A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND
NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED
MATHEMATICIANS

DE CLASS NOTES 1

A COLLECTION OF HANDOUTS ON
FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

CHAPTER 1

Introduction To Differential Equations

and Mathematical Modeling,

and a Technique for Solving

First Order Linear ODE's

1. Introduction to Differential Equations and Mathematical Modeling

2. Useful Characteristics of ODE's

3. Technique for Solving First Order Linear ODE's Using an Integrating Factor

SR 1. Brief Historical Remarks on Differential Equations

Handout #1   INTRODUCTION TO DIFFERENTIAL EQUATIONS   Professor Moseley
AND MATHEMATICAL MODELING

Our objective is to not only to learn how to solve differential equations, but to also underline **why we use the methods we choose**. That is, we not only want to learn methods and be able to apply them to problems where we are told they work, but also to know what methods apply to what problems and to underline the theory behind the methods. We are motivated by the fact that differential equations are used as mathematical models of scientific and other phenomena, particularly systems that change with time and space. To underline differential equations, we begin by asking some fundamental questions:

1)   What is a **differential equation**?
2)   How are **differential equations** different from **algebraic equations**?
3)   Where do **differential equations** come from?
4)   What do we mean by a **solution** to a **differential equation**?
5)   Can a **differential equation** have more than one **solution**?
6)   How does one obtain **solutions** to **differential equations**?

To answer these questions (and many others) completely will take time, particularly for the last one. However, a partial answer to most of them can be obtained by considering a simple example of an **applied math problem** with which you are familiar. Applied math problems are driven by a desire to get answers to specific questions in a given application area. For example:

QUESTION: If a ball is thrown up with an initial velocity of 30 ft per second, when will it come down? What will its velocity be when it comes down?

We use a five step procedure to solve any **applied math** or **application** problem:

Steps in Solving an Applied Math Problem

Step 1. **Understand** the required concepts from the **application area** where solution to a mathematical model will provide answers to the questions of interest.
Step 2. **Understand** the required concepts from **mathematics** that are needed to develop and solve the mathematical model.
Step 3. Use your mathematical knowledge and your knowledge of the application area to **develop a mathematical model** at an appropriate level of generality.
Step 4. Use your mathematical knowledge to **solve the mathematical model**.
Step 5. **Interpret your results**. This includes applying the specific data that originally motivated the development of the model and **answering the specific questions originally asked.**

To answer the specific questions originally asked requires all five steps, but to (partially) answer our questions about ODE's requires only the first three. Answers to the application questions asked must wait until Chapter (1-)5.

APPLICATION   **ONE DIMENSIONAL MOTION OF A POINT MASS**
Application Areas include Physics and Mechanical Engineering.

Step 1: UNDERSTAND THE CONCEPTS IN THE APPLICATION AREA. This step begins
with a description of the phenomenon to be modeled, including the **concepts** involved and the
"laws" it must follow. For one-dimensional motion we need the following concepts:
1) **Time**, 2) **Point mass** or particle, 3) **Position, Velocity,** and **Acceleration** in one dimensional
space, and 4) **Force** in one dimensional space. Now consider the following physical laws for the
motion of a point particle.

PHYSICAL LAW#1. (Newton's Second Law of Motion) At any given time, the net **force**
(magnitude and direction) on a **particle** (**point mass**) is equal to its **mass** times its **acceleration**.
(**F = MA**).

PHYSICAL LAW#2. (Force of Gravity) The magnitude of the force of gravity on a point mass
equals its mass times a constant (which we denote by g). Its direction is towards the center of the
earth.

It is assumed that you have been exposed to these concepts, these laws, and indeed, to the model
we will develop. Physical laws are referred to as **theoretical** if they are simply assumed as a
foundation for a general theory, much as axioms in mathematics are assumed (without proof).
They are referred to as **empirical** if they are found to describe particular experimental data
(without worrying about how they might fit into or be derived from any general theory). Both of
these laws could be described as either theoretical or empirical. However, since Newtonian
Mechanics can be used to predict virtually all macroscopic experimental results on earth, it is
thought of as theoretical and is used as a starting point for deriving many physical models in
science and engineering. However, it is just an approximation of the more general (and more
complicated) relativity theory which also works at speeds close to the speed of light. Similarly,
the (earthly) gravitational law for "small" objects on the surface of the earth approximates the
General Gravitation Law which can be applied to describe the motion of planets.

