A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 1

A COLLECTION OF HANDOUTS ON FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

CHAPTER 2

Some Theory, a Technique for Solving Separable Equations, Direction Fields, and More Theory

Read Sections 1.2, 2.1, 2.4, and 2.8 of Chapter 2 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.). **Pay** particular **attention** to the **concepts** of **general solution** and **integral curves** introduced on page 11 and solution introduced on page 20. Read the **existence** and **uniqueness** theorem on page 106.

We give more definitive answers to the questions:

1) What do we mean by an **Ordinary Differential Equation** (ODE)?
2) What do we mean by a **solution** to an Ordinary Differential Equation (ODE)?
3) What do we mean by the **general solution** of an ODE?
4) What do we mean by the **existence** of a solution to an ODE?
5) What do we mean by **uniqueness** of the solution of an IVP?
6) What do we mean by **explicit** and **implicit** solutions?

By an **n^{th} order ODE** we mean an equation of the form

$$F(x,y,y',...y^{(n)}) = 0 \quad \forall\, x \in I = (a,b) \tag{1}$$

where the n^{th} derivative must appear explicitly in the equation. Thus the equation may contain x (the independent variable), y (the dependent variable), and y' through $y^{(n-1)}$ (the first n-1 derivatives of the dependent variable) , but it must contain the n^{th} derivative explicitly. The order of the ODE determines the qualitative behavior of its family of solution. Specifically, the family should have n arbitrary parameters. For the most part, we consider ODE's where we can algebraically solve for the n^{th} derivative, that is, those of the form

$$y^n = f(x,y,y',...y^{(n-1)}) = 0 \quad \forall\, x \in I = (a,b). \tag{2}$$

DEFINITION. A **solution** to (1) on the open interval $I = (\alpha,\beta)$ is a **function** $y = \varphi(x)$ in $\Sigma = C^{(n)}(I)$ (i.e. a function $\varphi(x)$ for which the first n derivatives $\varphi'(x), \varphi''(x),...,\varphi^{(n)}(x)$ exist and are continuous on the open interval I) such that if it (and its first n derivatives) are substituted into Equation (1) we have equality $\forall x \in I$ (i.e. for all x such that $\alpha < x < \beta$). That is,

$$F(x,\varphi(x),\varphi'(x),...,\varphi^{(n)}(x)) = 0 \tag{3}$$

is an **identity** (i.e. (3) is true $\forall\, x \in I = (a,b)$).

If a family of functions given by a formula which contains, in addition to the independent variable x, n arbitrary constants satisfies (1) (i.e. each member of the family is a solution to (1)), then we refer to this formula as the **"general solution"** of Equation (1). Hence the "general solution" of (1) is a family of an **infinite number of solutions**. For **linear ODE's** with reasonable hypotheses on the coefficients, the **linear theory** assures that the formula for the **"general solution"** will give all of the **solutions** of Equation (1). However, for **nonlinear equations**, the formula obtained may not give all of the solutions. Hence, what we have called the "general solution" is sometimes referred to as the n^{th} **general integral** of the ODE. We will use the term **"general solution"**, but for **nonlinear** equations, we will put it in quotation marks since this **parametric formula** may not give all of the solutions to the ODE.

EXISTENCE of solutions to an ODE. Not all ODE's have solutions. Sometimes no solution exists. On the other hand, not all solutions to ODE's are **elementary functions**. That is, an ODE may have a solution which is not given by an algebraic formula (e.g. $3x^2 + 2x$), not an algebraic function, not an elementary function, or even have a name. (Recall that sin(x) is the name of a function that you learned about in trigonometry that is not an algebraic function; it is **transcendental**.) It easy to understand why if you recall that antiderivatives of elementary functions need not be elementary functions, and may not have names. (See Chapter 0-3 for a definition of algebraic, transcendental, and elementary functions.) Some non-elementary functions, mostly because of their importance in applications, have been given names (e.g., the error function and Bessel functions) and as a group are called **special functions**.

UNIQUENESS of solutions to an ODE. The **"general solution"** of an n^{th} order ODE will be a **family** of (an infinite number of) solutions because of the n arbitrary constants in the parametric formula. A particular solution from this family can be specified by requiring **side conditions** (e.g., the initial position and velocity) which can be used to evaluate the constants. If they are specified at one point, they are called **initial conditions** (IC's). If $n \geq 2$, they may be specified at more than one point and are then called **boundary conditions** (BC's). (We have already seen initial conditions; we consider boundary conditions later, after we consider second order ODE's).
2) **An Initial Value Problem** (IVP) or a **Boundary Value Problem**(BVP) consisting of an ODE and appropriate side conditions is called **well-posed** in a set theoretic sense if there exists exactly one solution. (An additional condition that is sometimes required for a problem to be well-posed concerns continuity with respect to the parameters involved.)

