DEFINITION #1. Let $B = \{ \vec{x}_1, \vec{x}_2, ..., \vec{x}_k \} \subseteq W \subseteq V$ where W is a subspace of the vector space V. Then B is a **basis** of W if

i) B is linearly independent

ii) B spans W (i.e. Span B = W)

To prove that a set B is a basis (or basis set or base) for W we must show both i) and ii). We already have a method to show that a set is **linearly independent**. To use DUD consider the vector equation

$$c_1 \vec{x}_1 + c_2 \vec{x}_2 + \cdots + c_k \vec{x}_k = \vec{0} \qquad (1)$$

in the unknown variables $c_1, c_2, ..., c_k$ and show that the trivial solution $c_1 = c_2 \cdots = c_k = 0$ is the only solution of (1). To show that B is a **spanning set** using DUD we must show that an arbitrary vector $\vec{b} \in W$ can be written as a linear combination of the vectors in B; that is we must show that the vector equation

$$c_1 \vec{x}_1 + c_2 \vec{x}_2 + \cdots + c_k \vec{x}_k = \vec{b} \qquad (2)$$

in the unknown variables $c_1, c_2, ..., c_k$ always has at least one solution.

EXAMPLE (THEOREM) #1. Show that $B = \{ [1,0,0]^T, [1,1,0]^T \}$ is a basis for $W = \{ [x, y, 0]^T : x,y \in \mathbf{R} \}$.

Solution. (proof) To show linear independence we solve

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{array}{r} c_1 + c_2 = 0 \\ c_2 = 0 \\ 0 = 0 \end{array} \quad \text{to obtain } c_1 = c_2 = 0$$

so that B is linearly independent.

ii) To show B spans W we let $\vec{x} = [x, y, 0] \in W$ (i.e., an arbitrary vector in W) and solve

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{array}{r} c_1 + c_2 = x \\ c_2 = y \\ 0 = 0 \end{array} \quad \text{to obtain}$$

$c_2 = y \Rightarrow c_1 = x - c_2 = x - y$.

Hence for any $\vec{x} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \in W$ we have $\vec{x} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = (x-y)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$;

that is, every vector in W can be written as a linear combination of vectors in B.
Hence B spans W and Span B = W.

Since B is a linearly independent set and spans W, it is a basis for W.

Q.E.D.

EXAMPLE (THEOREM) #2. B = $\{\hat{e}_1, ..., \hat{e}_n\}$ where $\hat{e}_i = [0,...,0,\underset{\longleftarrow i^{th}\text{ slot} \longrightarrow}{1},0,...,0]^T$ is a basis of $\mathbf{R}^n$.

THEOREM #3. Let B = $\{\vec{x}_1, \vec{x}_2, ..., \vec{x}_k\} \subseteq W \subseteq V$ where W is a subspace of the vector space V. Then B is a basis of W iff $\forall \vec{x} \in W$, $\exists!$ $c_1, c_2, ..., c_n$ such that $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k$.

The values of $c_1, c_2, ..., c_n$ associated with each $\vec{x}$ are called the coordinates of $\vec{x}$ with respect to the basis B = $\{\vec{x}_1, \vec{x}_2, ..., \vec{x}_n\}$. Given a basis, finding the coordinates of $\vec{x}$ for any given vector is an important problem..

Although a basis set is not unique, if there is a finite basis, then the number of vectors in a basis set isunique.

THEOREM #4. If B = $\{\vec{x}_1, \vec{x}_2, ..., \vec{x}_k\}$ is a basis for a subspace W in a vector space V, then every basis set for W has exactly k vectors.

DEFINITION #2. The number of vectors in a basis set for a subspace W of a vector space V is the **dimension** of W. If the dimension of W is k, we write **dim W = k**.

THEOREM #5. The dimension of $\mathbf{R}^n$ over $\mathbf{R}$ (and the dimension of $\mathbf{C}^n$ over $\mathbf{C}$) is n.

Proof idea. Exhibit a basis and prove that it is a basis. (See Example (Theorem) #2)

EXERCISES on Basis Sets and Dimension

EXERCISE #1. True or False.

_____ 1. If $B = \{\vec{x}_1, \vec{x}_2, ..., \vec{x}_n\} \subseteq V$ where is a vector space, then B is a basis of W if B is linearly independent and B spans W (i.e. Span B = W)

_____ 2. To show that B is a spanning set using DUD we must show that an arbitrary vector $\vec{b} \in W$ can be written as a linear combination of the vectors in B

_____ 3. To show that $B = \{\vec{x}_1, \vec{x}_2, ..., \vec{x}_n\} \subseteq V$ where V is a vector space is a spanning set we must show that for an arbitrary vector $\vec{b} \in W$ the vector equation $c_1 \vec{x}_1 + c_2 \vec{x}_2 + \cdots + c_n \vec{x}_n = \vec{b}$ in the unknown variables $c_1, c_2, ..., c_n$ always has at least one solution.

_____ 4. If $B = \{\vec{x}_1, \vec{x}_2, ..., \vec{x}_n\} \subseteq V$ where V is a vector space, then B is a basis of W iff $\forall \vec{x} \in W, \exists! \, c_1, c_2, ..., c_n$ such that $\vec{x} = c_1\vec{x} + c_2\vec{x} + \cdots + c_n\vec{x}$.

_____ 5. $B = \{ [1,0,0]^T, [1,1,0]^T \}$ is a basis for $W = \{ [x, y, 0]^T: x,y \in \mathbf{R} \}$.

_____ 6. If $B = \{\vec{x}_1, \vec{x}_2, ..., \vec{x}_n\} \subseteq W \subseteq V$ where W is a subspace of the vector space V and B is a basis of W so that $\forall \vec{x} \in W, \exists! \, c_1, c_2, ..., c_n$ such that $\vec{x} = c_1\vec{x} + c_2\vec{x} + \cdots + c_n\vec{x}$, then the values of $c_1, c_2, ..., c_n$ associated with each $\vec{x}$ are called the coordinates of $\vec{x}$ with respect to the basis B.

_____ 7. A basis set for a vector space is not unique.

_____ 8. If $B = \{\vec{x}_1, \vec{x}_2, ..., \vec{x}_n\}$ is a basis for a subspace W in a vector space V, then every basis set for W has exactly n vectors.

_____ 9. The number of vectors in a basis set for a vector space V is called the dimension of V.

_____ 10. If the dimension of V is n, we write dim V = n.

_____ 11. The dimension of $\mathbf{R}^n$ over $\mathbf{R}$.

_____ 12. The dimension of $\mathbf{C}^n$ over $\mathbf{C}$ is n.

EXERCISE #2. Show that $B = \{ [1,0,0]^T, [2,1,0]^T \}$ is a basis for $W = \{ [x, y, 0]^T: x,y \in \mathbf{R} \}$.

EXERCISE #3. Show that $B = \{\hat{e}_1, ..., \hat{e}_n\}$ where $\hat{e}_i = \underbrace{[0,...,0,1,0,...,0]^T}_{i^{th}\text{ slot}}$ is a basis of $\mathbf{R}^n$.

EXERCISE #4. Show that the dimension of $\mathbf{R}^n$ over $\mathbf{R}$ is n.

EXERCISE #5. Show that the dimension of $\mathbf{C}^n$ over $\mathbf{C}$ is n.