Step 2: UNDERSTAND THE REQUIRED CONCEPTS IN MATHEMATICS. The required
mathematics for all of the mathematical models developed in this course is as follows:

1. **High School Algebra**, including the solution of **algebraic equations** and inequalities,
2. **Calculus** including computation of **derivatives**, **antiderivatives**, and **definite integrals**,
3. **Solution techniques for differential equations** developed in this course, and
4. Concepts from **Linear Algebra** including solution of a **system of linear algebraic equations**.
   (Before we use linear theory, we will review it in Part 2 of these notes.)

Step 3. DEVELOP THE MATHEMATICAL MODEL. To develop a model for the one
dimensional motion of a ball, we make three assumptions

ASSUMPTION #1: The ball can be modeled as a **particle** or **point mass**. (We are only interested in the longitudinal motion and not the rotational motion of the ball. This helps simplify the model.)

ASSUMPTION #2: The ball is constrained to move in one direction, namely up and down. For this problem, we assume up is positive. The particle has one degree of physical motion. However, as we shall see later, there are **two state variables** so that the model actually has **two degrees of freedom**. (We learn later that the state space for the model is two dimensional.)

ASSUMPTION #3: **Air resistence** is negligible and we assume that the only force acting is **gravity**.

Next we **list** the **variables** and **parameters** used in our model (we may add to the list as the model is developed) and **draw a sketch** to help visualize the process.

Nomenclature:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x} = \text{ the acceleration of the particle.}$$

$$v = \frac{dx}{dt} = \dot{x} = \text{ the velocity of the particle (state variable \#2)}$$

x = the position of the particle (state variable #1)

t = time.

m = the mass of the particle.

F = the force of the particle (in the x direction).

g = acceleration due to gravity

F_g = the force of gravity

$\uparrow$ mass

$|$ $\bigcirc$ $|$ $F_g = -mg$

$|$ $\downarrow$

$|$

$|\_\_\_$ground level

$x = 0$

We set up a **coordinate system** where the positive direction for x is up with $x = 0$ being at ground level. Since the only **force** acting on the particle is the force due to **gravity** then

$$F = F_g = -mg. \tag{1}$$

The minus sign is required since we have selected up as the positive x direction and gravity acts in the downward direction (force is a vector quantity even though we are considering only one dimensional motion). Hence, if the **velocity** is positive, this means that the particle is moving up. If the **velocity** is negative, the particle is traveling down. If we had selected down as the positive direction, then the model would be different mathematically by a minus sign. Then
Step 4: SOLUTION OF THE MATHEMATICAL MODEL would yield different mathematical formulas for the position and velocity of the particle. However, the particle behaves the same for both models. The mathematical descriptions are just different. That is,
Step 5: INTERPRETATION OF THE MATHEMATICAL RESULTS will yield the same physical results, even though different mathematical **functions** would be used to describe the

physical behavior. Here g is the **acceleration due to gravity** with magnitude, 32 ft/sec² in English units. However, g is easier to write and we leave open the option to use metric units.

To develop the mathematical model we use the <u>physical laws</u> we have cited and the notation we have developed. Putting $F = ma$ or $m\,a = F$ and $F = F_g = -mg$ together we obtain

$$m\,\ddot{x} = -m\,g \quad \text{or} \quad \ddot{x} = -g. \tag{2}$$

With more or different forces acting, the **Ordinary Differential Equation** (ODE, since it involves only derivatives with respect to one **independent variable**, namely time t) is given by

$$m\,\ddot{x} = F(t, x, \dot{x}) = \text{sum of all forces acting on the particle} \tag{3}$$

which can be much more complicated and more difficult to solve. (Can you see why?) In order to uniquely determine the path of the particle and complete the <u>mathematical model</u>, we must also specify the particle's **initial position**, say $x(0) = x_0$, and **initial velocity**, say $\dot{x}(0) = v_0$. (We need initial values for both of our state variables.) Hence our **mathematical model** for the particle acted on only by gravity is given by the **Initial Value Problem** (IVP):

