We consider the (abstract) equation (<u>of the first kind</u>)

$$T(\vec{x}) = \vec{b} \qquad (\textbf{Nonhomogeneous}) \tag{1}$$

where T is a linear operator from the vector space V to the vector space W ($T:V \to W$). We view (1) as a mapping problem; that is, we wish to find those $\vec{x}$'s that are mapped by T to $\vec{b}$.

<u>THEOREM #1</u>. For the <u>nonhomogeneous</u> equation (1) there are three possibilities:
1) There are no solutions.
2) There is exactly one solution.
3) There are an infinite number of solutions.

<u>THEOREM #2</u>. For the <u>homogeneous</u> equation

$$T(\vec{x}) = \vec{0} \qquad (\textbf{Homogeneous}) \tag{2}$$

there are only two possibilities:
1) There is exactly one solution, namely $\vec{x} = \vec{0}$; that is the **<u>null space</u>** of T (i.e. the set of vectors that are mapped into the zero vector) is $N(T) = \{\vec{0}\}$.
2) There are an infinite number of solutions. If the null space of T is finite dimensional, say has dimension $k \in \mathbf{N}$, then the **general solution** of (2) is of the form

$$\vec{x} = c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k = \sum_{i=1}^{k} c_i \vec{x}_i \tag{3}$$

where $B = \{\vec{x}_1, \cdots, \vec{x}_k\}$ is a basis for N(T) and c_i, i=1,...,k are arbitrary constants.

<u>THEOREM #3</u>. The nonhomogeneous equation (1) has at least one solution if $\vec{b}$ is contained in the **range space** of T, R(T), (the set of vectors $\vec{w} \in W$ for which there exist $\vec{v} \in V$ such that $T[\vec{v}] = \vec{w}$). If this is the case, then the **general solution** of (1) is of the form

$$\vec{x} = \vec{x}_p + \vec{x}_h \tag{4}$$

where $\vec{x}_p$ is a **<u>particular</u>** (i.e. any specific) **solution** to (1) and $\vec{x}_h$ is the **general** (e.g. a parametric formula for all) **solution(s)** of (2). If N(T) is finite dimensional then

$$\vec{x} = \vec{x}_p + \vec{x}_h = \vec{x}_p + c_1 \vec{x}_1 + \cdots + c_k \vec{x}_k = \vec{x}_p + \sum_{i=1}^{k} c_i \vec{x}_i \tag{5}$$

where B = { $\vec{x}_1,\cdots,\ \vec{x}_k$ } is a basis of N(T). For the examples, we assume some previous knowledge of **determinants** and **differential equations**. Even without this knowledge, you should get a feel for the theory. And if you lack the knowledge, you may wish to reread this handout after obtaining it.

EXAMPLE 1 OPERATORS DEFINED BY MATRIX MULTIPLICATION

We now apply the general linear theory to operators defined by matrix multiplication. We look for the unknown column vector $\vec{x} = [x_1,x_2,\cdots,x_n]^T$. (We use the **transpose** notation on a **row vector** to indicate a **column vector** to save space and trees.) We consider the operator $T[\vec{x}] = A\vec{x}$ where A is an m×n matrix.

THEOREM 4. If $\vec{b}$ is in the column (range) space of the matrix (operator) A, then the **general solution** to the **nonhomogeneous system of algebraic equation(s)**

$$\underset{mxn}{A}\ \underset{nx1}{\vec{x}} = \underset{mx1}{\vec{b}} \tag{6}$$

can be written in the form

$$\vec{x} = \vec{x}_p + c_1\vec{x}_1 + \cdots + c_k\vec{x}_k = \vec{x}_p + \sum_{i=1}^{k} c_i\vec{x}_i \tag{7}$$

where $\vec{x}_p$ is a **particular** (i.e. any) **solution** to (6) and

$$\vec{x}_h = c_1\vec{x}_1 + \cdots + c_k\vec{x}_k = \sum_{i=1}^{k} c_i\vec{x}_i \tag{8}$$

is the general solution (i.e. a parametric formula for all solutions) to the **complementary homogeneous equation**

$$\underset{mxn}{A}\ \underset{nx1}{\vec{x}} = \underset{mx1}{\vec{0}} \tag{9}$$

Here B = { $\vec{x}_1,\cdots,\vec{x}_k$ } is a basis for the null space N(T) (also denoted by N(A)) which has dimension k. All of the vectors $\vec{x}_p,\ \vec{x}_1,\cdots,\vec{x}_k$ can be founded together using the computational technique of Gauss Elimination. If N(T) = { $\vec{0}$ }, then the unique solution of $\underset{mxn}{A}\ \underset{nx1}{\vec{x}} = \underset{mx1}{\vec{b}}$ is $\vec{x}_p$ (and the unique solution to $\underset{mxn}{A}\ \underset{nx1}{\vec{x}} = \underset{mx1}{\vec{0}}$ is $\vec{x}_h = \underset{nx1}{\vec{0}}$).