EXPLICIT AND IMPLICIT SOLUTIONS. As we have seen, the general solution of a first order linear ODE can always be found **explicitly**. However, for many nonlinear problems it is not easy or even possible to obtain explicitly the family of functions which gives the "general solution" (e.g., $y = f(x;c)$). For this reason, sometimes an **equation** for a **collection of curves** (e.g., circles) that **implicitly** define **functions** that are solutions to the first order equation $y' = f(x,y)$ will be given as $g(x,y;c) = 0$ and often as $g(x,y) = c$.

THEOREM. If $c > 0$, then all of the functions $y = y(x)$ defined on the open interval $(-\sqrt{c}, \sqrt{c})$ by the equation

$$y^2 + x^2 = c \qquad (4)$$

(this is a family of curves, specifically they are circles) are solutions of

$$y' = -x/y \tag{5}$$

<u>Proof.</u> To prove that all of the functions defined by the family of curves in (3) are all solutions, we must **substitute** these into (4). However, since we do not have the solution explicitly, (the problem is nonlinear and the solutions are given implicitly), we use a different technique from that used for linear equations. We compute the derivative y' of these functions using **implicit differentiation**. Let y be a function defined by the equation $y^2 + x^2 = c$. Then $2yy' + 2x = 0$ implies $2yy' = -2x$ so that $y' = -x/y$. Hence any function y defined implicitly by (3) satisfies the nonlinear ODE (4).

QED

Note that the family of circles $y^2 + x^2 = c$ implicitly defines two solutions for each value of c.

$$y_1 = \sqrt{c^2 - x^2} \quad \text{and} \quad y_2 = -\sqrt{c^2 - x^2} \tag{6}$$

We accept $y^2 + x^2 = c$ as the "general" solution of (4). Sometimes the term **<u>first</u> <u>integral</u>** is used since we have not shown that every solution of (4) is obtainable by specifying a value of c. Note that the **interval of validity** (i.e., the domain of the solution) depends on the parameter c. Note that the differential equation

$$y^2 y' + x y = 0 \tag{7}$$

has not only the solutions given by (6) but also $y_3 = 0$. There is no value for the parameter c in (4) that gives this solution so that it is not represented by this family of curves.

Recall the first order Initial Value Problem (IVP):

IVP

ODE $$\frac{dy}{dx} = f(x,y) \quad (1)$$

IC $$y(x_0) = y_0 \quad (2)$$

Since we are considering the mathematical solution to a mathematical problem, we choose to use y as a function of x and we allow the **initial condition** (IC) to be at an arbitrary point. It is perhaps better referred to as a **side condition** (SC) since an interpretation of x as time is not required and in deed may cause false conclusions. We require that all of our logic must be mathematical and not temporal or spacial.

We consider three problem solving contexts: Calculus, Classical, and Modern.

Calculus. In this context, f(x,y) is specified explicitly in terms of elementary functions. Algebra and calculus are then used to obtain an (infinite) parametric family of solutions to the ODE, one for each value of an (integration) constant, in terms of elementary functions Then the particular solution that satisfies the side condition is obtained by substituting these values into the formula. This context can be expanded to allow special functions and indeed to allow antiderivatives of any elementary function. It can be further expanded to allow general forms where we are assured that for any specific f(x,y) having this form, a particular algorithm will work. An example is when f(x,y) has the (linear) form $f(x,y) = -p(x)\,y + g(x)$ where $p,g \in C(I)$, then we know a procedure that will solve the problem. In addition to algebraic operations, the formula for the solution involves antiderivatives of functions that involve p and g. We say that the linear problem has been solved in the **general context** or up to qradrature (i.e., up to finding antiderivatives of certain functions).

A difficulty can result from the lack of consideration for the number of solutions; that is, what about existence and uniqueness? How do we know that what we have is a solution and that it is the only one? If a solution technique results in a parametric family of solutions, they can be checked by substituting into the ODE. After the constant has been evaluated using the initial condition, it can be checked by substituting the initial value. However, what about uniqueness? Is this the only solution? For a linear equation, the solution process itself provides a proof of existence and uniqueness since each step is reversible. A sequence of equivalent equations show that all solutions to the ODE are given by the parametric family obtained.. It assumes that a solution exists and goes through a sequence of equivalent problems (or properties) that the solution function must satisfy, ending with the family that gives a formula (i.e., a collection of names) for a parametric family of solution functions (e.g., sin(x) + c or an algebraic formula defining the function including an arbitrary constant). Since all steps are reversible, all of these functions are solutions to the ODE. Similarly for the steps in finding the arbitrary constant. However, this is not the case for nonlinear equations.

