Handout #4 | POSSIBILITIES FOR LINEAR ALGEBRAIC EQUATIONS | Professor Moseley

We consider the three possible outcomes of applying Gauss elimination to the system of scalar algebraic equations:

$$\begin{array}{l} a_{11}\ x_1\ +\ a_{12}\ x_2 + \cdots + a_{1n}\ x_n = b_1 \\ a_{21}\ x_1\ +\ a_{22}\ x_2 + \cdots + a_{2n}\ x_n = b_2 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}\ x_1\ +\ a_{m2}\ x_2 + \cdots + a_{mn}\ x_n = b_m \end{array} \tag{1}$$

THEOREM. There exists three possibilities for the system (1):
1. There is no solution.
2. There exists exactly one solution.
3. There exists an infinite number of solutions

We have illustrated the case where there exists exactly one solution. We now illustrate the cases where there exists no solution and an infinite number of solutions.

EXAMPLE. Solve

$$\begin{array}{ll} \text{E1} & x + \ \ y + \ z = b_1 \\ \text{E2} & 2x + 2y + 3z = b_2 \\ \text{E3} & 4x + 4y + 8z = b_3 \end{array} \tag{2}$$

if $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has the values 1) $\vec{b} = \vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and 2) $\vec{b} = \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Solution: Forward Step. Reduce the augmented matrix

$$\begin{array}{r} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 2 & 3 & b_2 \\ 4 & 4 & 8 & b_3 \end{array}\right] \Rightarrow \begin{array}{r} \\ \\ R_3 - 4R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 4 & b_3 - 4b_1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - 4b_2 + 4b_1 \end{array}\right]$$

as $R_3 - 4R_2 \Rightarrow b_3 - 4b_1 - 4(b_2 - 2b_1) = b_3 - 4b_2 + 4b_1$

This completes the forward elimination step. We see that (2) has a solution if and only if $b_3 - 4b_2 + 4b_1 = 0$ or $4b_1 - 4b_2 + b_3 = 0$. This means that the range of the operator T associated with the matrix A is the plane in $\mathbf{R}^3$ through the origin whose equation in x,y,z variables is

P: $4x - 4y + z = 0$ where $P \subseteq \mathbf{R}^3$. If $\vec{b} = \vec{b}_1 = [1,1,1]^T$ then $4(1) - 4(1) + 1 = 1 \neq 0$ and hence we have no solution. If $\vec{b} = \vec{b}_2 = [1,1,0]^T$ then $4(1) - 4(1) + 0 = 0$ so there exists a solution. In fact there exists an infinite number of solutions. The pivots are already ones. We finish the Gauss-Jordan process for $\vec{b} = \vec{b}_2 = [1,1,0]^T$. Our augmented matrix is

$$R_1 - R_2 \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{rl} x + y & = 2 \\ z & = -1 \\ 0 & = 0 \end{array} \quad \begin{array}{l} x = 2 - y \\ \Rightarrow z = -1 \\ \end{array}$$

We can now write a (parametric) formula for the infinite number of solutions.

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 - y \\ y \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

It will become clear why one wishes to writw the solution in this form after you learn to solve differential equations.

<u>CONVENIENT CHECKS</u>. There are convenient checks for the vectors that appear in the formula for the infinite number of solutions. Satisfying these checks does not guarantee a correct solution. However, failing these checks does guarantee an error.

Since we claim to have all solutions to $\underset{3x3}{A}\,\underset{3x1}{\vec{x}} = \underset{3x1}{\vec{b}}$, letting y = 0, we see that $\vec{x}_p = [2, 0, -1]^T$ should be a (particular) solution to $A\vec{x} = \vec{b}$. (There are an infinite number of solutions.)

$$\begin{array}{rl} u + v + w = 1 & \Rightarrow\ 2 + 0 - 1 = 1 \\ 2u + 2v + 5w = 1 & \Rightarrow\ 2(2) + 2(0) + 5(-1) = -1 \\ 4u + 4v + 5w = 0 & \Rightarrow\ 4(2) + 4(0) + 8(-1) = 0 \end{array}$$

Hence this part of the formula for the solution checks. Careful analysis indicates that $\vec{x}_1 = [-1, 1, 0]^T$ should be a solution to the complementary (homogeneous) equation $A\vec{x} = \vec{0}$.

$$\begin{array}{rl} u + v + w = 0 & \Rightarrow\ -1 + 1 + 0 = 0 \\ 2u + 2v + 5w = 0 & \Rightarrow\ 2(-1) + 2(1) + 5(0) = 0 \\ 4u + 4v + 8w = 0 & \Rightarrow\ 4(-1) + 4(1) + 8(0) = 0 \end{array}$$

Hence this part of the formula for the solution also checks.

