A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS
TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

LINEAR CLASS NOTES:
A COLLECTION OF HANDOUTS FOR
REVIEW AND PREVIEW
OF LINEAR THEORY
INCLUDING FUNDAMENTALS OF
LINEAR ALGEBRA

CHAPTER 5

Linear Operators,

Span, Linear Independence,

Basis Sets, and Dimension

1. Linear Operators

2. Spanning Sets

3. Linear Independence of Column Vectors

4. Basis Sets and Dimension

In this handout, we preview **linear operator** theory. The most important examples of linear operators are differential and integral operators and operators defined by matrix multiplication. These arise in many applications. **Lumped parameter systems** (e.g., linear circuits and mass spring systems) give rise to **discrete operators** defined on finite dimensional vector spaces (e.g., $\mathbf{R}^n$). Differential and integral equations (e.g., Maxwell's equations and the Navier-Stokes equation) are used to model **distributed (continuum) systems** and require infinite dimensional vector spaces. These give rise to differential and integral operators on **function spaces**.

Even without covering any topics in differential equations, your background in calculus should be sufficient to see how discrete and continuous operators are connected as linear operators on a vector space.

A function or map T from one vector space V to another vector space W is often call an **operator**. If we wish to think geometrically (e.g., if V and W are R^2 or R^3) rather than algebraically we might call T a transformation.

DEFINITION 1. Let V and W be vector spaces over the same field **K**. An operator $T:V \to W$ is said to be **linear** if $\forall \vec{x}, \vec{y} \in V$ and scalars $\alpha,\beta \in \mathbf{K}$, it is true that

$$T(\alpha\,\vec{x} + \beta\,\vec{y}) = \alpha\,T(\vec{x}) + \beta\,T(\vec{y}). \qquad (1)$$

THEOREM 1. Let V and W be vector spaces over the same field **K**. An operator $T: V \to W$ is linear if and only if the following two properties hold:

i) $\vec{x}, \vec{y} \in V$ implies $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (2)

ii) $\alpha \in \mathbf{K}$ and $\vec{x} \in V$ implies $T(\alpha\vec{x}) = \alpha\,T(\vec{x})$. (3)

EXAMPLE 1 Let the **operator** $T:\mathbf{R}^n \to \mathbf{R}^m$ be defined by **matrix multiplication** of the column vector $\vec{x}$ by the $m\times n$ matrix A; that is, let

$$T(\vec{x}) =_{df} \underset{m\times n}{A}\ \underset{n\times 1}{\vec{x}} \qquad (4)$$

where $\underset{mxn}{\vec{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \in \mathbf{R}^n$ and $\underset{mxn}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m1} & a_{m21} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix} \in \mathbf{R}^{m\times n}$. Then T is a linear operator.

EXAMPLE 2 Let I = (a,b). The operator $D:C^1(I,\mathbf{R}) \to C(I,\mathbf{R})$ defined by

$$D(f) =_{df} \frac{df}{dx} \tag{5}$$

where $f \in C^1(I,\mathbf{R}) = \{f:I \to \mathbf{R}: \frac{df}{dx}$ exists and is continuous on I$\}$ and $C(I, \mathbf{R}) = \{f:I \to \mathbf{R}: f$ is continuous on I$\}$. Then D is a linear operator. We may restrict D to $A(I, \mathbf{R}) = \{f:I \to \mathbf{R}: f$ is analytic on I$\}$ so that $D:A(I, \mathbf{R}) \to A(I, \mathbf{R})$ maps a vector space back to itself.

DEFINITION #2. Let T:V→W be a mapping from a set V to a set W. The set $R(T) = \{y \in W$: there exists an $x \in V$ such that $y = T(x)\}$ is called the **range** of T. If W has an additive structure (i.e., a binary operation which we call addition) with an additive identity which we call 0, then the set $N(T) = \{x \in V: T(x) = 0\}$ is called the **null set** of T (or nullity of T).

If T is a linear operator from a vector space V to another vector space W, we can say more.