Another difficulty is that these solution processes (e.g., for nonlinear problems) may result in implicit, rather than explicit descriptions of the functions (i.e., curves rather than functions). Thus the interval of validity (i.e., the domain of the function we seek) is not self evident and must be determined on a problem by problem basis.

For these reasons and others, we need a second context.

Classical. Instead of specifying f(x,y) explicitly, we simply require it to satisfy certain conditions and then show that there exist exactly one solution in a particular function class. This context can be further subdivided based on the function class.

Classical I : Sufficient conditions are given so that there is exactly one solution in $C^1(I)$ where $x_0 \in I=(a,b)$ (see Chapter 0-3 for the notation for sets of functions).
Classical II : Sufficient conditions are given so that there is exactly one solution in $A(I)$ where $x_0 \in I=(a,b)$ (see Chapter 0-3 for the notation for sets of functions).

Modern. This is the same as classical except that the problem is reformulated to allow "weak" solutions, that is, things that, strictly speaking, are not functions. For example, solutions may be considered to be **distributions** or **equivalence classes** of functions (Look up the definition of an equivalence class in any good abstract algebra textbook)..

In addition to the reasons sighted above, if the traditional context does not yield an explicit solution in terms of elementary functions, it is very useful to know that exactly one solution exist to an initial value problem. If we know that the problem is well-posed, numerical techniques such as **finite differences** and **finite elements** can then be used to find approximate solutions that can be shown to be "close" to the actual solution. This requires that the set of functions (or **function space**) where we look for solutions be equipped with a **topology**, or at least a **metric**. If we do not know that exactly one solution exists, there is no guarantee that the approximate solution obtained has any relevance to the problem. A metric gives the "distance" between the approximate solution and the exact solution and hence an estimate of how good the approximate solution is.

We end with a "proof" (i.e. an outline of a proof since all details are not included) that if f(x,y) is infinitely differentiable, then in the Classical II context, there is at most one solution to the IVP problem (i.e. the problem has the **uniqueness property**). This "proof" also provides an infinite recursive solution algorithm. Since the proof is the same, we first extend our context to allow both the **independent and the dependent variables** to be complex. The concepts of <u>derivative</u> and <u>analytic function</u> can be extended to the <u>complex plane</u>. In this setting, the open interval I is replaced by a **domain** D (an open connected set). The terms **open** and **connected** have technical definitions in complex analysis (and indeed in general topology), but these coincide with (but are more restrictive than) their general "dictionary" definitions. Although D is the domain of a complex function of a complex variable, since the use of the word domain in complex analysis (and in multi-variate calculus) is more restrictive then its general use as the set of things that get mapped by a function, we must be sure we understand the difference between these two uses of the word domain. For convenience, we require D to be an open simply connected set. (The term **simply connected** also has a technical definitions in complex analysis.) The nice thing is that all of the calculus formulas you know for elementary functions of a real variable extend to complex variables. Often, in complex analysis, the word analytic is replaced by the word **holomorphic** and x is replaced by z and y by w so that w = w(z). The holomorphic functions on D are denoted by H(D) (see Chapter 0-3).

ODE $$\frac{dw}{dz} = f(z, w) \qquad (3)$$

IVP

IC $$w(z_0) = w_0 \qquad (4)$$

THEOREM. If f(z,w)∈$C^\infty(\mathbf{C}^2,\mathbf{C})$, then there exists at most one holomorphic (and hence analytic if z and w are real variables in the IVP defined by (1) and (2)) solution to the problem Prob(H(D), (3) and (4) (i.e., to the problem of finding a holomorphic solution to (3) and (4)) where D is a simply-connected domain in **C**, (see Chapters 0-3 and 1-3 as well as an introductory text in complex variables).