EXERCISES on Possibilities for Linear Algebraic Equations

<u>EXERCISE #1</u>. True or False.

_____ 1. There exists three possibilities for the system $\underset{mxn}{A}\,\underset{nx1}{\vec{x}} = \underset{mx1}{\vec{b}}$.

_____ 2. One possibility for the system $\underset{mxn}{A}\,\underset{nx1}{\vec{x}} = \underset{mx1}{\vec{b}}$ is that there is no solution.

_____ 3. One possibility for the system $\underset{m\times n}{A}\,\underset{n\times1}{\vec{x}} = \underset{m\times1}{\vec{b}}$ is that there exists exactly one solution.

_____ 4. One possibility for the system $\underset{m\times n}{A}\,\underset{n\times1}{\vec{x}} = \underset{m\times1}{\vec{b}}$ is $\exists$ an infinite number of solutions.

<u>EXERCISE #2</u>. Solve

$$\begin{aligned} x + y + z &= b_1 \\ 2x + 2y + 3z &= b_2 \\ 4x + 4y + 8z &= b_3 \end{aligned}$$

if $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has the values 1) $\vec{b} = \vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and 2) $\vec{b} = \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

<u>EXERCISE #3</u>. Solve

$$\begin{aligned} x + y + z &= b_1 \\ 2x + 2y + 3z &= b_2 \\ 4x + 4y + 4z &= b_3 \end{aligned}$$

if $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has the values 1) $\vec{b} = \vec{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and 2) $\vec{b} = \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

Handout #2 WRITING SOLUTIONS OF LINEAR ALGEBRAIC EQUATIONS Professor Moseley

Consider the system of linear equations

$$\underset{m\times n}{A}\ \underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}} \qquad (1)$$

Suppose we apply Gauss-Jordan elimination so that

$$\left[\underset{m\times n}{A}\,\middle|\,\underset{m\times 1}{\vec{b}}\right] \overset{GE}{\Rightarrow} \left[\underset{m\times n}{U}\,\middle|\,\underset{m\times 1}{\vec{c}}\right] \overset{GJE}{\Rightarrow} \left[\underset{m\times n}{R}\,\middle|\,\underset{m\times 1}{\vec{d}}\right]$$

where U is an upper triangular matrix obtained using Gauss Elimination, $\vec{c}$ is the result of applying these same operations to $\vec{b}$, R is the **<u>row- reduced-echelon</u> form** of A obtained using **Gauss - Jordan elimination**, and $\vec{d}$ is the result of applying these same operations to $\vec{b}$. This means that the matrix $\left[\underset{m\times n}{R}\,\middle|\,\underset{m\times 1}{\vec{d}}\right]$ has zeros above as well as below the pivots. Recall that there all three cases: 1) No solution, 2) Exactly one solution (a unique solution), and 3) An infinite number of solutions.

1) If there is a row in $\left[\underset{m\times n}{R}\,\middle|\,\underset{m\times 1}{\vec{d}}\right]$ where the entries in R are all zero and the entry in $\vec{d}$ is nonzero (this corresponds to an equation of the form $0 = d_i \neq 0$). Since this is not true, the system has no solution, and you conclude by writing: No Solution.

2) If there are $m - n$ rows of all zeros (including the elements in $\vec{d}$) and so n pivots (dependent variables) then each variable in $\vec{x}$ is determined uniquely and there is exactly one solution. Conclude by giving the solution explicitly in **both scalar and vector form**.

3) If there are an infinite number of solutions we proceed as follows:

a) Separate the variables (i.e. the columns) into two groups. Those corresponding to pivots are the **<u>dependent (basic) variables</u>**. Those not corresponding to pivots are the **<u>independent (free) variables</u>**.

b) Write the equations that correspond to the augmented matrix $[R \mid \vec{d}\,]$.

c) Solve these equations for the dependent variables in terms of the independent variables. (This is easy if GJE is used).

d) Replace the dependent variables in the vector $\vec{x} = [x_1, \ldots, x_n]^T$ to obtain $\vec{x}$ in terms of the independent variables only.

e) Write the infinite number of solutions $\vec{x}$ in the paramentric form $\vec{x} = \vec{x}_p + \sum_{i=1}^{k} c_i \vec{x}_i$

where the c_i's are the independent variables. Thus we see that there is a solution for each set of paramenter values.

