A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

LINEAR CLASS NOTES: A COLLECTION OF HANDOUTS FOR REVIEW AND PREVIEW OF LINEAR THEORY INCLUDING FUNDAMENTALS OF LINEAR ALGEBRA

CHAPTER 9

More on

Matrix Inverses

1. Re-introduction to Matrix Inverses

2. Computation Using Gauss Elimination

4. Formula for a 2x2 Matrix

Recall that if A and B are square, then we can compute both AB and BA. Unfortunately, these may not be the same.

THEOREM #1. If n >1, then there exists A, B$\in\mathbf{R}^{n\times n}$ such that AB $\neq$ BA. Thus matrix multiplication is **not commutative**.

Thus AB=BA is not an identity. Can you give a **counter example** for n=2? (i.e. an example where AB $\neq$ BA.)

DEFINITION #1. For square matrices, there is a **multiplicative identity element**. We define the n×n matrix I by

$$\underset{mxn}{I} = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

One's down the diagonal. Zero's everywhere else.

THEOREM #2. We have $\underset{nxn}{A}\underset{nxn}{I} = \underset{nxn}{I}\underset{nxn}{A} = \underset{nxn}{A} \quad \forall A \in \mathbf{K}^{nxn}$

DEFINITION #2. If there exists B such that AB = I., then B is a **right (multiplicative) inverse** of A. If there exists C such that CA = I., then C is a **left (multiplicative) inverse** of A. If AB = BA = I, then B is a (**multiplicative) inverse** of A and we say that A is **invertible**. If B is the only matrix with the property that AB = BA = I, then B is the **inverse** of A. If A has a unique inverse, then we say A is **nonsingular** and denote its inverse by A^{-1}.

THEOREM #3. Th identity matrix is its own inverse.

Later we show that if A has a right and a left inverse, then it has a unique inverse. Hence we prove that A is invertible if and only if it is nonsingular. Even later, we show that if A has a right (or left) inverse, then it has a unique inverse. Thus, even though matrix multiplication is not commutative, a right inverse is always a left inverse and is indeed the inverse. Some matrices have inverses; others do not. Unfortunately, it is usually not easy to look at a matrix and determine whether or not it has a (multiplicative) inverse.

THEOREM #4. There exist A, B$\in\mathbf{R}^{n\times n}$ such that A$\neq$I is invertible and B has no inverse.

INVERSE OPERATION. If B has a right and left inverse then it is a unique inverse ((i.e., $\exists\ B^{-1}$ such that $B^{-1}B = BB^{-1} = I$) and we can define **Right Division** AB^{-1} and **Left Division** $B^{-1}A$ of A by B (provided B^{-1} exists). But since matrix multiplication is not commutative, we do not know that these are the same. Hence $\frac{A}{B}$ is not well defined since no indication of whether we mean left or right division is given.

EXERCISES on Re-introduction to Matrix Inverses

EXERCISE #1. True or False.

_____ 1. If A and B are square, then we can compute both AB and BA.
_____ 2. If n >1, then there exists A, $B \in \mathbf{R}^{n\times n}$ such that $AB \neq BA$.
_____ 3. Matrix multiplication is not commutative.
_____ 4. AB=BA is not an identity.
_____ 5. For square matrices, there is a multiplicative identity element, namely the n×n matrix I,

given by $$I_{nxn} = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}.$$

_____ 6. $\forall\ A \in \mathbf{K}^{nxn}$ we have $A_{nxn} I_{nxn} = I_{nxn} A_{nxn} = A_{nxn}$
_____ 7. If there exists B such that AB = I., then B is a right (multiplicative) inverse of A.