THEOREM #2 Let T:V→W be a linear operator from a vector space V to a vector space W. The range of T, $R(T) = \{\vec{y} \in W$: there exists an $\vec{x} \in V$ such that $\vec{y} = T(\vec{x})\}$, is a subspace of W and the null set of T, $N(T) = \{\vec{x} \in V: T(\vec{x}) = 0\}$ is a subspace of V.

We rename these sets.

DEFINITION #3. Let T:V→W be a linear operator from a vector space V to another vector space W. The set $R(T) = \{\vec{y} \in W$: there exists an $\vec{x} \in V$ such that $\vec{y} = T(\vec{x})\}$ is called the **range space** of T and the set $N(T) = \{\vec{x} \in V: T(\vec{x}) = \vec{0}\}$ is called the **null space** of T. .

EXERCISES on Linear Operator Theory

<u>EXERCISE #1</u>. True or False.

_____ 1. A linear circuit is an example of a lumped parameter system.

_____ 2. Amass/spring system is an example of a lumped parameter system.

_____ 3. A function or map T from one vector space V to another vector space W is often call an operator.

_____ 4. An operator T:V → W is said to be linear if $\forall \vec{x}, \vec{y} \in V$ and scalars $\alpha,\beta\in\mathbf{K}$, it is true that $T(\alpha\vec{x} + \beta\vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y})$.

_____ 5. An operator T: V → W is linear if and only if the following two properties hold: i) $\vec{x}, \vec{y}$ ε V implies $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and ii) $\alpha\in\mathbf{K}$ and $\vec{x}$ ε V implies

$T(\alpha\vec{x}) = \alpha T(\vec{x})$.

_____ 6. The operator $T:\mathbf{R}^n \to \mathbf{R}^m$ defined by $T(\vec{x}) = \underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}}$ is a linear operator.

_____ 7. The operator $D:C^1(a,b) \to C(a,b)$ be defined by $D(f) =_{df} \dfrac{df}{dx}$ is a linear operator.

_____ 8. C(a,b) ={f:(ab) → **R**:f is continuous}.

_____ 9. $C^1(a,b) = \{f:(a,b) \to \mathbf{R}: \dfrac{df}{dx}$ exists and is continuous$\}$.

_____ 10. If T:V→W , then the set R(T) = {y∈ W: there exists an x∈V such that y = T(x) } is called the range of T.

_____ 11. If T:V→W and W has an additive structure (i.e., a binary operation which we call addition) with an additive identity which we call 0, then the set N(T) = {x∈V: T(x) = 0} is called the null set of T (or nullity of T).

_____ 12. If T:V→W is a linear operator from a vector space V to a vector space W, then the range of T, $R(T) = \{\vec{y} \in W$: there exists an $\vec{x} \in V$ such that $\vec{y} = T(\vec{x})\}$, is a subspace of W

_____ 14.If T:V→W is a linear operator from a vector space V to a vector space W, then the null set of T, $N(T) = \{\vec{x} \in V: T(\vec{x}) = 0\}$ is a subspace of V.

_____ 15. If T:V→W is a linear operator from a vector space V to another vector space W, then the set $R(T) = \{\vec{y} \in W$: there exists an $\vec{x} \in V$ such that $\vec{y} = T(\vec{x})\}$ is called the range space of T.

_____ 16. If T:V→W is a linear operator from a vector space V to another vector space W, then the set $N(T) = \{\vec{x} \in V: T(\vec{x}) = \vec{0}\}$ is called the null space of T.

DEFINITION #1. If $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ are vectors in a vector space V and $\alpha_1,\alpha_2,\cdots,\alpha_n$ are scalars, then

$$\alpha, \ \vec{x}_1 + \alpha_2 \vec{x}_2 + \cdots + \alpha_n \vec{x}_n = \sum_{i=1}^{n} \alpha_i \vec{x}_1$$

is called a **linear combination** of the vectors. (It is important to note that a linear combination allows only a finite number of vectors.)