Alternately we see that $\vec{x}_p$ can be obtained by letting all the c_i's (i.e. the independent variables) be zero. If $\vec{b} = \vec{0}$ then $\vec{x}_p$ can be taken to be $\vec{0}$. The above procedure will in fact make it zero.) Hence the vectors $\vec{x}_I$ can be obtained by letting $\vec{b} = \vec{0}$ and letting each independent variable be one with the rest of the independent variables equal to zero.

<u>EXAMPLE#1</u>. Recall that the solution process for

$$\begin{aligned} 2x + y + z &= 1 && (1)\\ 4x + y &= -2 && (2)\\ -2x + 2y + z &= 7 && (3) \end{aligned}$$

using Gauss-Jordan elimination (right to left) on augmented matrices is

$$\begin{array}{r} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & 1 & 0 & -2 \\ -2 & 2 & 1 & 7 \end{array}\right] \Rightarrow \begin{array}{r} \\ \\ R_3 + 3R_2 \end{array}\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 3 & 2 & 8 \end{array}\right] \Rightarrow$$

$$\begin{array}{r} 1/2\, R_1 \\ -R_2 \\ -1/4\, R_3 \end{array}\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{array}\right] \Rightarrow \begin{array}{r} R_1 - 1/2R_3 \\ R_2 - 2R_3 \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \begin{array}{r} R_1 - 1/2R_2 \\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow$$

$$\boxed{\begin{array}{ll} x = -1 & \\ y = 2 & \text{Scalar form} \\ z = 1 & \end{array}} \qquad \vec{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \text{ Vector form.}$$

<u>EXAMPLE#2</u>. Recall that the solution process for

$$\begin{array}{lrl} \text{E1} & x + y + z &= b_1 \\ \text{E2} & 2x + 2y + 3z &= b_2 \\ \text{E3} & 4x + 4y + 8z &= b_3 \end{array} \qquad (2)$$

if $\vec{b} = \vec{b}_1 = [1,1,1]^T$ using Gauss elimination (only the forward sweep is needed) on augmented matrices is

$$\begin{array}{r}\\ R_2-2R_1\\ R_3-4R_1\end{array}\left[\begin{array}{ccc|c}1&1&1&1\\2&2&3&1\\4&4&8&1\end{array}\right]\Rightarrow \begin{array}{r}\\ \\ R_3-4R_2\end{array}\left[\begin{array}{ccc|c}1&1&1&1\\0&0&1&-1\\0&0&4&-3\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}1&1&1&1\\0&0&1&-1\\0&0&0&1\end{array}\right]\Rightarrow \begin{array}{r}x+y+z=1\\ z=-1\\ 0=1\end{array}$$

The last equation (0=1) is not true so we see that the equations are **inconsistent** and that there is no solution. We write **NO SOLUTION**.

On the other hand, if $\vec{b}=\vec{b}_1=[1,1,0]^T$ using Gauss-Jordan elimination on augmented matrices is

$$\begin{array}{r}\\ R_2-2R_1\\ R_3-4R_1\end{array}\left[\begin{array}{ccc|c}1&1&1&1\\2&2&3&1\\4&4&8&0\end{array}\right]\Rightarrow \begin{array}{r}\\ \\ R_3-4R_2\end{array}\left[\begin{array}{ccc|c}1&1&1&1\\0&0&1&-1\\0&0&4&-3\end{array}\right]\Rightarrow R_1-R_2\left[\begin{array}{ccc|c}1&1&1&1\\0&0&1&-1\\0&0&0&0\end{array}\right]\Rightarrow R_1-R_2\left[\begin{array}{ccc|c}1&1&0&2\\0&0&1&-1\\0&0&0&0\end{array}\right]$$

$$\Rightarrow \begin{array}{r}x+y\quad=2\\ z=-1\\ 0=0\end{array}\Rightarrow \begin{array}{l}x=2-y\\ z=-1\end{array}$$

We can now write a (parametric) formula for the infinite number of solutions.

$$\begin{array}{l}x=2-y\\ z=-1\end{array}=\begin{bmatrix}2-y\\ y\\ -1\end{bmatrix}=\begin{bmatrix}2\\0\\-1\end{bmatrix}+y\begin{bmatrix}-1\\1\\0\end{bmatrix}\ \text{Vector Form}\qquad \begin{array}{l}x=1+y\\ (y=y)\ \text{Scalar Form}\\ z=-1\end{array}$$