IVP
$$\text{ODE} \qquad \ddot{x} = -g \tag{4}$$
$$\text{IC's} \qquad x(0) = x_0, \qquad \dot{x}(0) = v_0. \tag{5}$$

The usual strategy for solving an IVP is to first find all solutions of the ODE. There will be an infinite number of solutions involving one or more arbitrary (integration) constants. We then apply the IC's to evaluate these constants. You already know how to solve this ODE, simply integrate twice. However, at this point we are not interested in learning how to solve differential equations. That will come soon enough. Currently we are interested in

① establishing an interest in studying differential equations,
② learning that a **solution** to an ODE is a **function** (and not a number), and
③ learning how to determine if a given <u>function</u> is or is not a <u>solution</u> to a particular ODE (e.g.,the one developed as part of this model).

<u>SOLUTIONS TO THE ODE</u> Some solutions to the ODE (4) are given by the following:

$$x_1 = -\tfrac{1}{2}\,gt^2 \tag{6}$$
$$x_2 = -\tfrac{1}{2}\,gt^2 + t + 3 \tag{7}$$
$$x_3 = -\tfrac{1}{2}\,gt^2 + c_1\,t + c_2 \tag{8}$$

Actually, x_3 is not just one <u>solution</u> (function) but a **parametric representation** of a whole **family of solutions** (functions) since c_1 and c_2 are arbitrary constants.

EXISTENCE PROBLEM: CHECKING THAT A GIVEN FUNCTION IS A SOLUTION. How do we know that the functions given by (6) and (7) and each function in the family of functions given by (8) are solutions? Just as we can check that $x = -2$ is a solution of $x^2 = 4$ by simply computing x^2, we can show that x_1 satisfies (4) by computing its second derivative:

$$x_1 = -\tfrac{1}{2}\, g\, t^2, \qquad \dot{x}_1 = -g\, t, \qquad \ddot{x}_1 = -g$$

Similarly you can compute the second derivatives of x_2 and x_3 to show that they are solutions to (4). Note that even though x_3, because of the parameters c_1 and c_2, represents an **infinite number of functions**, we can use theorems from calculus to compute the first and second derivatives of the whole (infinite) **family of functions** at the same time. Hence we can check that each function in this infinite family of functions is a solution. This computation proves the following **existence theorem**:

THEOREM (EXISTENCE). Each member of the family of functions given by $x = -\tfrac{1}{2} g t^2 + c_1 t + c_2$ is a solution to the ODE $\ddot{x} = -g$

UNIQUENESS PROBLEM: Not only is each member of the whole **family of functions** given by (8) a solution of the ODE (4), but the **converse** is true. It is easily shown by integrating twice (using theorems from calculus) that any solution of the ODE (4) must be a member of the family of functions given by x_3 in Equation (8). Thus the solution process solves the uniqueness problem (i.e., limits the set where all solutions must reside).

THEOREM #2. If x is a solution of $\ddot{x} = -g$, then x is in the family of functions given by $x = -\tfrac{1}{2} g t^2 + c_1 t + c_2$.

GENERAL SOLUTION: Hence x_3 given in Equation (8) is the **general solution** of the ODE given in Equation (4) (i.e., gives all of the **infinite number** of solutions and only solutions).

CLASS EXERCISE: What are the values of c_1 and c_2 which give x_1 and x_2?

The constants c_1 and c_2 are simply the constants of integration. If the force is simply a function of time (e.g. a constant) then this is always possible.

However, the force is often a function not only of time t but also of distance (space) x and velocity $v = \dot{x}$. This makes the solution process more difficult. Can you explain why? Suppose, instead of (4) we wish to solve $m\ddot{x} = -c\,\dot{x} - kx + g(t)$ which is the equation for the displacement of a point mass hanging on a spring from its equilibrium position where there is damping (see Chapter 3-4). This equation is **linear** (we will explain what linear means shortly) and is usually written as

$$m\ddot{x} + c\dot{x} + kx = g(t).$$

Similar to checking that a <u>number</u> is a solution of an <u>algebraic equation</u> (e.g., that x=2 is a solution to $x^2 + 3x - 4 = 6$), we can check that a <u>function</u> is a <u>solution</u> to an ODE by **substituting** it into the ODE. Suppose our spring problem is

$$\ddot{x} + 5\dot{x} + 6x = 5\cos t - 5\sin t = 5\sqrt{2}\,\cos(t - \pi/4) \tag{9}$$