Proof idea. If there exists a solution w∈H(D) to Prob(H(D), (3) and (4)), then it has a Taylor series given by

$$w(z) = \sum_{n=1}^{\infty} \frac{w^{(n)}(z_0)}{n!}(z - z_0)^n \qquad (5)$$

where

$$w(z_0) = w_0 \text{ and } w'(z_0) = f(z_0, w_0). \qquad (6)$$

Recall that the constants $w^{(n)}(z_0)$, n = 0, 1, 2, 3, ..., define the function w. That is, if I know the function w (i.e the rule assigning a value w(z) to each z in the domain D) and I know that w is holomorphic (i.e. analytic), then (at least in theory) I can compute all of the constants $w^{(n)}(z_0)$, n = 0, 1, 2, ... and conversely, if I know all of the constants $w^{(n)}(z_0)$, n = 0, 1, 2, 3, ..., then the rule given by (5) defines the function w. Equations (3) and (4) explain how to find $w(z_0)$ and $w'(z_0)$. The remaining constants, $w^{(n)}(z_0)$, n = 2, 3, ..., can be computed recursively (and hence are shown to be unique, assuming, as we have, that they exist), starting with $w''(z_0)$ as follows:

$$w''(z) = \frac{\partial f(z,w)}{\partial z} + \frac{\partial f(z,w)}{\partial w}\frac{dw(z)}{dz}, \qquad w''(z_0) = \frac{\partial f(z_0,w_0)}{\partial z} + \frac{\partial f(z_0,w_0)}{\partial w}\frac{dw(z_0)}{dz}$$

This process can be repeated to obtain all of the constants $w^{(n)}(z_0)$, n = 3, 4, 5 ... ..

Q.E.D.

A difficulty is that the radius of convergence of the holomorphic function defined by (5) using the computed constants might be zero yielding the result that no solution exists. Another difficulty is that the solution process is infinite and hence can only be carried out in part. However, you should recall that the polynomial

$$w_A(z) = \sum_{n=1}^{N} \frac{w^{(n)}(z_0)}{n!}(z - z_0)^n \tag{7}$$

where N is chosen sufficiently large will approximate the holomorphic function w close to $z = z_0$.

Read Sections 2.2 and 2.4 of Chapter 2 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima seventh ed.) again. Pay particular attention to the form given on page 40 for the general first order ODE. Be able to determine when an equation is separable and how to solve it.

The general first order ODE

$$dy/dx = f(x,y) \qquad (1)$$

can be written in the form

$$M(x,y) + N(x,y)\ dy/dx = 0 \qquad (2)$$

in many different ways. For example, by letting N(x,y) = 1 and M(x,y) = - f(x,y). If this can be accomplished so that M is only a function of x and N is only a function of y, we say that the ODE is **separable:**

$$M(x) + N(y)\ dy/dx = 0. \qquad (3)$$

Using differentials and proceeding informally we write (3) as

$$M(x)\, dx = -N(y)\, dy \qquad (4)$$

and "integrate" both sides. More formally we can integrate (3) to obtain

$$\int [M(x) + N(y)\, dy/dx]\, dx = c \qquad (5)$$

$$\int M(x)\, dx + \int [N(y)\, dy/dx]\, dx = c. \qquad (6)$$

THEOREM #1. If F(x) is any antiderivative of M(x) and G(y) is any antiderivative of N(y), then

$$g(x,y) = F(x) + G(y) = c. \qquad (7)$$

defines implicitly a family of solution curves for (3). If, on some interval, G has an inverse function G^{-1}, then (7) can be solved explicitly to obtain $y = G^{-1}(-F(x) + c)$.

EXAMPLE #1: Solve (i.e., find an implicit solution of) the IVP: $\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}$, y(0) = 1

Solution: $\int \frac{1+2y^2}{y}\, dy = \int \cos x\, dx$

$$\ell n|y| + y^2 = \sin x + c$$

Hence $G(y) = \ln|y| + y^2$ and $F(x) =)\sin x$. Since G(y) is not readily invertible, we can not solve for y explicitly, but must be content with an implicit solution. The "general solution"or first integral is a family of curves rather than a family of functions. Applying the initial condition y = 1 when x = 0 to this family of curves, we obtain $\ln 1 + 1 = \sin(0) + c$ so that c = 1. Hence the particular curve that goes through the point (0,1) is

$$\ln *y* + y^2 = \sin x + 1.$$

WRITTEN EXERCISES on Technique for Solving Separable Differential Equations

EXERCISE #1. Write a formal statement of Theorem #1. Include continuity conditions on M(x) on an open interval I = (a,b) and N(y) on an open interval J = (c,d). Can you determine a priori an interval of validity for the entire family of solution curves? Why or why not? Can you add additional conditions on N(y) so that G(y) is invertible? Supposing that such a condition is possible, what is the interval of validity of the entire family of solutions?