EXERCISES on Writing Solutions of Linear Algebraic Equations

EXERCISE #1. Suppose the Gauss process yields

$$\begin{aligned} x + y + z &= b_1 \\ 2y + 3z &= b_2 \\ 8z &= b_3 \end{aligned}$$

If $\vec{b} = [b_1, b_2, b_3]^T = \vec{b}_1 = [1,1,1]^T$, give the solution and the solution set. What is the difference in these two questions. How many solutions are there? The solution should be written as ______.
If $\vec{b} = [b_1, b_2, b_3]^T = \vec{b}_1 = [0,0,0]^T$, give the solution and the solution set. What is the difference in these two questions. How many solutions are there? The solution should be written as ______.

EXERCISE #2. Suppose the Gauss-Jordan process yields $\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 - b_2 \\ 0 & 1 & 0 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$

If $\vec{b} = [b_1, b_2, b_3]^T = \vec{b}_1 = [1,1,1]^T$, give the solution and the solution set. What is the difference in these two questions. How many solutions are there? The solution should be written as ______.
If $\vec{b} = [b_1, b_2, b_3]^T = \vec{b}_1 = [1,-1,0]^T$, give the solution and the solution set. What is the difference in these two questions. How many solutions are there? The solution should be written as _____.

DEFINITION #1. An mxm matrix is said to be an elementary matrix if it can be obtained from the mxm identity matrix $\underset{m\times m}{I}$ by means of a single ERO.

THEOREM #1. Let e(A) denote the matrix obtained when the ERO e is applied to the matrix A. Now let E = e(I). Then for any mxn matrix A, we have e(A) = EI. That is, the effect of an ERO e on a matrix A can be obtain by multiplying A by E where E = e(I). Thus E is called the **matrix representation** of e.

THEOREM#2. All elementary matrices are invertible. If E is an elementary matrix that represents the ERO e with ERO inverse e^{-1}, then E^{-1} is just the matrix representation of e^{-1}.

Suppose we apply Gauss-Jordan elimination to $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$ so that

$$\left[\underset{m\times n}{A}\,\middle|\,\underset{m\times 1}{\vec{b}}\right] \overset{GJE}{\Rightarrow} \left[\underset{m\times n}{R}\,\middle|\,\underset{m\times 1}{\vec{d}}\right]$$

via the sequence of ERO's $e_1, e_2, ..., e_n$ and $E_i = e_i(I)$ as well as $E_i^{-1} = e_i^{-1}(I)$ for i=1, 2, 3, ..., n. then

$$E_n\,E_{n-1} \cdots E_2\,E_1\,A\,\vec{x} = R\,\vec{x} = E_n\,E_{n-1} \cdots E_2\,E_1\,\vec{b} = \vec{d}$$

That is, we can solve the vector equation by repeatedly multiplying the vector (or matrix) equation $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$ by matrix representations of the appropriate ERO's until we obtain $R\,\vec{x} = \vec{d}$. If we are in the unique solution case, then $n \le m$, the first n row of R are the nxn identity matrix I and we obtain $\vec{x} = \vec{d}_r$ where $\vec{d}_r$ is the nx1 vector containing the first n components of $\vec{d}$ whose remaining m − n components are all zeros. If not, we must examine R and $\vec{d}$ more closely to determine which case we are in. The main point is that instead of working with the scalar equations or the augmented matrices, we may solve the vector (or matrix) equation $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$ by multiplying it successively by (invertible) elementary matrices to obtain an equivalent form of the vector equation where the unique solution is readily available, we know immediately that there is no solution, or we may easily find a parametric form for all of the infinite number of solutions.

EXERCISES on Elementary Matrices and Solving the Vector Equations

<u>EXERCISE #1</u>. True or False

_____ 1. An mxm matrix is said to be an elementary matrix if it can be obtained from the mxm identity matrix I_{mxm} by means of a single ERO

_____ 2. If e(A) denotes an ERO applied to the matrix A and let E = e(I), then for any mxn matrix A, we have e(A) = EI.

_____ 3. All elementary matrices are invertible.

_____ 4. If E is an elementary matrix that represents the ERO e with ERO inverse e^{-1}, then E^{-1} is just the matrix representation of e^{-1}.