EXAMPLE #1. Consider the system of linear algebraic equations:

$$\begin{aligned} 2x + y + z &= 1 \\ 4x + y \quad &= -3 \\ x - 2y - z &= 0 \quad . \end{aligned}$$

This set of **scalar equations** can be written as the **vector equation**

$$x \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

where the left hand side (LHS) is a linear combination of the (column vectors whose components come from the) columns of the coefficient matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ 1 & -2 & -1 \end{bmatrix} .$$

If we generalize Example #1, we obtain:

THEOREM #1. The general system

$$\underset{nxn}{A} \ \underset{nx1}{\vec{x}} = \underset{nx1}{\vec{b}} \qquad (1)$$

has a solution if and only if the vector $\vec{b}$ can be written as a linear combination of the columns of A. That is, $\vec{b}$ is in the range space of $T(\vec{x}) = A\vec{x}$ if and only if $\vec{b}$ can be written as a linear combination of the (column vectors whose components come from the) columns of the coefficient matrix. A.

DEFINITION #2. Let S be a (finite) subset of a subspace W of a vector space V. If every vector in W can be written as a linear combination of (a finite number of) vectors in S, then S is said to **span** W or to form a **spanning set** for W. On the other hand, if S is any (finite) set of vectors, the **span** of S, written Span(S), is the set of all possible linear combinations of (a finite number of) vectors in S.

THEOREM #2. For any (finite) subset S of a vector space V, Span (S) is a subspace of V.

THEOREM #3. If (a finite set) S is a spanning set for W, then Span S = W.

EXAMPLES. Consider the following subsets of $\mathbf{R}^3$.

1. $S = \{[1,0,0]^T\} = \{\hat{\imath}\}$. Then Span (S) = $\{x\,\hat{\imath}: x \in \mathbf{R}\}$ = x - axis.
2. $S = \{[1,0,0]^T, [0,1,0]^T\} = \{\hat{\imath}, j\}$. Then Span (S) = $\{x\,\hat{\imath}+y\,\hat{\jmath}: x, y \in \mathbf{R}\}$ = the xy plane.
3. $S = \{[1,0,0]^T, [1,1,0]^T\} = \{\hat{\imath}, \hat{\imath}+\hat{\jmath}\}$. Then Span (S) = xy plane.
4. $S = \{[1,0,0]^T, [0,1,0]^T, [1,1,0]^T\} = \{\hat{\imath}, \hat{\jmath}, \hat{\imath}+\hat{\jmath}\}$. Then Span (S) = xy plane.

DEFINITION #3. For any matrix $\underset{m\times n}{A}$, the span of the set of column vectors is the **column space** of A. The usual notation for the column space is R(A) and sometimes R_A. Normally we will use R(A). The reason for this notation is that when we think of the operator $T(\vec{x})$ defined by (matrix) multiplication of the matrix $\underset{m\times n}{A}$ by the column vector $\vec{x} = [x_1,\cdots,x_n]^T$, we see that T maps vectors $\vec{x} \in \mathbf{R}^n$ into vectors $\vec{y} = A\vec{x} \in \mathbf{R}^m$, the column space R(A) is seen to be the **range** (space) of T.

COMMENT. Consider the general system (1) above. Theorem #1 can now be rephrased to say that (1) has a solution if and only if $\vec{b}$ is in the column space of A (i.e. in the range of the operator T).

DEFINITION 4. For any matrix $\underset{m\times n}{A}$, the span of the set of row vectors is the **row space** of A. The usual notation for the row space is $R(A^T)$ and sometimes R_{A^T} since the row space of A is the column space of A^T. Normally we will use $R(A^T)$.

If we think of the operator $T^T(\vec{x})$ defined by (matrix) multiplication of the matrix $\underset{m\times n}{A}^T$ by the column vector $\vec{y} = [y_1,\cdots,y_m]^T$, we see that A^T maps vectors $\vec{y} \in \mathbf{R}^m$ into vectors $\vec{x} = A^T\vec{y} \in \mathbf{R}^n$, the row space $R(A^T)$ is seen to be the **range** (space) of A^T.

