1-2} = y^{-1}$ where $g:(0,\infty) \to (0,\infty)$ so that $y = g^{-1}(v) = v^{-1}$ where $g^{-1}:(0,\infty) \to (0,\infty)$. Hence $y' = -v^{-2}\,(dv/dx)$. Substituting into the ODE we get

$$-v^{-2}\frac{dv}{dt} - x^{-1}v^{-1} = v^{-2}\frac{\sin x}{x}$$
$$\frac{dv}{dt} + x^{-1}v = -\frac{\sin x}{x} \quad , \quad \text{which is linear.}$$

$p(x) = 1/x$, $\int p(x)dx = \int (1/x)dx = \ln(x) + c$. Leting c=0 we obtain

$\mu = e^{\ln(x)} = x$. Multiplying the ODE by μ we obtain

$$x\frac{dv}{dt} + v = -\sin x$$

$$\frac{d(vx)}{dt} = -\sin x$$

$$xv = -\int \sin x\, dx = \cos x + c$$

$$v = \frac{\cos x + c}{x}$$

Since $y = v^{-1}$ we have $y = \dfrac{x}{\cos x + c}$

EXERCISES on Technique for Solving Bernoulli Equations

EXERCISE #1. Write a formal statement of Theorem #1. Include continuity conditions on p(x) and q(x) on an open interval I = (a,b). Can you determine a priori an interval of validity for the entire family of solution curves? Why or why not? What about the domain of the family of functions v? How are the domains of the y's and the v's related?

EXERCISE #2. Proof Theorem #1. Hint: Show that substitution always converts the general Bernoulli equation into a linear equation.

EXERCISE #3. Solve (i.e., find the family of solution curves of)

ODE's-I-3 TECHNIQUE FOR SOLVING
Handout #4 HOMOGENEOUS DIFFERENTIAL EQUATIONS Professor Moseley

Read Section 2.9 of Chapter 2 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.). Be able to solve a homogeneous equation. Know the two uses of the word homogeneous and what they mean.

First order ODE's of the form

$$dy/dx = f(x,y) = F(y/x) \quad (\text{or } F(x/y)) \tag{1}$$

are called **homogeneous**. (This is a different use of the term homogeneous than the way it is used with regard to **linear** ODE's. The terminology is unfortunate. However, it is standard so we are stuck with it. Once you get use to it, it should not be confusing.) We know a substitution that will change (1) into a separable equation. Hence if an equation of the form (1) is already separable (e.g., $y' = y/x$) we do not classify it as homogeneous. However, we will allow the linear equation $y' = y/x + 3$ to be called homogeneous even though the usefulness of converting it to a separable equation instead of just solving it as a linear equation is questionable.

THEOREM #1. The substitution $v = y/x$ will convert a homogeneous equation into a separable ODE.

Proof. To solve ODE's of this form we use the substitution $v = g(y;x) = y/x$ where $g:\mathbf{R}\to\mathbf{R}$ where $x \in D_2 = \{x \in \mathbf{R} : x \neq 0\}$. Hence $y = g^{-1}(v;x) = xv$ and $dy/dx = x\, dv/dx + v$. Hence (1) becomes

$$x\, dv/dx + v = F(v) \quad \text{or} \quad x\, dv/dx = F(v) - v \tag{2}$$

which is separable. Q.E.D.

EXAMPLE. $y' = \dfrac{y^2 + 2xy}{x^2}$ $\quad \left(\dfrac{y}{x}\right)^2 + 2\left(\dfrac{y}{x}\right)$

$v = y/x \Rightarrow y = vx \Rightarrow y' = xv' + v$

$xv' + v = v^2 + 2v \Rightarrow \quad x\dfrac{dv}{dx} \quad = v^2 + \quad \displaystyle\int \frac{dv}{v^2 + v} \quad \int \frac{dx}{x} \quad =$

$$\frac{1}{v^2 + v} = \frac{A}{v} + \frac{B}{v+1}$$

$1 = A(v+1) + Bv$

1 = Av + A + Bv
0 = (A-1) + (A+B)v

$$\left.\begin{array}{r} A = 1 \\ A + B = 0 \end{array}\right\} \Rightarrow A = 1, B = -1$$

$$\int \frac{dv}{v^2 + v} = \int \frac{dv}{v} - \int \frac{dv}{v+1} = \ln|v| - \ln|v+1| + c$$

so

$$\ln|v| - \ln|v+1| = \ln|x| + \ln|c| \Rightarrow \ln\left|\frac{v}{cx(v+1)}\right| = 0$$

$$\Rightarrow \frac{v}{v+1} = \pm cx = c_1 x \Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+1} = c_1 x \Rightarrow \frac{y}{y+x} = c_1 x \quad y - c_1 xy - c_1 x^2 = 0 \Rightarrow y = \frac{c_1 x^2}{1 - c_1 x}$$

$$\left|\frac{v}{cx(v+1)}\right| = 1 \Rightarrow \frac{v}{cx(v+1)} = \pm 1 \Rightarrow \frac{v}{(v+1)} = \pm cx = c_1 x \Rightarrow \frac{\frac{y}{x}}{(\frac{y}{x}+1)} = c_1 x \Rightarrow \frac{y}{y+x} = c_1 x$$

$$\Rightarrow \frac{v}{v+1} = \pm cx = c_1 x \Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+1} = c_1 x \Rightarrow \frac{y}{y+x} = c_1 x \quad y - c_1 xy - c_1 x^2 = 0 \Rightarrow y = \frac{c_1 x^2}{1 - c_1 x}$$

$$y = c_1 x(y + x) \Rightarrow y - c_1 xy = c_1 x^2 \Rightarrow y = \frac{c_1 x^2}{1 - c_1 x}$$