_____ 8. If there exists C such that CA = I., then C is a left (multiplicative) inverse of A.
_____ 9. If AB = BA = I, then B is a multiplicative inverse of A and we say that A is invertible.
_____ 10. If B is the only matrix with the property that AB = BA = I, then B is the inverse of A.
_____ 11. If A has a unique inverse, then we say A is nonsingular and denote its inverse by A^{-1}.
_____ 12. The identity matrix is its own inverse.
_____ 13. If A has a right and a left inverse, then it has a unique inverse.
_____ 14. A is invertible if and only if it is nonsingular.
_____ 15. If A has a right (or left) inverse, then it has a unique inverse.
_____ 16. Even though matrix multiplication is not commutative, a right inverse is always a left inverse.
_____ 17. The inverse of a matrix is unique.
_____ 18. Some matrices have inverses; others do not.
_____ 19. It is usually not easy to look at a matrix and determine whether or not it has a (multiplicative) inverse.
_____ 20. There exist A, $B \in \mathbf{R}^{n\times n}$ such that $A \neq I$ is invertible and B has no inverse

EXERCISE #2. Let $\alpha = 2$, $A = \begin{bmatrix} 1+i & 1-i \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ i & 1+i \end{bmatrix}$. Compute the following:

$\overline{A}$ = ___________. A^T =____________. A^* = ____________. αA = ____________.

A+B =____________. AB =_____________.

EXERCISE #3. Let $\alpha = 3$, $A = \begin{bmatrix} i & 1-i \\ 0 & 1+i \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ i & 1+i \end{bmatrix}$. Compute the following:

$\overline{A}$ = ___________. A^T =____________. A^* = ____________. αA = ____________.

A+B =____________. AB =_____________.

EXERCISE #4. Solve $\underset{2x2}{A}\ \underset{2x1}{\vec{x}} = \underset{2x1}{\vec{b}}$ where $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

EXERCISE #5. Solve $\underset{2x2}{A}\ \underset{2x1}{\vec{x}} = \underset{2x1}{\vec{b}}$ where $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

EXERCISE #6 Solve $\underset{2x2}{A}\ \underset{2x1}{\vec{x}} = \underset{2x1}{\vec{b}}$ where $A = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

We give an example of how to compute the inverse of a matrix A using Gauss-Jordan elimination (or Gauss-Jordan reduction). The procedure is to augment A with the identity matrix. Then use Gauss-Jordan to convert A into I. Magically, I is turned into A^{-1}. Obviously, if A is not invertible, this does not work.

<u>EXAMPLE</u>#1. Use Gauss-Jordan reduction to compute A^{-1} where $A=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$.

$$\begin{matrix} \\ R_2+(1/2)R_1 \\ \\ \\ \end{matrix}\left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \begin{matrix} \\ \\ R_3+(2/3)R_2 \\ \\ \end{matrix}\left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \begin{matrix} \\ \\ \\ R_4+(3/4)R_3 \end{matrix}\left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 1/3 & 2/3 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 1/3 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 5/4 & 1/4 & 1/2 & 3/4 & 1 \end{array}\right]$$

We now divide by the pivots to make them all one.

$$\begin{matrix} 1/2R_1 \\ 2/3R_2 \\ 3/4R_3 \\ 5/4R_4 \end{matrix}\left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 1/3 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 5/4 & 1/4 & 1/2 & 3/4 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & -1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & -3/4 & 1/4 & 1/2 & 3/4 & 0 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right]$$

We now make zeros above the pivots.

$$\begin{matrix} \\ \\ R_3+3/4R_4 \\ \\ \end{matrix}\left[\begin{array}{cccc|cccc} 1 & -1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & -3/4 & 1/4 & 1/2 & 3/4 & 0 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right] \Rightarrow \begin{matrix} \\ R2+2/3R_3 \\ \\ \\ \end{matrix}\left[\begin{array}{cccc|cccc} 1 & -1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2/5 & 4/5 & 6/5 & 3/5 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right]$$

$$\Rightarrow \begin{matrix} R_1+1/2R_2 \\ \\ \\ \\ \end{matrix}\left[\begin{array}{cccc|cccc} 1 & -1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 3/5 & 6/5 & 4/5 & 2/5 \\ 0 & 0 & 1 & 0 & 2/5 & 4/5 & 6/5 & 3/5 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & -1/2 & 0 & 0 & 4/5 & 3/5 & 2/5 & 1/5 \\ 0 & 1 & -2/3 & 0 & 3/5 & 6/5 & 4/5 & 2/5 \\ 0 & 0 & 1 & 0 & 2/5 & 4/5 & 6/5 & 3/5 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 4/5 & 3/5 & 2/5 & 1/5 \\ 3/5 & 6/5 & 4/5 & 2/5 \\ 2/5 & 4/5 & 6/5 & 3/5 \\ 1/5 & 2/5 & 3/5 & 4/5 \end{bmatrix}$. We will check.