Recall that all of the coefficient matrices for the associated linear systems of algebraic equations obtained in the process of doing Gauss elimination are row equivalent. Hence they all have the same row space. However, they do not have the same column space.

EXERCISES on Spanning Sets

EXERCISE #1. True or False.

_____ 1. If $\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n$ are vectors in a vector space V and $\alpha_1,\alpha_2,\cdots,\alpha_n$ are scalars, then

$$\alpha, \ \vec{x}_1 + \alpha_2 \ \vec{x}_2 + \cdots + \alpha_n \ \vec{x}_n = \sum_{i=1}^{n} \alpha_i \ \vec{x}_1$$ is a linear combination of the vectors

_____ 2. A linear combination allows only a finite number of vectors.

_____ 3. The linear algebraic equations: $2x + y + z = 1$, $4x + y = -3$, $x - 2y - z = 0$, is a system of scalar equations.

_____ 4. The coefficient matrix for the system of linear algebraic equations: $2x + y + z = 1$,

$4x + y = -3$, $x - 2y - z = 0$ is $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$.

_____ 5. The linear algebraic equations: $2x + y + z = 1$, $4x + y = -3$, $x - 2y - z = 0$ can be

written as the vector equation $x\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$.

_____ 5. The left hand side (LHS) of the vector equation $x\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ is a linear combination of the column vectors whose components come from

the columns of the coefficient matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$.

_____ 6. The general system $\underset{n\times n}{A} \ \underset{n\times 1}{\vec{x}} = \underset{n\times 1}{\vec{b}}$ has a solution if and only if the vector $\vec{b}$ can be written as a linear combination of the columns of A.

_____ 7. $\vec{b}$ is in the range space of $T(\vec{x}) = A\vec{x}$ if and only if $\vec{b}$ can be written as a linear combination of the column vectors whose components come from the columns of the coefficient matrix A.

_____ 8. If S be a finite subset of a subspace W of a vector space V and every vector in W can be written as a linear combination of a finite number of vectors in S, then S is said to span W or to form a spanning set for W.

_____ 9. If S be a finite subset of a subspace W of a vector space V, then the span of S, written Span(S), is the set of all possible linear combinations of a finite number of vectors in S.

_____ 10. For any finite subset S of a vector space V, Span (S) is a subspace of V.

_____ 11. If S be a finite subset of a subspace W of a vector space V and S is a spanning set for W, then Span S = W.

_____ 12. For any matrix $\underset{m\times n}{A}$, the span of the set of column vectors is the column space of A.

_____ 13. The usual notation for the column space is R(A) and sometimes R_A.

_____ 14. The reason that R(A) is the column space of A is that when we think of the operator $T(\vec{x})$ defined by matrix multiplication of the matrix $\underset{m\times n}{A}$ by the column vector $\vec{x} = [x_1,\cdots,x_n]^T$, we see that T maps vectors $\vec{x} \in \mathbf{R}^n$ into vectors $\vec{y} = A\vec{x} \in \mathbf{R}^m$ which is the column space R(A)

_____ 14. The column space R(A) is the range space of T.

_____ 15. $\underset{n\times n}{A}\ \underset{n\times 1}{\vec{x}} = \underset{n\times 1}{\vec{b}}$ has a solution if and only if $\vec{b}$ is in the range of the operator $T(\vec{x}) = A\vec{x}$.

_____ 16. $\underset{n\times n}{A}\ \underset{n\times 1}{\vec{x}} = \underset{n\times 1}{\vec{b}}$ has a solution if and only if $\vec{b}$ is in the column space of A.

_____ 17. For any matrix $\underset{m\times n}{A}$, the span of the set of row vectors is the row space of A.

_____ 18. The usual notation for the row space is $R(A^T)$ and sometimes R_{A^T} since the row space of A is the column space of A^T.

