A SERIES OF CLASS NOTES TO INTRODUCE LINEAR AND NONLINEAR PROBLEMS
TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

LINEAR CLASS NOTES:
A COLLECTION OF HANDOUTS FOR
REVIEW AND PREVIEW
OF LINEAR THEORY
INCLUDING FUNDAMENTALS OF
LINEAR ALGEBRA

CHAPTER 4

Introduction to Solving

Linear Algebraic Equations

Handout #1 SYSTEMS, SOLUTIONS, AND ELEMENTARY EQUATION OPERATIONS Prof. Moseley

In high school you learned how to solve two linear (and possibly nonlinear) equations in two unknowns by the elementary algebraic techniques of **addition** and **substitution** to eliminate one of the variables. These techniques may have been extended to three and four variables. Also, **graphical** or **geometric techniques** may have been developed using the intersection of two lines or three planes. Later, you may have been introduced to a more formal approach to solving a system of m equations in n unknown variables.

$$\begin{array}{l} a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n = b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{m1}\, x_1 + a_{m2}\, x_2 + \cdots + a_{mn}\, x_n = b_m \end{array} \tag{1}$$

We assume all of the **scalars** a_{ij}, x_i, and b_j are elements in a **field K**. (Recall that a **field** is an abstract algebra structure that may be defined informally as a number system where you can add, subtract, multiply , and divide. Examples of a field are **Q**, **R**, and **C**. However, **N** and **Z** are not fields. Unless otherwise noted, the scalars can be assumed to be real or complex numbers. i.e., **K** = **R** or **K** = **C**.) The formal process for solving **m linear algebraic equations in n unknowns** is called **Gauss Elimination**. We need a formal process to prove that a solution (or a parametric expression for an infinite number of solutions) can always be obtained in a finite number of steps (or a proof that no solution exists), thus avoiding the pitfall of a "circular loop" which may result from ad hoc approaches taken to avoid fractions. Computer programs using variations of this algorithm avoid laborious arithmetic and handle problems where the number of variables is large. Different programs may take advantage of particular characteristics of a category of linear algebraic equations (e.g., banded equations). Software is also available for **iterative techniques** which are not discussed here. Another technique which is theoretically interesting, but only useful computationally for very small systems is **Cramer Rule** which is discussed in Chapter 7.

DEFINITION #1. A **solution** to (1) is an n-tuple (finite sequence) $x_1, x_2, ..., x_n$ in $\mathbf{K}^n$ (**e.g., $\mathbf{R}^n$** or $\mathbf{C}^n$) such that all of the equations in (1) are true. It can be considered to be a row vector $[\, x_1, x_2, ..., x_n \,]$ or as a column vector $[\, x_1, x_2, ..., x_n \,]^T$ using the transpose notation. When we later formulate the problem given by the scalar equatiuons (1) as a matrix or "vector" equation, we will need our unknown "vector" to be a column vector, hence we use column vectors. If we use column vectors, the **solution set** for (1) is the set

$S = \{ \vec{x} = [\, x_1, x_2, ..., x_n \,] \in \mathbf{K}^n$: when $x_1, x_2, ..., x_n$ are substituted into (1), all of the equations in (1) are satisfied}.

DEFINITION #2. Two systems of linear algebraic equations are **equivalent** if their solution sets are equal (i.e., have the same elements).

As with a single algebraic equation, there are algebraic operations that can be performed on a system that yield a new equivalent system. We also refer to these as **equivalent equation operations** (EEO's). However, we will restrict the EEO's we use to three **elementary equation operations**.

DEFINITION #3. The **Elementary Equation Operations (EEO's)** are
1. Exchange two equations
2. Multiply an equation by a nonzero constant.
3. Replace an equation by itself plus a scalar multiple of another equation.

The hope of elimination using EEO's is to obtain, if possible, an equivalent system of n equations that is **one-way coupled** with new coefficients a_{ij} as follows:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad & \quad \vdots \\
a_{nn}x_n &= b_n
\end{aligned}
\tag{2}
$$

where $a_{ii} \neq 0$ for i = 1, 2, ...,n. Since these equations are only one-way coupled, the last equation may then be solved for x_n and substituted back into the previous equation to find x_{n-1}. This process may be continued to find all of the x_i's that make up the unique (vector) solution. Note that this requires that $m \geq n$ so that there are at least as many equations as unknowns (some equations may be redundant) and that all of the diagonal coefficients a_{ii} are not zero. This is a very important special case. The nonzero coefficients a_{ii} are called **pivots**.