$$AA^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4/5 & 3/5 & 2/5 & 1/5 \\ 3/5 & 6/5 & 4/5 & 2/5 \\ 2/5 & 4/5 & 6/5 & 3/5 \\ 1/5 & 2/5 & 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence $A^{-1} = \begin{bmatrix} 4/5 & 3/5 & 2/5 & 1/5 \\ 3/5 & 6/5 & 4/5 & 2/5 \\ 2/5 & 4/5 & 6/5 & 3/5 \\ 1/5 & 2/5 & 3/5 & 4/5 \end{bmatrix}$ is indeed the inverse of A.

<u>EXAMPLE</u>#2. For a 2x2 we can do the computation in general. This means that we can obtain a formula for a 2x2. We can do this for a 3x3, but the result is not easy to remember and we are better off just using Gauss-Jordan for the particular matic of interest. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. For the 2x2, we will assume that $a \neq 0$. We leave it to the exercises to show that the formula that we will derive for $a \neq 0$ also works for $a = 0$. Proceeding we first get zeros below the pivots.

$$R_2 - c/aR_1 \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{c}{a}b & -\frac{c}{a} & 1 \end{array}\right] \text{ or } \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad - cb}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

Next we make the pivots all one. We now assume that $\det(A) = ad - bc \neq 0$ so that the matrix is nonsingular.

$$\begin{array}{c} 1/aR_1 \\ a/(ad-bc)R_2 \end{array} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-cb}{a} & -\frac{c}{a} & 1 \end{array}\right] \Rightarrow R_1 - b/aR_2 \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} - \frac{b}{a}\frac{c}{ad-bc} & -\frac{b}{a}\frac{a}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{ad-(bc-bc)}{a(ad-bc)} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{a(ad-bc)} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} \frac{d}{a(ad-bc)} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

EXERCISES on Computation Using Gauss-Jordan Elimination

<u>EXERCISE #1</u>. Using the formula $A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, compute A^{-1} when $A = \begin{bmatrix} 6 & 1 \\ 11 & 2 \end{bmatrix}$

EXERCISE #2. Using the formula $A^{-1} = \; = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, compute A^{-1} where A= $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

EXERCISE #3. Using Gauss-Jordan elimination, compute A^{-1} where A= $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

EXERCISE #4. Using Gauss-Jordan elimination, compute A^{-1} where A= $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

EXERCISE #5. Let A= $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$. Without using the formula $A^{-1} = \; = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ use Gauss-Jordan to show that $A^{-1} = \; = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & 0 \end{bmatrix}$. Thus you have proved that the formula works even when a = 0.

EXERCISE #6. Using the formula $A^{-1} = \; = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, compute A^{-1} where A= $\begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$.

EXERCISE #7. Compute A^{-1} if $A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$

EXERCISE #8. Compute A^{-1} if $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

EXERCISE #9. Compute A^{-1} if $A = \begin{bmatrix} 1 & i \\ i & 2 \end{bmatrix}$

EXERCISE #10. Compute A^{-1} if $A = \begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}$

EXERCISE #11. Compute A^{-1} if $A = \begin{bmatrix} 0 & 5 \\ 0 & -1 \end{bmatrix}$

EXERCISE #12. Compute A^{-1} if $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

EXERCISE #13. Compute A^{-1} if $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

EXERCISE #14. Using Gauss elimination, compute A^{-1} where A= $\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

EXERCISE 15. Using Gauss elimination, compute A^{-1} where A= $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

EXERCISE #16. Using Gauss elimination, compute A^{-1} where A= $\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

EXERCISE #17. Using Gauss elimination, compute A^{-1} where A= $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$