_____ 19. If we think of the operator $T^T(\vec{x})$ defined by (matrix) multiplication of the matrix $\underset{m\times n}{A}^T$ by the column vector $\vec{y} = [y_1,\cdots,y_m]^T$, we see that A^T maps vectors $\vec{y} \in \mathbf{R}^m$ into vectors $\vec{x} = A^T\vec{y} \in \mathbf{R}^n$, the row space $R(A^T)$ is seen to be the range space of A^T.

DEFINITION #1. Let V be a vector space. A finite set of vectors $S = \{\vec{x}_1, \ldots, \vec{x}_k\} \subseteq V$ is **linearly independent** (ℓ.i.) if the only set of scalars $c_1, c_2, \ldots, c_k$ which satisfy the (homogeneous) vector equation

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{0} \qquad (1)$$

is $c_1 = c_2 = \cdots = c_n = 0$; that is, (1) has only the trivial solution. If there is a set of scalars not all zero satisfying (1) then S is **linearly dependent** (ℓ.d.).

It is common practice to describe the vectors (rather than the set of vectors) as being linearly independent or linearly dependent. Although this is technically incorrect, it is wide spread, and hence we accept this terminology. Since we may consider (1) as a linear homogeneous equation with unknown vector $[\, c_1, c_2, \ldots, c_n] \in \mathbf{R}^n$, if (1) has one nontrivial solution, then it in fact has an infinite number of nontrivial solutions. As part of the standard procedure for showing that a set is **linearly dependent** directly using the definition (DUD) you must exhibit one (and only one) such non trivial solution. To show that a set is **linearly independent** directly using the definition (DUD) you must show that the only solution of (1) is the trivial solution (i.e. that all the c_i's must be zero). Before looking at the application of this definition to column vectors, we state four theorems.

THEOREM #1. Let V be a vector space. If a finite set of vectors $S = \{\vec{x}_1, \ldots, \vec{x}_k\} \subseteq V$ contains the zero vector, then S is linearly dependent.

THEOREM #2. Let V be a vector space. If $\vec{x} \neq \vec{0}$, then $S = \{\vec{x}\} \subseteq V$ is linearly independent.

Proof. To show S is linearly independent we must show that $c_1 \vec{x} = \vec{0}$ implies that $c_1 = 0$. But by the zero product theorem, if $c_1 \vec{x} = \vec{0}$ is true then $c_1 = 0$ or $\vec{x} = \vec{0}$. But by hypothesis $\vec{x} \neq \vec{0}$. Hence $c_1 \vec{x} = \vec{0}$ implies $c_1 = 0$. Hence $S = \{\vec{x}\}$ where $\vec{x} \neq \vec{0}$ is linearly independent.

Q.E.D.

THEOREM #3. Let V be a vector space and $S = \{\vec{x}, \vec{y}\} \subseteq V$. If either $\vec{x}$ or $\vec{y}$ is the zero vector, then S is linearly dependent.

THEOREM #4. Let V be a vector space and $S = \{\vec{x}, \vec{y}\} \subseteq V$ where $\vec{x}$ and $\vec{y}$ are nonzero vectors. Then S is linearly dependent if and only if one vector is a scalar multiple of the other.

Although the definition is stated for an abstract vector space and hence applies to any vector space and we have stated some theorems in this abstract setting, in this section we focus on column vectors in $\mathbf{R}^n$ (or $\mathbf{C}^n$ or $\mathbf{K}^n$). Since we now know how to solve a system of linear algebraic equations, using this procedure, we can develop a "procedure" to show that a finite set in $\mathbf{R}^n$ (or $\mathbf{C}^n$ or $\mathbf{K}^n$) is linearly independent. We also show how to give sufficient conditions to show that a finite set in $\mathbf{R}^n$ (or $\mathbf{C}^n$ or $\mathbf{K}^n$) is linearly dependent.