Although other operations on the system of equations can be derived, the three EEO's in Definition #3 are sufficient to find a one-way coupled equivalent system, if one exists. If sufficient care is not taken when using other operations such as replacing an equation by a linear combination of the equations, a new system which is not equivalent to the old one may result. Also, if we restrict ourselves to these three EEO's, it is easier to develop computational algorithms that can be easily programed on a computer. Note that applying one of these EEO's to a system of equations (the Original System or OS) results in a new system of equations (New System or NS). Our claim is that the systems OS and NS have the same solution set.

THEOREM #1. The new system (NS) of equations obtained by applying an elementary equation operation to a system (OS) of equations is equivalent to the original system (OS) of equations.

Proof idea. By Definition#2 we need to show that the two systems have the same solution set.

EEO # 1 Whether a given $\vec{x} = [x_1, x_2, ..., x_n]$ is a solution of a given scalar equation is "clearly" not dependent on the order in which the equations are written down. Hence whether a given $\vec{x} = [x_1, x_2, ..., x_n]$ is a solution of all equations in a system of equations is not dependent on the order in which the equations are written down

EEO # 2 If $\vec{x} = [x_1, x_2, ..., x_n]$ satisfies.

$$a_{i1}x_1 + ka_{i2}x_2 + \cdots + a_{in}x_n = b_i \tag{3}$$

then by the theorems of high school algebra, for any scalar k it also satisfies

$$ka_{i1}x_1 + ka_{i2}x_2 + \cdots + ka_{in}x_n = kb_{i.} \tag{4}$$

The converse can be shown if $k \neq 0$ since k will have a multiplicative inverse.

EEO # 3. If $\vec{x} = [x_1, ..., x_n]^T$ satisfies each of the original equations, then it satisfies

$$a_{i1}x_1 + \cdots + a_{in}x_n + k(a_{j1}x_1 + \cdots + a_{jn}x_n) = b_i + kb_j. \tag{5}$$

Conversely, if (5) is satisfied and

$$a_{j1}x_1 + \cdots + a_{jn}x_n = b_j \tag{6}$$

is satisfied then (5) - k (6) is also satisfied. But this is just (3).

QED

<u>THEOREM #2</u>. Every EEO has in inverse EEO of the same type.

Proof idea. EEO # 1. The inverse operation is to switch the equations back.
EEO # 2. The inverse operation is to divide the equation by $k(\neq 0)$; that is, to multiply the equation by 1/k.
EEO # 3. The inverse operation is to replace the equation by itself minus k (or plus $-k$) times the previously added equation (instead of plus k times the equation).

QED

EXERCISES on Systems, Solutions, and Elementary Equation Operations

<u>EXERCISE #1</u>. True or False.

_____ 1. The formal process for solving m linear algebraic equations in n unknowns is called Gauss Elimination

_____ 2. Another technique for solving n linear algebraic equations in n unknowns is Cramer Rule

_____ 3. A solution to a system of m linear algebraic equations in n unknowns is an n-tuple (finite sequence) $x_1, x_2, ..., x_n$ in $\mathbf{K}^n$ (e.g., $\mathbf{R}^n$ or $\mathbf{C}^n$) such that all of the equations are true.

Gauss elimination can be used to solve a system of m linear algebraic equations over a field **K** in n unknown variables $x_1, x_2, ..., x_n$ in a finite number of steps.

$$\begin{array}{ccccccccc} a_{11}\ x_1 & + & a_{12}\ x_2 & + \cdots + & a_{1n}\ x_n & = & b_1 \\ a_{21}\ x_1 & + & a_{22}\ x_2 & + \cdots + & a_{2n}\ x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}\ x_1 & + & a_{m2}\ x_2 & + \cdots + & a_{mn}\ x_n & = & b_m \end{array} \qquad (1)$$

Recall

<u>DEFINITION #1</u>. A **solution** of to (1) is an n-tuple $x_1, x_2, ..., x_n$ which may be considered as a (row) vector $[\ x_1, x_2, ..., x_n\ ]$ (or column vector) in $\mathbf{K}^n$ (usually **K** is **R** or **C**) such that all of the equations in (1) are true. The **solution set** for (1) is the set $S = \{\ [\ x_1, x_2, ..., x_n\ ] \in \mathbf{K}^n$: all of the equations in (1) are satisfied$\}$. That is, the solution set is the set of all solutions. The Σ set is the set where we look for solutions. In this case it is $\mathbf{K}^n$. Two systems of linear algebraic equations are **equivalent** if their solution set are the same (i.e., have the same elements).