THEOREM 5. If n = m, then we consider two cases (instead of three) for equation (6):

1) det A ≠ 0 so that A is **nonsingular**; then the matrix A has a unique inverse, A^{-1} (which is almost never computed), and for any $\vec{b} \in \mathbf{R}^m$, $\underset{nxn}{A}\ \underset{nx1}{\vec{x}} = \underset{nx1}{\vec{b}}$ always has the unique solution

$\vec{x} = A^{-1}\vec{b}$. Thus the operator $T(\vec{x}) = A\vec{x}$ is one-to-one and onto so that any vector $\vec{b}$ is always in the range space R(A) and only the vector $\vec{x} = A^{-1}\vec{b}$. maps to it. Again, the matrix A defines an operator that is a one-to-one and onto mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$ (or $\mathbf{C}^n$ to $\mathbf{C}^n$).

2) det A = 0 so that A is **singular**; then either there is no solution or if there is a solution, then there are an infinite number of solutions. Whether there is no solution or an infinite numbers of solutions depends on $\vec{b}$, specifically, on whether $\vec{b}\ \varepsilon R(A)$ or not. The operator defined by tha matrix A is not one-to-one or onto and the dimension of N(A) is greater than or equal to one.

EXAMPLE 2 LINEAR DIFFERENTIAL EQUATIONS

To avoid using x as either the independent or dependent variable, we look for the unknown function u (dependent variable) as a function of t (independent variable). We let the domain of u be I = (a,b) and think of the function u as a vector in an (infinite dimensional) vector (function) space.

THEOREM 6. If g is in the range space R(L) of the linear differential operator L (i.e. g ε R(L)) then the **general solution to the nonhomogeneous equation**

$$L[\,u(t)\,] = g(t) \qquad \forall\, t \in I \tag{10}$$

can be written in the form

$$u(t) = u_p(t) + u_h(t) \tag{11}$$

where u_p is a **particular solution** to (10) and u_h is the **general solution to the homogeneous equation**

$$L[\,u(t)\,] = 0 \qquad \forall\, t \in I \tag{12}$$

Special cases:

1) $L[\,u(t)\,] = u'' + p(t)u' + q(t)u$. Second Order Scalar Equation.
For this case, we let I = (a,b) and L:$\mathcal{A}$(I,R) →$\mathcal{A}$(I,R). It is known that the dimension of the null space is two so that

$$u_h(t) = c_1u_1(t) + c_2u_2(t).$$

2) $L[\,u(t)\,] = p_o(t)\dfrac{d^n u}{dt^n} + \cdots + p_n(t)\,u(t)$ n^{th} Order Scalar Equation.

Again we let I = (a,b) and L:$\mathcal{A}$(I,R) →$\mathcal{A}$(I,R). For this case, the dimension of the null space is n so that

$$u_h(t) = c_1 u_1(t) + \cdots + c_n u_n(t) = \sum_{i=1}^{n} c_i u_i(t).$$

3) $L[\vec{u}(t)] = \dfrac{d\vec{u}}{dt} - \underset{n\times n}{P}(t)\vec{u}(t)$ First Order System ("Vector" Equation)

Again we let I = (a,b), but now $L: A(I,\mathbf{R}^n) \to A(I,\mathbf{R}^n)$ where $A(I,\mathbf{R}^n) =$ $\{\vec{u}(t): I \to R^n\}$;

that is the set of all time varying "vectors". Here the word "vector" means an n-tuple of functions. We replace (10) with

$$L[\vec{u}(t)] = \vec{g}(t)$$

and (12) with

$$L[\vec{u}(t)] = \vec{0}.$$

Then

$$\vec{u}(t) = \vec{u}_p(t) + \vec{u}_h(t)$$

where

$$\vec{u}_h(t) = c_1 \vec{u}_1(t) + \cdots + c_n \vec{u}_n(t)$$ (i.e. the null space is n dimensional).