Since our ODE is **linear**, an easy technique for substituting into the ODE is illustrated below:

<u>CHECK</u> that each and every member of the <u>family of functions</u> given by

$$x = \sin t + c_1\,e^{-2t} + c_2\,e^{-3t} \tag{10}$$

is a solution of the ODE (9):

<u>Coefficient,</u>   <u>Function or derivative</u>
      6) $x = \sin t + c_1\,e^{-2t} + c_2\,e^{-3t}$
      5) $\dot{x} = \cos t - 2\,c_1\,e^{-2t} - 3\,c_2\,e^{-3t}$
      1) $\ddot{x} = -\sin t + 4\,c_1\,e^{-2t} + 9\,c_2\,e^{-3t}$
_____

$\ddot{x} + 5\dot{x} + 6x = (-1 + 6)\sin t + 5\cos t + (4 - 10 + 6)\,c_1 + (9 - 15 + 6)\,c_2 = 5\cos t - 5\sin t$

so each member of the family of functions given by x in is a solution to the ODE (9). Note again that even though x, because of the parameters c_1 and c_2, represents an **infinite number of functions**, we can use theorems from calculus to compute the first and second derivatives of the whole **family of functions** at the same time. Hence we can check that a whole **family of functions** are solutions. This computation proves the following existence theorem:

<u>THEOREM #3</u>. Each member of the family of functions (10) is a solution to the ODE (9)

<u>GENERAL SOLUTION OF AN ODE</u> Although it is not easy to prove, we will eventually learn that not only is each member of the whole **family of functions** given by (9) a solution of the ODE (10), but the **converse** is true. That is, it can be shown that **all solutions** of (9) are members of the family of solutions given by (10). Hence x given by (10) is in fact the **general solution** of (9) (i.e., it gives all solutions of (9)).

<u>WHAT KINDS OF THINGS ARE SOLUTIONS TO ODE'S</u>. It is important to emphasize that the "unknown" that we solve for when solving an <u>ordinary differential equation</u> (ODE) is a **function** (or **family of functions**) instead of a <u>number</u> (or <u>set of numbers</u>) as when solving elementary <u>algebraic equations</u>. Just as it is important to know the set of numbers where solutions of an algebraic equation are to be found (e.g., in **R** or **C**), it is important to know the set of functions where solutions of a differential equation are to be found. Since our examples involve the second derivative, solutions must be functions with at least two derivatives and we

look for solutions in the set of functions denoted by $C^2(\mathbf{R})$. (See Chapter 3 of the Remedial Class Notes for notation for sets of functions.)

NOTATION FOR FUNCTIONS. Since we begin by studying solution techniques for differential equations and consider applications later, we begin by using the notation standard for variables and functions used in algebra and calculus and let $y = f(x)$. However, unless it is essential to distinguish between the function f as a rule to compute the dependent variable y we usually make no such distinction and use the "engineering" notation $y = y(x)$. Hence for derivatives we use:

$$y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2y}{dx^2}, \qquad y''' = \frac{d^3y}{dx^3}.$$

Derivatives of order higher than 3 may be indicated in three ways. One is by Roman numerals (both upper and lower case are used). The second method is by integers in parentheses (in order to distinguish them from exponents). A third way is to use Roman numerals in parentheses. Although redundant, this does insure that a distinction between powers and derivatives and is acceptable. Thus

$$y^{iv} = y^{IV} = y^{(4)} = y^{(iv)} = y^{(IV)}$$

all indicate the fourth derivative. Since for applications the independent variable is often time, dynamical systems people like to use t for the independent variable. Hence we will also use

$$y' = \dot{y} = \frac{dy}{dt}, \qquad y'' = \ddot{y} = \frac{d^2y}{dt^2}, \qquad y''' = \dddot{y} = \frac{d^3y}{dt^3}.$$

The context should tell you the meaning of y', y'', and y'''. (If x or t is not specified in problems, you may use either. However, if it is specified, you **must** use the one specified. Points will be deducted on **every problem** on exams where you have used the wrong independent variable. Learn to **proof read** your work. As engineers and scientists, this is very important.)