EXERCISES on Technique for Solving Homogeneous Differential Equations

<u>EXERCISE #1</u>. Write a formal statement of Theorem #1. Include continuity conditions on the function F(v) and its derivatives on an open interval I = (a,b). Can you determine a priori an interval of validity for the entire family of solution curves? Why or why not? What about the domain of the family of functions v? How are the domains of the y's and the v's connected?

<u>EXERCISE #2</u>. Proof Theorem #1. Hint: Show that substation always converts the general Homogeneous equation into a separable equation.

<u>EXERCISE #3</u>. Solve (i.e., find the family of solution curves of)

ODE’s-I-3 A CLASSIFICATION OF TECHNIQUES
Handout #5 FOR SOLVING FIRST ORDER ODE'S Professor Moseley

Read Section 2.10 of Chapter 2 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.). Be able to determine all first order methods which apply to any first order ODE. Be able to solve using all applicable methods.

We have covered **five techniques** for solving first order ode's.
(1) First order Linear - Integrating factor = $\mu = \exp(\int p(x)\, dx)$
(2) Separable
(3) Exact
(4) Substitutions
(a) Bernoulli's Equation (but not linear, use $v = y^{1-n}$)
(b) Homogeneous (but not separable, use $v = y/x$).

Recall that since the substitution $v = y^{1-n}$ converts a Bernoulli equation ($y = f(x)$) into a linear equation, we do not classify any linear equation as a Bernoulli equation even if it has the right form (i.e., if $n = 0$ or $n =1$). Similarly, since the substitution $v = y/x$ converts a homogeneous equation into a separable equation, we do not classify any separable equation as a homogeneous equation even if it has the right form. Even with these restrictions, these techniques are not mutually exclusive. A linear problem may be separable. Separable problems can be placed in an exact form. Indeed some problems are both Bernoulli and homogeneous. It should also be noted that for homework the method to be used on a problem was dictated by the section developing that technique. Coming up with the technique for solving a real world model is up to the modeler (or his consultant). In choosing a method for solution, we first note that First Order Linear and Bernoulli equations distinguish between the independent and dependent variables. Separable, Exact, and Homogeneous do not. We also note that all first order ODE’s are not solvable by these techniques. Indeed, some are not solvable by any known technique. We develop a classification scheme for an ODE in the variables x and y. Appropriate changes can be made if other variables (e.g., y and t) are used.

1) First order linear (y as a function of x).- Integrating factor = $\mu = \exp(\int p(x)\, dx)$
2) First order linear (x as a function of y).- Integrating factor = $\mu = \exp(\int p(y)\, dy)$

3) Separable.
4) Exact Equation (Must be exact in one of the two forms discussed in class).
5) Bernoulli, but not linear (y as a function of x). Use the substitution $v = y^{1-n}$ (or $y = v^{1/(1-n)}$).
6) Bernoulli, but not linear (x as a function of y). Use the substitution $v = x^{1-n}$ (or $v = x^{1/(1-n)}$).
7) Homogeneous, but not separable. Use the substitution $v = y/x$ ($y=xv$) or $v = x/y$ ($x=y/v$).
8) None of the above techniques works.

EXERCISES on A Classification of Techniques for Solving First Order ODE's

<u>CLASS EXERCISE</u>. Classify the following first order ODE's according to the solution technique(s) which can be used to solve them. Choose one or more from the list above. Recall that some equations may fall into more than on classification since some equations may be solved by more than one technique (e.g. an equation may be linear and exact).

1. $y' = e^x/(2y)$
2. $y' = (e^y x)/(e^y + x^2 e^y)$
3. $(1+x)dy + 3y\,dx = 0$
4. $y' = \sqrt{1+y}$
5. $x\,dy - (y + \sqrt{xy}\,dx = 0$
6. $x' = [x - t\cos^2(x/t)]/t$
7. $\dfrac{dx}{dy} - x\ln y = y^2$ Hint: Let y be the independent variable.
8. $y' + (2/x)y = (\cos x)/x^2$
9. $(2\,xy + e^y)dx + (x^2 + xe^y)dy = 0$
10. $(3x^2\ell n\, x + x^2 - y)dx - x\,dy = 0$

<u>EXERCISE</u> Solve all equations by <u>all</u> techniques which apply. Recall that we do not use a substitution (e.g.,Bernoulli equation with n=0 or n=1 as a Bernoulli equation) if the problem is already linear or separable. There is no advantage to using a substitution to convert a linear or separable equation to a linear or separable equation.