PROCEDURE. To determine if a set $S = \{\vec{x}_1,\ldots,\vec{x}_k\} \subseteq V$ is linearly independent or linearly dependent, we first write down the equation (1) and try to solve. If we can show that the only solution is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$, then we have shown **directly using the definition** (DUD) of linear independence that S is linearly independent. On the other hand (OTOH), if we can exhibit a nontrivial solution, then we have shown **directly using the definition** of linear dependence that S is linearly dependent. We might recall that the linear theory assures us that if there is one nontrivial solution, that there are an infinite number of nontrivial solutions. However, to show that S is linearly dependent directly using the definition, it is not necessary (or desirable) to find all of the nontrivial solutions. Although you could argue that once you are convinced (using some theorem) that there are an infinite number of solutions, then we do not need to exhibit one, this will not be considered to be a proof directly using the definition (as it requires a theorem). Thus to prove that S is linearly dependent directly using the definition of linear dependence, **you must exhibit one nontrivial solution**. This will help you to better understand the concepts of linear independence and linear dependence. We apply these "procedures" to finite sets in $\mathbf{R}^n$ (or $\mathbf{C}^n$). For $S = \{\vec{x}_1,\ldots,\vec{x}_k\} \subseteq \mathbf{R}^n$ (or $\mathbf{C}^n$) (1) becomes a system of n equations (since we are in $\mathbf{R}^n$ (or $\mathbf{C}^n$)) in k unknowns (since we have k vectors). (This can be confusing when applying general theorems about m equations in n unknowns. However, this should not be a problem when using DUD on specific problems.)

EXAMPLE #1. Determine (using DUD) if $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is linearly independent.

Solution. (This is not a yes-no question and a proof is required). Assume

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

and (try to) solve. The vector equation (2) is equivalent to the two scalar equations (since we are in $\mathbf{R}^2$) in three unknowns (since S has three vectors).

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ c_1 + 3c_2 + 2c_3 &= 0 \end{aligned}$$

(3)

Simple systems can often be solved using ad hoc (for this case only) procedures. But for complicated systems we might wish to write this system in the form $A\vec{c} = \vec{0}$ and use Gauss elimination (on a computer). Note that when we reduce A we do not have to augment. Why?

For this example $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Since the system is homogeneous we can solve by reducing A without augmenting (Why?).

$$R_1 - R_2 \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 + 2c_2 = c_3 = 0 \\ c_2 + c_3 = 0 \end{matrix} \Rightarrow \begin{matrix} c_1 = -2c_2 - c_3 = 2c_3 - c_3 = c_3 \\ c_2 = -c_3 \end{matrix}$$

Hence the general solution of (2) is $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_3 \\ -c_3 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Hence there are an infinite number of solutions. They are the vectors in the subspace $W = \{ \vec{x} \in R^3 : \vec{x} = c\,[2, -1, 1]^T$ with $c \in \mathbf{R}\}$. Since there is a nontrivial solution, S is linearly dependent. However, we <u>must</u> exhibit one nontrivial solution which we do by choosing $c_1=1$, $c_2 = -1$, and $c_3 = 1$. Hence we have

$$(1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (4)$$

Since we have exhibited a nontrivial linear combination of the vectors in S, (4) alone proves that S is a linearly dependent set in the vector space $\mathbf{R}^2$ QED

It may be easy to guess a nontrivial solution (if one exists). We call this method the Clever Ad Hoc (CAH) method. You may have noted that the first and third vectors in the previous example add together to give the second vector. Hence the coefficients c_1, c_2, and c_3 could have been easily guessed.

<u>EXAMPLE #2</u>. Determine if $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ is linearly independent.

<u>Solution</u>. (Again this is not a yes-no question and a proof is required) Note that since

$3\begin{bmatrix}1\\2\\-1\end{bmatrix}=\begin{bmatrix}3\\6\\-3\end{bmatrix}$ we have

$$(3)\begin{bmatrix}1\\2\\-1\end{bmatrix}+(-1)\begin{bmatrix}3\\6\\-3\end{bmatrix}+(0)\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}. \qquad (5)$$

Since we have exhibited a nontrivial linear combination of the vectors in S, (5) alone proves that S is a linearly dependent set in the vector space $\mathbf{R}^3$ Hence S is linearly dependent. Q.E.D.