EXAMPLES. Other examples of ordinary differential equations (ODE's) are:
  1) $y' + x^2 y = 0$
  2) $(y')^2 + x\,y' + 4 = 0$
  3) $y^{IV} + \sin x\ y' + y = \ln x$
  4) $y'' + p(x)y' + q(x)y = g(x)$ where p, q, and g are continuous functions of the independent variable x in some open interval $I = (\alpha, \beta)$.

SCANTRON EXERCISES on the reading assignment for Introduction to Differential Equations

EXERCISE #1. True of False. History of Differential Equations.

_____1. Isaac Newton (1642-1727) was one of the "inventors" of calculus.

_____2. Gottfried Wilhelm Leibniz (1646-1716) was one of the "inventors" of calculus.

_____3. The brothers Jakob (1654-1705) and Johann (1667-1748) did much to develop methods
          of solving differential equations.

_____4. Leonhard Euler (1741-1766) identified the condition for exactness of first order ODE's.

_____5. Joseph-Louis Lagrange (1736-1813) showed that the general solution of an nth linear
          homogeneous ordinary differential equation is a linear combination of n linearly
          independent solutions.

Ordinary Differential Equations (ODE's) have certain characteristics. These characteristics often determine the method of solution. We classify ODE's using these characteristics. It is important that you learn what these characteristics are and how to recognize them.

ORDER. The **order** of an ODE is the order of the **highest derivative** that appears in the equation. Recall that derivatives higher than 3 may be indicated in different ways. Thus $y^{iv} = y^{IV} = y^{(4)} = y^{(iv)} = y^{(IV)}$ all indicate the fourth derivative.

LINEARITY. The **general** n^{th} order ODE of the form

$$F(x,y,y',...,y^{(n)}) = g(x) \tag{1}$$

is said to be **linear in y** if F is a linear function of the "variables" $y, y',...,y^{(n)}$. Hence the **general n^{th} order linear equation** is given by

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)}+...+a_1(x)\, y'+a_0(x)\, y = g(x). \tag{2}$$

For example, the **general form for a first order linear ODE** (after dividing by $a_1(x)$)is

$$y' + p(x)\, y = g(x). \tag{3}$$

where $p(x) = a_0(x)/a_1(x)$ and $g(x)$ is redefined to be $g(x)/a_1(x)$. The **general form for a second order linear equation** can be written as

$$y'' + p(x)\, y' + q(x)\, y = g(x). \tag{4}$$

CLASS EXERCISE. Determine the **order** of the following ODE's.
1. $y' + 3x^2\, y = 15$ 4. $(y''')^2 + y^2 = xe^x$
2. $(y')^2 + 3(\sin x)\, y = x^2$ 5. $y^{(4)} + 3x\, y = 17$
3. $y'' + 3y^3 = x^3$ 6. $y'' + 3y = 3e^x$

CLASS EXERCISE. Determine which of the ODE's above are **linear**.

ODE's-I-2   TECHNIQUE FOR SOLVING FIRST ORDER LINEAR

The **general** first order ODE is of the form

$$F(x,y,y') = g(x) \tag{1}$$

We only consider first order ODE's that can be put into the **standard general form** given by

$$y' = f(x,y) \tag{2}$$

where we have solved for the first derivative. To get you started with learning solution processes for first order ODE's, we begin with first order linear:

$$y' = -p(x)\, y + g(x) \tag{3}$$

where

$$f(x,y) = -p(x)\, y + g(x). \tag{4}$$

The solution strategy is similar to solving algebraic equations. But for an ODE we wish to isolate the unknown function(s) on one side of the equation. Then the solution function (actually a family of an infinite number of functions since we will have an arbitrary parameter obtained as an integration constant) will be on the other side of the equation. To do this, we put (3) in the **standard general form** for a **first order linear** ODE in y given by

$$y' + p(x)\, y = g(x). \tag{5}$$

For reasons that will become apparent when we consider systems of first order ODE's, some texts use the alternate general form

$$y' = a(x)\, y + g(x) \tag{6}$$

so that there is a sign change as well as a notational change from what we use here.

