A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

LINEAR CLASS NOTES:
A COLLECTION OF HANDOUTS FOR
REVIEW AND PREVIEW
OF LINEAR THEORY
INCLUDING FUNDAMENTALS OF
LINEAR ALGEBRA

CHAPTER 7

Introduction to

Determinants

Rather than give a fairly complicated definition of the determinant in terms of minors and cofactors, we focus only on two methods for computing the determinant function det:$\mathbf{R}^{n\times n}\to\mathbf{R}$ (or det:$\mathbf{C}^{nxn}\to\mathbf{C}$). Let A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The we define det(A) = ad−bc. Later we will show that $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & -b \\ -c & d \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$. For $A\in\mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$), we develop two methods for computing det(A): Laplace Expansion and Gauss Elimination

But first, we give (without proof) several properties of determinants that aid in their evaluation.

THEOREM. Let $A\in\mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$). Then

1. (ERO's of type 1) If B is obtained from A by exchanging two rows, then det(B) = − det(A).
2. (ERO's of type 2) If B is obtained from A by multiplying a row of A by a≠0, then det(B) = a det(A).
3. (ERO's of type 3) If B is obtained from A by replacing a row of A by itself plus a scalar multiple of another row, then det(B) = det(A).
4. If U is the upper triangular matrix obtained from A by Gauss elimination (forward sweep) using only ERO's of type 3, then det(A) = det(U)
5. If $U\in\mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) is upper triangular, then det(U) is equal to the product of the diagonal elements.
6. If A has a row (or column) of zeros, then det(A) = 0..
7. If A has two rows (or columns) that are equal, then det(A) = 0..
8. If A has one row (column) that is a scalar multiple of another row (column), then det(A) = 0..
9. det(AB) =det(A) det(B)..
10. If det(A) ≠ 0, then $\det(A^{-1}) = 1/\det(A)$.
11. $\det(A^T) = \det(A)$.

EXERCISES on Introduction of Computation of Determinants

EXERCISE #1. True or False.

_____ 1. If A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then det(A) = ad−bc

_____ 2.If A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$.

_____ 3.If $A\in\mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$), there are (at least) two methods for computing det(A).

_____ 4. If $A\in\mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$), Laplace Expansionis one method for computing det(A)

_____ 5. 3.If $A\in\mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$), use of Gauss Elimination is one method for computing det(A).

_____ 6. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and B is obtained from A by exchanging two rows, then det(B) = − det(A).

_____ 7. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and B is obtained from A by multiplying a row of A by $a \neq 0$, then det(B) = a det(A).

_____ 8. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and B is obtained from A by replacing a row of A by itself plus a scalar multiple of another row, then det(B) = det(A).

_____ 9. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and U is the upper triangular matrix obtained from A by Gauss elimination (forward sweep) using only ERO's of type 3, then det(A) = det(U)

_____ 10. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and $U \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) is upper triangular, then det(U) is equal to the product of the diagonal elements.

_____ 11. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and A has a row of zeros, then det(A) = 0.

_____ 12. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and A has a column of zeros, then det(A) = 0.

_____ 13. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and A has two rows that are equal, then det(A) = 0.

_____ 14. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and A has two columns that are equal, then det(A) = 0.

_____ 15. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and A has one row that is a scalar multiple of another row, then det(A) = 0.

_____ 16. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and A has one column that is a scalar multiple of another column, then det(A) = 0.

_____ 17. If $A, B \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) then det(AB) =det(A) det(B)..

_____ 18. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$) and det(A) ≠ 0, then $\det(A^{-1}) = 1/\det(A)$.

_____ 19. If $A \in \mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$), thjen $\det(A^T) = \det(A)$.

EXERCISE #2. Compute det A where $A = \begin{bmatrix} 0 & 5 \\ 0 & -1 \end{bmatrix}$

EXERCISE #3. Compute det A where $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

EXERCISE #4. Compute det A where $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

EXERCISE #5. Compute det A where $A = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}$

EXERCISE #6. Compute A^{-1} if $A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$

EXERCISE #7. Compute A^{-1} if $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

EXERCISE #8. Compute A^{-1} if $A = \begin{bmatrix} 1 & i \\ i & 2 \end{bmatrix}$

EXERCISE #9. Compute A^{-1} if $A = \begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}$

EXERCISE #10. Compute A^{-1} if $A = \begin{bmatrix} 0 & 5 \\ 0 & -1 \end{bmatrix}$

EXERCISE #11. Compute A^{-1} if $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

EXERCISE #12. Compute A^{-1} if $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

We give an example of how to compute a determinant using Laplace expansion..