EXERCISE #2. Proof Theorem #1. Hint: Justify the steps in the discussion prior to the statement of the theorem.

EXERCISE #3. Solve (i.e., find the family of solution curves of). If possible, solve for y in terms of x.
a) $y' = (x^2 + 1)\, y$ b) $y\,(x^2 + 1)\, y' = 1$ c) $y' = \sin(x/y)$ d) $y' = (y^2 + y)(x^2 + x)$

Read Chapter 1 and Sections 2.1) 2.4 of Chapter 2 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, sixth ed.) again. Pay particular attention to the direction fields given on pages 7,8,9,22, and 35. Be able to sketch isoclines direction fields and integral curves (solutions to specific IVPs)

DIRECTION FIELDS We illustrate how to use isoclines and a direction field to provide qualitative information about a solution to a first order ODE.

EXAMPLE #1. First, use **isoclines** to draw the **direction field** for $y' = y + x$ (Note $f(x,y) = y + x$). Then draw the **integral curves** associated with the following IVP's:

(1) $y' = y + x$	(2) $y' = y + x$	(3) $y' = y + x$	(4) $y' = y + x$	(5) $y' = y + x$
$y(0) = 0$	$y(0) = 1$	$y(0) = 2$	$y(0) = -1$	$y(0) = -2$

Solution. We first sketch some **isoclines** (curves where solutions all have the same slope). Let $f(x,y) = p$ = constant where we initially choose $p = -2, -1, 0, 1, 2$. We sketch these curves on a Cartesian coordinate axes on the next page. First let $p = -2$. We get $y + x = -2$ or $y = -x - 2$ which is a (straight) line that we can sketch. Next, letting $p = -1$ and 0, we note that $y + x = -1$ implies $y = -x - 1$ and $y + x = 0$ implies $y = -x$. There is a pattern! In general, $y + x = p \Rightarrow y = -x + p$ which is always a (straight) line with slope -1. Changing p simply changes the y-intercept. (In general isoclines can be any family of curves, e.g., lines, parabolas, ellipses, hyperbolas, sine curves, etc.) We sketch all of these curves and , noting the pattern, also sketch the isoclines for p =3,4, and -3. On the next page we have draw the direction field by sketching "tic" marks with the appropriate slope on the isoclines. A computer could draw the direction field by simply drawing "tic" marks with the appropriate slope at the points with integer coordinates and not worry about isoclines (see the textbook). But if we are drawing the direction field by hand, the isoclines for $p = -2, -1, 0, 1, 2$ are generally the most helpful. Other isoclines may be helpful as indicated by the particular problem. Since we saw a pattern that made them easy to draw, on the next page we have drawn the isoclines for p =3, 4, and -3 as well as those for $p = -2, -1, 0, 1, 2$. However if they are drawn by hand, differences between the slopes of "tic" marks for p = 2, 3, and 4 are hard to discern.

Now consider the solutions of the following IVP's. Solutions are also called **integral curves** of the ODE.

(1) $y' = y + x$	(2) $y' = y + x$	(3) $y' = y + x$	(4) $y' = y + x$	(5) $y' = y + x$
$y(0) = 0$	$y(0) = 1$	$y(0) = 2$	$y(0) = -1$	$y(0) = -2$

We use the direction field to sketch in these three integral curves on the sketch on the next page and label them (1), (2), and (3). Note that since the problem is linear with $p(x) = -1$ and

$g(x) = x$. Since $p,g \in C(\mathbf{R})$, the interval of validity for all solutions is **R**. As is indicated by our choice of initial conditions, we can obtain all integral curves (i.e., all solutions) by varying the initial condition at x = 0.

The isoclines, the direction field, and the integral curves (i.e., solutions) are sketched on the next page. We now check this work by computing the exact solutions. Since the problem is linear, we may find the general solution (i.e., the entire family of all solutions to this ODE).