Note that the form of Equation (5) prescribes the way f(x,y) will be a function of y but not x. That is, p(x) and g(x) are <u>arbitrary functions</u> of x (except for continuity constraints). Hence, recognizing a first order linear ODE in y means recognizing how f(x,y) is a function of y. (We can also consider first order linear ODE's in x where x is a function of y.) The reason we put p(x) y on the left hand side with y' will become clear from our solution technique which will isolate the unknown function on the left side of the equation and from our later study of <u>mapping problems for linear operators on vector spaces</u>.

<u>SOLUTION TECHNIQUE</u>. To find the **general solution** of a first order linear ODE in y, we follow the technique used to solve algebraic equations. That is, we use equivalent equation operations (EEO's) to obtain a sequence of equivalent equations (all having the same solution) and eventually isolate the unknown function on the left side of the equation. We first put the

equation in the standard form and identify the function p(x). Next we find <u>an</u> **Integrating Factor** (IF) μ, <u>all of which</u> are defined by the formula

$$IF = \mu(x) = \exp(\int^x p(t)\, dt) = e^{\int p(x)\, dx} \tag{7}$$

That is, first find the **family of antiderivatives** of p(x). (Both notations for the **general antiderivative** of a function will be used.) Since we need only <u>one integrating factor</u> and not the <u>entire family,</u> <u>only one antiderivative is needed</u>. Hence we usually take the integration constant c to be zero (c = 0). We then apply the exponential function to this function, that is, we use the antiderivative selected as the exponent of e. Next we multiply both sides of the ODE by the integrating factor. (This is an EEO.) To see how we may then isolate the unknown (family of) function(s), we consider a nine step process for obtaining and checking the general solution for some examples.

EXAMPLE 1) $y' + 2y = e^{-x}$ (ODE) $\tag{8}$
                   $y(0) = 3$ (IC) $\tag{9}$

<u>**SOLUTION**</u> (Giving a step by step procedure for solving):

<u>Step 1</u>:   **Identify p(x)**: "Clearly" $p(x) = 2$ (a(x) = -2).

<u>Step 2</u>:   **Compute the integrating factor μ:** $\int p(x)\, dx = \int 2\, dx = 2x + c$
                 Choosing c = 0   so that   IF = $\mu = e^{2x}$.

<u>Step 3</u>:   **Multiply both sides of the ODE by the integrating factor IF = μ:**
                 $e^{2x}(y' + 2y) = e^{2x}(e^{-x})$

                 $y'e^{2x} + y(2e^{2x}) = e^{x}$ The left hand side (LHS) will always be $\dfrac{d(y\mu)}{dx}$.

<u>Step 4</u>:   **Replace the left hand side of the ODE with this term:**
                 $$\frac{d(y\, e^{2x})}{dx} = e^{x}$$

<u>Step 5</u>:   **Check that your integrating factor is correct**: You should always check this
                 out. It will help you to locate computation errors. You will receive no part credit
                 on exams if your answer is wrong and you have failed to check your integrating
                 factor. (Also, checking your integrating factor avoids confusion if the alternate
                 form y' = a(x) y + g(x) is used.) Using the <u>product rule</u> we obtain:

                 $$\frac{d(y\, e^{2x})}{dx} = y'\, e^{2x} + y\, (2\, e^{2x})$$

                 Compare this to the left hand side (LHS) of the ODE to see if they are the same.

If not, recheck your computation. Specifically check to see if you have identified $p(x)$ correctly. Since we see that it is correct, we return to the ODE:

$$\frac{d(y\ e^{2x})}{dx} = e^x$$

Step 6:   **Integrate both sides of the equation**:

$$y\,e^{2x} = \int e^x\,dx = e^x + c$$

Note that we have used the **fundamental theorem of calculus** to obtain the left hand side (LHS).