EXAMPLE. Compute det(A) where $A=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ using Laplace expansion..

Solution: Expanding in terms of the first row we havc

$$\det(A) = \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$+(0)\begin{vmatrix} -1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{vmatrix} - (0)\begin{vmatrix} -1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{vmatrix}$$

so the last two 3x3's are zero. Hence expanding the first remaining 3x3 in terms of the first row and the second interms of the first column we have

$$\det(A) = 2\left[\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} + (0)\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix}\right] - (-1)\left[\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (0)\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} + (0)\begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix}\right]$$

$= 2[(4-1) + (-2)] + [4 - 1] = 2 + 3 = 5$

EXERCISES on Computation Using Laplace Expansion

EXERCISE #1. Using Laplace expansion, compute det A where A= $\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

EXERCISE #2. Using Laplace expansion, compute det A where A= $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}$

EXERCISE #3. Using Laplace expansion, compute det A where A= $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

EXERCISE #4. Using Laplace expansion, compute det A where A= $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$

EXERCISE #5. Using Laplace expansion, compute det A where A= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$

EXERCISE #6. Using Laplace expansion, compute det A where A= $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

We give an example of how to compute a determinant using Gauss elimination.

EXAMPLE. Compute det(A) where $A=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ using Gauss elimination.

Recall

THEOREM. Let $A\in\mathbf{R}^{n\times n}$ (or $\mathbf{C}^{n\times n}$). Then

3. (ERO's of type 3) If B is obtained from A by replacing a row of A by itself plus a scalar multiple of another row, then det(B) = det(A).

4. If U is the upper triangular matrix obtained from A by Gauss elimination (forward sweep) using only ERO's of type 3, then det(A) = det(U).

$$R_2+(1/2)R_1\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \Rightarrow R_3+(2/3)R_2\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow R_3+(3/4)R_2\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

Since only ERO's of Type 3 were used, we have det(A) = det(U) = 2(3/2)(4/3)(5/4) = 5.

EXERCISES on Computation Using Gauss Elimination

EXERCISE #1. Using Gauss elimination, compute det A where $A=\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

EXERCISE #2. Using Gauss elimination, compute det A where $A=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}$

EXERCISE #3. Using Gauss elimination, compute det A where $A=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

Cramer's rule is a method of solving $A\vec{x} = \vec{b}$ when A is square and the determinant of A which we denote by $D \neq 0$. The good news is that we have a formula. The bad news is that, computationally, it is not efficient for large matrices and hence is never used when n > 3. Let A be an nxn matrix and $A = [a_{ij}]$. Let $\vec{x} = [x_1, x_2, ..., x_n]^T$ and $\vec{b}$ be nx1 column vectors and $\vec{x} = [x_i]$ and $\vec{b} = [b_i]$. Now let A_i be the matrix obtained from A by replacing the ith column of a by the column vector $\vec{b}$. Denote the determinant of A_i by D_i. Then $x_i = D_i/D$ so that $\vec{x} = [x_1, x_2, ..., x_n]^T = [D_i / D]^T$.

EXERCISES on Introduction to Cramer's Rule

EXERCISE #1. Use Cramer's rule to solve $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

EXERCISE #2. Use Cramer's rule to solve $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 2i & -3 \\ 3 & 5i \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

EXERCISE #3.Use Cramer's rule to solve $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 2 & 1 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$

EXERCISE #4. Use Cramer's rule to solve

$x_1 + x_2 + x_3 + x_4 = 1$

$x_1 + 2x_2 + x_3 = 0$

$x_3 + x_4 = 1$

$x_2 + 2x_3 + x_4 = 1$