$y' - y = x$

$u = e^{-x}$

$d/dx(y\,e^{-x}) = xe^{-x}$

$y\,e^{-x} = -xe^{-x} - e^{-x} + c$

$y = -x - 1 + c\,e^{x}$

$I = \int xe^{-x}dx = xe^{-x} + \int e^{-x}dx$

$u = x \qquad dv = e^{-x}$

$du = dx \qquad v = -e^{-x}$

$I = -xe^{-x} - e^{-x} + c$

Applying the general initial condition $y(0) = y_0$ we obtain $y_0 = -1 + c$ so that $c = y_0 + 1$. Hence we obtain

$$y = -(x+1) + (y_0 + 1)\,e^{x}$$

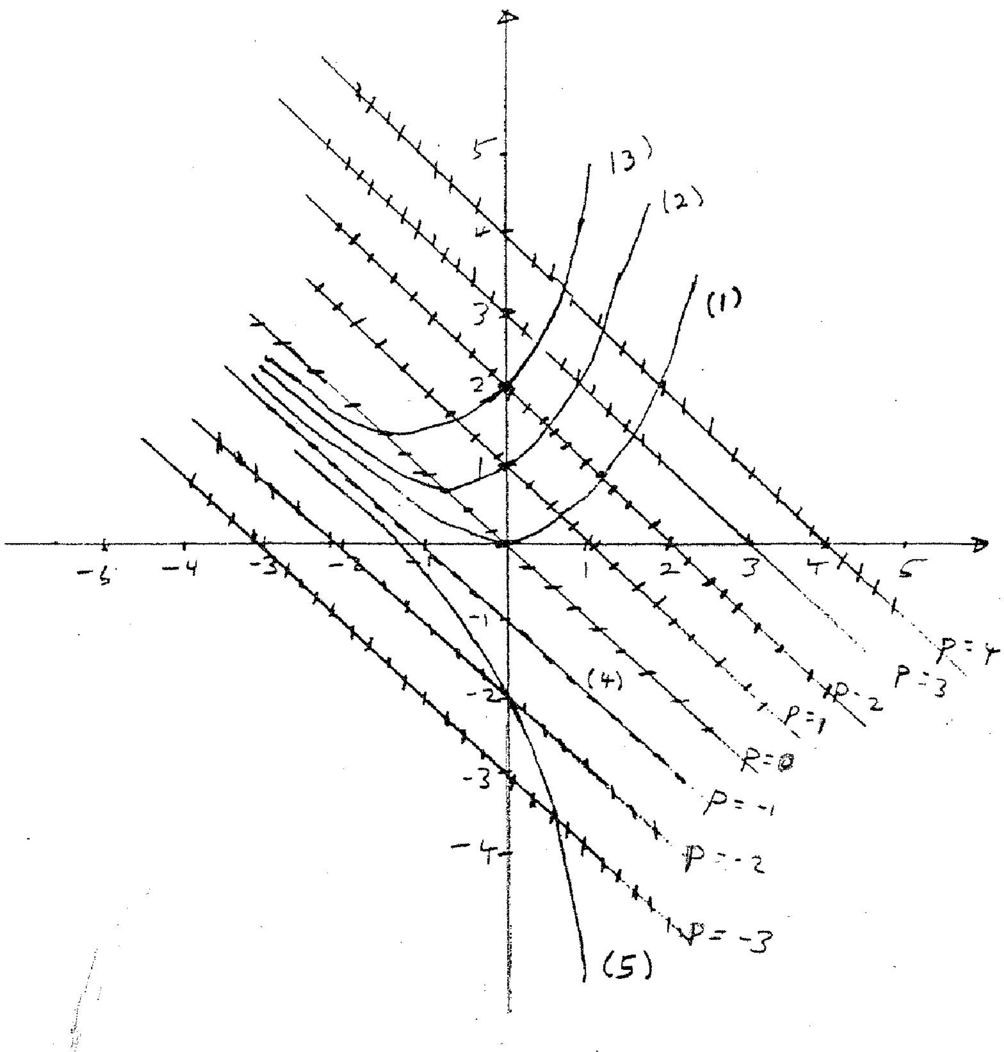
For the five IVP's given we obtain the Table:

	$y_1(x)$	$y_2(x)$	$y_3(x)$	$y_4(x)$	$y_5(x)$
y_0	0	1	2	−1	−2
c	1	2	3	0	−1

Hence we have the solutions:

$y_1 = -(x + 1) + e^{x}$,
$y_2 = -(x + 1) + 2e^{x}$,
$y_3 = -(x + 1) + 3e^{x}$,
$y_4 = -(x + 1)$,
$y_5 = -(x + 1) - e^{x}$.

Check that the solutions we sketched are consistent with these algebraic formulas



Direction Field for $y' = y + x$

Ch. 2 Pg. 13

Read Introduction and Sections 2.1 and 2.4 of Chapter 2 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.) again. Think about and learn the process (algorithm) for solving a first order linear ODE. Again think about the concept of general solution and the existence and uniqueness theorem on page 65. Avoid learning the formulas. **LEARN THE PROCESS**.

