A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS
TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

LINEAR CLASS NOTES:
A COLLECTION OF HANDOUTS FOR
REVIEW AND PREVIEW
OF LINEAR THEORY
INCLUDING FUNDAMENTALS OF
LINEAR ALGEBRA

CHAPTER 6

Preview of the

Theory of Abstract Linear Equations

of the First Kind

1. Linear Operator Theory

2. Introduction to Abstract Linear Mapping Problems

In this handout, we review our preview of **linear operator** theory in the previous chapter. The most important examples of linear operators are differential and integral operators and operators defined by matrix multiplication. These arise in many applications. **Lumped parameter systems** (e.g., linear circuits and mass spring systems) have a finite number of state variables and give rise to **discrete operators** defined by matrices on finite dimensional vector spaces such as $\mathbf{R}^n$. Differential and integral equations (e.g., Maxwell's equations and the Navier-Stokes equation) are used to model **distributed (continuum) systems** having an infinite number of state variables and require infinite dimensional vector spaces (i.e., **function spaces**). That is, we have differential and integral operators on function spaces.

Even without covering any topics in differential equations, your background in calculus should be sufficient to understand discrete and continuous operators as linear operators on vector spaces.

A function or map T from one vector space V to another vector space W is often called an **operator**. If we wish to think geometrically (e.g., if V and W are $\mathbf{R}^2$ or $\mathbf{R}^3$) rather than algebraically we might call T a transformation.

DEFINITION 1. Let V and W be vector spaces over the same field **K**. An operator $T:V \to W$ is said to be **linear** if for all $\vec{x}, \vec{y} \in V$ and scalars α,β, it is true that

$$T(\alpha \vec{x} + \beta \vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y}). \qquad (1)$$

THEOREM 1. Let V and W be vector spaces over the same field **K**. An operator $T: V \to W$ is linear if and only if the following two properties are true:

i) $\vec{x}, \vec{y} \;\varepsilon\; V$ implies $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (2)

ii) α a scalar and $\vec{x} \;\varepsilon V$ implies $T(\alpha \vec{x}) = \alpha T(\vec{x})$. (3)

EXAMPLE 1 Let the **operator** $T:\mathbf{R}^n \to \mathbf{R}^m$ be defined by **matrix multiplication** of the column vector $\vec{x}$ by the $m\times n$ matrix A; that is, let

$$T(\vec{x}) =_{df} \underset{m x n}{A} \; \underset{n x 1}{\vec{x}} \qquad (4)$$

where $\underset{nx1}{\vec{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \in \mathbf{R}^{nx1} = \mathbf{R}^n$ and $\underset{mxn}{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdot & \cdot & \cdot & a_{2,n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m,1} & a_{m,1} & \cdot & \cdot & \cdot & a_{m,n} \end{bmatrix} \in \mathbf{R}^{m\times n}$.

Then T is a linear operator.

EXAMPLE 2 Let I = (a,b). The operator $D:C^1(I,\mathbf{R}) \to C(I,\mathbf{R})$ be defined by

$$D(f) =_{df} \frac{df}{dx} \qquad (5)$$

where $f \varepsilon C^1(I,R) = \{f:I \to \mathbf{R}: \frac{df}{dx}$ exists and is continuous on I$\}$ and $C(I,R) = \{f:I \to \mathbf{R} : f$ is continuous on I$\}$. Then D is a linear operator. We may restrict D to $\mathcal{A}(I, \mathbf{R}) = \{f:I \to \mathbf{R}:f$ is analytic on I$\}$ so that $D:\mathcal{A}(I, \mathbf{R}) \to \mathcal{A}(I, \mathbf{R})$ maps a vector space back to itself.

DEFINITION #2. Let $T:V \to W$ be a mapping from a set V to a set W. The set $R(T) = \{ y \in W$: there exists an $x \in V$ such that $y = T(x) \}$ is called the **range** of T. If W has an additive structure (i.e., a binary operation which we call addition) with an additive identity which we call 0, then the set $N(T) = \{x \in V: T(x) = 0\}$ is called the **null set** of T (or nullity of T).

If T is a linear operator from a vector space V to another vector space W, we can say more.

THEOREM #2 Let $T:V \to W$ be a linear operator from a vector space V to a vector space W. The range of T $R(T) = \{ \vec{y} \in W$: there exists an $\vec{x} \in V$ such that $\vec{y} = T(\vec{x}) \}$, is a subspace of W and the null set of T, $N(T) = \{ \vec{x} \in V: T(\vec{x}) = 0\}$ is a subspace of V.

We rename these sets.

DEFINITION #3. Let $T:V \to W$ be a linear operator from a vector space V to another vector space W. The set $R(T) = \{ \vec{y} \in W$: there exists an $\vec{x} \in V$ such that $\vec{y} = T(\vec{x}) \}$ is called the **range space** of T and the set $N(T) = \{ \vec{x} \in V: T(\vec{x}) = 0\}$ is called the **null space** of T. .