Step 7:   **Solve for y**: Hence $y = (e^x + c)e^{-2x} = e^{-x} + ce^{-2x} = ce^{-2x} + e^{-x}$.
is the general solution of the ODE (8). Note that the process of isolationg the unknown function(s) on the LHS proves that all solutions are of the form $y = ce^{-2x} + e^{-x}$.

Since all steps are reversible (i.e.they are EEO's) we have that all functions of the form
$y = ce^{-2x} + e^{-x}$ are solutions. It is useful to check that all solutions of the form $y = ce^{-2x} + e^{-x}$ are solutions.

Step 8:   **Check that these functions are all solution:** We can check directly that functions of the
form $y = ce^{-2x} + e^{-x}$ are solutions by using the previously described technique to substitute back into the linear ODE.
2) $y = ce^{-2x} + e^{-x}$
1) $y' = -2ce^{-2x} - e^{-x}$
_____
$y' + 2y = 2e^{-x} - e^{-x} = e^{-x}$

Step 9:   **Apply the Initial Conditions (IC's)** If an initial condition is given so that we have an initial value problem (IVP), we apply it to the general solution to obtain a value for the constant c

$$y(0) = y\big|_{x=0} = 3 \ .$$

Substituting $y = 3$ and $x = 0$ into (10) we obtain

$$3 = c\,e^{-2(0)} + e^{-0} \ \Rightarrow\ 3 = c + 1 \ \Rightarrow\ c = 2.$$

Hence the unique solution to the initial value problem (IVP) posed by (7) and (8) is

$$y = 2e^{-2x} + e^{-x}.$$ (10)

EXAMPLE 2. $y' = 2xy + x$ ODE (11)
$\qquad\qquad\qquad y(0) = 1$ IC (12)

SOLUTION (This time we number the steps without identifying what they are):

Step 1: We must first put the ODE in the standard form: $y' - 2xy = x$
$\qquad$ "Clearly" $p(x) = -2x$ $(a(x) = 2x)$.

Step 2 $\int p(x)\,dx = \int -2x\,dx = -x^2 + c,$ $\qquad$ Choosing $c = 0 \Rightarrow \mu =$

Step 3: $e^{-x^2}(y' - 2xy) = e^{-x^2}x$ or $y'e^{-x^2} + ye^{-x^2}(-2x) = e^{-x^2}x$

Step 4: $\Rightarrow \dfrac{d}{dx}\left(ye^{-x^2}\right) = e^{-x^2}x$

Step 5: Note $\dfrac{d}{dx}\left(ye^{-x^2}\right) = y'e^{-x^2} + ye^{-x^2}(-2x)$ ✔

Step 6: Returning to the problem,

$$ye^{-x^2} = \int e^{-x^2}x\,dx = -\frac{1}{2}\int e^{-x^2}(-2x)\,dx = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + c = -\frac{1}{2}e^{-x^2} + c$$

Step 7: $y = -\dfrac{1}{2} + ce^{x^2}$ or $y = ce^{x^2} - \dfrac{1}{2}$ (14)

Step 8: Checking $\qquad\qquad -2x)$ $y = ce^{x^2} - \dfrac{1}{2}$

$\qquad\qquad\qquad\qquad\qquad 1)$ $y' = ce^{x^2}(2x)$

$\qquad\qquad\qquad\qquad$ _____

$\qquad\qquad y' - 2xy = ce^{x^2}(2x) - 2x(ce^{x^2} - \dfrac{1}{2}) = x$ ✔

Step 9: $\qquad$ Since $\quad$ we substitute $y = 1$ and $x = 0$ into (14) to obtain

$\qquad\qquad 1 = ce^0 - (1/2)$ so that $c = 3/2$ and hence $y = \dfrac{3}{2}e^{x^2} - \dfrac{1}{2}$

What happens if $g(x) = 1$ instead of x ?

WRITTEN EXERCISES on Technique for Solving First Order Linear ODE's Using an Integrating Factor

1. Solve a) $y' + 2y = 3$ b) $2y' + y = x$ c) $y' - 2x\,y = 1$ d) $y' + (1/x)y = 3 \cos 2x$

2. Solve a) $2y' + y = 3x$ b) $xy' + 2y = \sin x$ c) $(1 + x^2)y' + 4x\,y = (1 + x^2)^{-2}$