The general **initial value problem** (IVP) for first order linear ODE is given by .

IVP

ODE $$y' + p(x)y = g(x) \tag{1}$$

IC $$y(x_0) = y_0 \tag{2}$$

THEOREM #1. (General Solution of the First Order Linear ODE) Suppose that $p,g \in C(I)$ where $I=(\alpha,\beta)$ and let

$$\mu(x) = e^{\int^x p(s)ds} . \tag{3}$$

Then the **general solution** of (1) (i.e. the family of all solutions of (1) in the function space $C^1(I) = \{y:I\to R: y' \text{ exists and is continuous for all } x\in I\}$) is given by

$$y(x) = y_p(x) + y_c(x) = y_p(x) + c\, y_1(x) \tag{4}$$

where

$$y_p(x) = \frac{1}{\mu(x)}\int^x \mu(s)g(s)ds \tag{5}$$

is a (i.e. any) **particular solution** of (1) (selected by the choice of the integration constant),

$$y_c(x) = c\,\frac{1}{\mu(x)} = c\, e^{-\int^x p(s)ds} = c\, y_1(x) \tag{6}$$

is the **general solution** (i.e. family of all solutions) of the associated **homogeneou**s (the use of the word homogeneous is different here from its use in Chapter 1-3) or **complementary equation**

$$y' + p(x)\, y = 0. \tag{7}$$

and

$$y_1(x) = \frac{1}{\mu(x)} = e^{-\int^x p(s)ds} \tag{8}$$

(You will learn later that B = $\{y_1\}$ where $y_1(x) = 1/\mu(x)$ is a basis of the null space of the linear operator $L[y] = y' + p(x)\,y$ where L maps the vector space $C^1(I)$ into the vector space $C(I)$.)

Note that Equations (5) and (6) can be considered as "formulas" for y_p and y_c. **Do not use these**. Learn to solve first order linear equations by using the integrating factor. Attempts to use these "formulas" will receive little or no part credit if precisely the correct answer is not obtained. **Learn the process**. No credit will be given for using even a slightly incorrect formula.

DEFINITION. A function $f{:}I\to\mathbf{R}$ is **analytic** at $x_0\in I = (a,b)$ if there exists a $\delta>0$ such that its Taylor series converges to f in $(x_0-\delta, x_0+\delta)$. We say that f is **analytic** on I if it is analytic at each point in I. (Recall that f is continuous on I if it is continuous at every point in I.) Let $A(I)$ denote the set of all functions that are analytic on I = (a,b).

THEOREM #2. Suppose I=(a,b) is an open interval. Then $A(I)\subseteq C^1(I)\subseteq C(I)$.

That is, analytic functions are "nicer" than functions with continuous derivatives which in turn are "nicer" than continuous functions. The fact that we have parametric formulas for all solutions in terms of the integrals of p(x) and $\mu(x)\,g(x)$ gives us the following **regularity result**: If $p,g\in A(I)$, then the solutions to (1) are not only in $C^1(I)$, but that they are analytic on I.

THEOREM #3. Suppose that $p,g\in A(I)$ where $I=(\alpha,\beta)$. Then the results of Theorem#1 are still true and in fact all solutions are in $A(I)$.

We turn now to the IVP:

THEOREM #4. (Existence and Uniqueness of the Solution to the IVP for a First Order Linear ODE) If $x_0 \in I = (\alpha,\beta)$ and $p,g \in C(I)$ (i.e. the functions p and q in (1) are continuous on the open interval (α,β)), then $\exists!\ \varphi(x)$ (i.e. there exists a unique function $y=\varphi(x)$) that satisfies (1) and (2) (i.e. the **initial value problem** (IVP) consisting of the ODE (1) and the IC (2) where $y_0 \in \mathbf{R}$ is an arbitrarily prescribed value of the function at x_0). The solution is given by:

$$y(x) = y_p(x) + y_c(x) = y_p(x) + y_0\, y_1(x) \tag{8}$$

where

$$y_p(x) = \frac{1}{\mu(x)}\int_{x_0}^{x} \mu(s)g(s)\,ds \tag{9}$$

$$y_1 = \frac{1}{\mu(x)} = e^{-\int_{x_0}^{x} p(s)\,ds} \tag{10}$$

$$\mu(x) = e^{\int_{x_0}^{x} p(s)\,ds} \tag{11}$$

If $p,g\in A(I)$, then the solution is likewise in $A(I)$.