We give a final example.

<u>EXAMPLE #3.</u> Determine if $S=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right\}$ is linearly independent.

<u>Solution</u>. Since $\begin{bmatrix}1\\1\end{bmatrix}$ is not a multiple of $\begin{bmatrix}1\\2\end{bmatrix}$ we assume

$$c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\Rightarrow\begin{matrix}c_1+c_2=0\\c_1+2c_2=0\end{matrix}\Rightarrow A\vec{c}=\vec{0}$$

where

$$A=\begin{bmatrix}1&1\\1&2\end{bmatrix} \text{ and } \vec{c}=\begin{bmatrix}c_1\\c_2\end{bmatrix}. \text{ Hence}$$

$$R_1-R_2\begin{bmatrix}1&1\\1&2\end{bmatrix}\Rightarrow\begin{bmatrix}1&1\\0&1\end{bmatrix}\Rightarrow\begin{matrix}c_1+c_2=0\\c_2=0\end{matrix}\Rightarrow c_1=c_2=c_2=0$$

Since we have proved that the trivial linear combination where $c_1 = c_2 = 0$ is the only linear combination of the vectors in S that gives the zero vector in $\mathbf{R}^2$ (i.e., we have proved that S is a linearly independent set.

EXERCISES on Linear Independence (of Column Vectors)

EXERCISE #1. True or False.

_____ 1. If V be a vector space and $S = \{\vec{x}_1, \ldots, \vec{x}_k\} \subseteq V$, then S is linearly independent (ℓ.i.) if the only set of scalars $c_1, c_2, \ldots, c_k$ which satisfy the (homogeneous) vector equation $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{0}$ is $c_1 = c_2 = \cdots = c_n = 0$.

_____ 2. If V be a vector space and $S = \{\vec{x}_1, \ldots, \vec{x}_k\} \subseteq V$, then S is linearly independent (ℓ.i.) if the only set of scalars $c_1, c_2, \ldots, c_k$ which satisfy the (homogeneous) vector equation $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{0}$ is the trivial solution.

_____ 3. If V be a vector space and $S = \{\vec{x}_1, \ldots, \vec{x}_k\} \subseteq V$, then S is linearly dependent (ℓ.d.) if there is a set of scalars not all zero satisfying $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_k\vec{x}_k = \vec{0}$.

_____ 4. If V is a vector space and $S = \{\vec{x}_1, \ldots, \vec{x}_k\} \subseteq V$ contains the zero vector, then S is linearly dependent.

_____ 5. If V be a vector space, $\vec{x} \neq \vec{0}$ and $S = \{\vec{x}\} \subseteq V$, then S is linearly independent.

_____ 6. If $c_1\vec{x} = \vec{0}$ is true then by the zero product theorem either $c_1 = 0$ or $\vec{x} = \vec{0}$. _____

_____ 7. If V is a vector space and $S = \{\vec{x}, \vec{y}\} \subseteq V$ and either $\vec{x}$ or $\vec{y}$ is the zero vector, then S is linearly dependent.

_____ 8. If V is a vector space and $S = \{\vec{x}, \vec{y}\} \subseteq V$ where $\vec{x}$ and $\vec{y}$ are nonzero vectors, then S is linearly dependent if and only if one vector is a scalar multiple of the other.

EXERCISE #2. Use the procedure given above to determine (using DUD) if $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ is linearly dependent or linearly independent or neither. Thus you must explain completely.

EXERCISE #3. Use the procedure given above to determine (using DUD) if $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ is linearly dependent or linearly independent or neither. Thus you must explain completely.

EXERCISE #4. Use the procedure given above to determine (using DUD) if $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is linearly dependent or linearly independent or neither. Thus you must explain completely.

EXERCISE #5. Prove Theorem #1.

EXERCISE #6. Prove Theorem #3.

EXERCISE #7. Prove Theorem #4.