COMMENTS. The **discontinuities** in p(x) and g(x) (may but do not have to) produce discontinuities in the solutions to Equation (1). This will determine the **interval of validity** of the solution (i.e., the largest open interval I on which the solution is valid). The theory guarantees that if p(x) and g(x) are at least continuous on I = (α,β) then the interval of validity of the solution (i.e. the domain of the function which is the solution) will at least contain I. For linear problems, the interval of validity for the entire family of functions is the same (i.e. independent of the integration constant). However, for nonlinear problems, the interval of validity may very well depend on the constant in the formula.

The concepts of derivative and analytic function can be extended to the complex plane. In this setting, the open interval I is replaced by a **domain** D (an open connected set). Although D is the domain of a complex function of a complex variable, the use of the word domain in complex analysis (and in multi variate calculus) is more restrictive then its general use as the set of things that get mapped by a function. D must be an open connected set. (The terms **open** and **connected** have technical definitions in complex analysis (and indeed in general topology), but these coincide with their general "dictionary" definitions.) The nice thing is that all of the calculus formulas you know for elementary functions extend to complex variables. Often, in complex analysis, the word analytic is replaced bt the word **holomorphic** and x is replaced by z and y by w so that w = f(z). The holomorphic functions on D are dented by *H*(D). If p,g∈*H*(D), then all of the solutions to (1) are given by (3) and are in *H*(D). When p(z) and g(z) are elementary functions, they may have **isolated singularities** where they become infinite (e.g., p(z) = sec z or when the denominator becomes zero, p(z)=1/z). These produce isolated singularities in the solutions.

Read Sections 2.4 and 2.8 of Chapter 2 of text (Elem. Diff. Eqs. and BVPs by Boyce and Diprima, seventh ed.). Pay particular attention to the existence and uniqueness theorem on page 106. Try to understand the concepts of general solution and implicit and explicit solutions. How is the linear and nonlinear case different?

For the general (possibly nonlinear) first order ODE we consider the IVP

$$\text{IVP}\quad \begin{cases} \text{ODE} & y' = f(x,y) \\ \text{IC} & y(x_o) = y_o \end{cases} \qquad (1)$$

(i.e. only those where we can solve for y' explicitly). It is not always easy to obtain the **general solution** of the ODE as a family of functions in the form $y = f(x;c)$ (e.g. $y = y_p(x) + c\, y_1(x)$ for the linear case) with one parameter. Instead we often obtain a family of curves in the form $g(x,y;c) = 0$ (e.g., $g(x,y) = c$) where a section of the **curve** which is not vertical will provide a solution over some **interval of validity**. A particular curve can be selected by requiring the initial condition. Under certain conditions the existence and uniqueness of the solution to the IVP can be asserted (see the text). The "interval of validity" for a nonlinear ODE (i.e., the domain of the solution function) is much more complicated than it is for a linear ODE and is usually found for each problem separately after a solution curve has been found.

IMPLICIT SOLUTIONS FOR NONLINEAR EQUATIONS

EXAMPLE #1. The functions defined by the curves (hyperbolas)

$$y^2 - x^2 + cx = 0 \qquad (\text{or } y^2 = x^2 - cx \quad \text{or} \quad -(y^2 - x^2)/x = c\ ) \qquad (2)$$

satisfy the ODE

$$2xy\, y' = x^2 + y^2 \qquad (\text{or} \quad y' = (x^2 + y^2)/(2xy)\ ). \qquad (3)$$

The implicit description of a family of curves given by (2) is usually preferable to the explicit description

$$y = \pm\sqrt{x^2 - cx} \qquad \text{for } x^2 - cx \geq 0. \qquad (4)$$

The domain for these functions is

$$D = \{x \in \mathbf{R} : x^2 - cx \geq 0\} = \{x \in \mathbf{R} : x(x - c) \geq 0\}$$
$$= \{x \in \mathbf{R} : (x \geq 0 \text{ and } x \geq c) \text{ or } (x \leq 0 \text{ and } x \leq c)\}.$$

To check that $g(x,y) = 0$ provides solutions for a nonlinear problem is more complicated. **Implicit differentiation** of (3) yields

$$2y\,y' - 2x + c = 0 \qquad \text{(so that } 2yy' = 2x - c). \tag{5}$$

Hence

$$2xyy' = x(2x - c) = 2x^2 - cx = x^2 + (x^2 - cx) = x^2 + y^2. \tag{6}$$

Hence all of the curves $y^2 - x^2 + cx = 0$ satisfy the ODE $2xy\,y' = x^2 + y^2$ at points on the curves where y' exists.