Recall also the three **elementary equation operations** (EEO's) that can be used on a set of linear equations which do not change the solution set.
1. Exchange two equations
2. Multiply an equation by a nonzero constant.
3. Replace an equations by itself plus a scalar multiple of another equation.

Although other operations on the system of equations can be derived, if we restrict ourselves to these operations, it is easier to develop computational algorithms that can be easily programed on a computer.

<u>DEFINITION #2</u>. The <u>**coefficient matrix**</u> is the array of coefficients for the system (not including the right hand side, RHS).

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_{m1} & a_{m21} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix} \qquad (2)$$

DEFINITION #3. The **augmented (coefficient) matrix** is the coefficient matrix augmented by the values from the right hand side (RHS).

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & & x_n \end{array} \\ \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \end{array} \begin{array}{l} \\ \text{Represents the first equation} \\ \text{Represents the second equation} \\ \\ \text{Represents the } m^{th} \text{ equation} \end{array} \tag{3}$$

Given the coefficient matrix A and the right hand side ("vector") $\vec{b}$, we denote the associated augmented matrix as $[A|\vec{b}]$. (Note the slight abuse of notation since A actually includes the brackets. This should not cause confusion.)

ELEMENTARY ROW OPERATIONS. All of the information contained in the equations of a linear algebraic system is contained in the augmented matrix. Rather than operate on the original equations using the elementary equation operations (EEO's) listed above, we operate on the augmented matrix using **Elementary Row Operations** (ERO's).

DEFINITION #4. We define the **Elementary Row Operations (ERO's)**

1. Exchange two rows (avoid if possible since the determinant of the new coefficient matrix changes sign).
2. Multiply a row by a non zero constant (avoid if possible since the determinant of the new coefficient matrix is a multiple of the old one).
3. Replace a row by itself plus a scalar multiple of another row (the determinant of the new coefficient matrix is the same as the old one).

There is a clear one-to-one correspondence between systems of linear algebraic equations and augmented matrices including a correspondence between EEO's and ERO's. We say that the two structures are **isomorphic**. Using this correspondence and the theorems on EEO's from the previous handout, we immediately have the following theorems.

THEOREM #1. Suppose a system of linear equations is represented by an augmented matrix which we call the original augmented matrix (OAM) and that the OAM is operated on by an ERO to obtain a new augmented matrix (NAM). Then the system of equations represented by the new augmented matrix (NAM) has the same solution set as the system represented by the original augmented matrix (OAM).

THEOREM #2. Every ERO has an inverse ERO of the same type.

DEFINITION #5. If A and B are $m \times n$ (coefficient or augmented) matrices over the field F, we say that B is row-equivalent to A if B can be obtained from A by finite sequence of ERO's.

THEOREM #3. Row-equivalence is an **equivalence relation** (Recall the definition of equivalence relation given in the remedial notes or look up it up in a Modern Algebra text).

THEOREM #4 If A and B are m×n augmented matrices which are row-equivalent, then the systems they represent are equivalent (i.e., have the same solution set).

Proof idea. Suppose $A = A_0 \to A_1 \to \cdots \to A_k = B$. Using induction and Theorem #1, the two systems can be shown to have the same solution set.

DEFINITION #6. If A is a m×n (coefficient or augmented) matrices over the field F, we call the first nonzero entry in each row its **leading entry**. We say that A is in **row-echelon form** if:
1. All entries below a leading entry are zero,
2. For i =2, ..., m, the leading entry in row i is to the right of the leading entry in row (i−1),
3. All rows containing only zeros are below any rows containing nonzero entries.
If, in addition,
4. The leading entry in each row is one,
5. All entries above a leading entry are zero,
then A is in **reduced-row-echelon** form (or **row-reduced-echelon** form).
If A is in row-echelon form (or reduced-row-echelon form) we refer to the leading entries as **pivots**.

For any matrix A, Gauss Elimination (GE) will always obtain a row-equivalent matrix U that is in row-echelon form. Gauss-Jordan Elimination (GJE) will yield a row-equivalent matrix R that is in reduced-row-echelon form. $[A|\vec{b}] \overset{\text{GE}}{\Rightarrow} [U|\vec{c}] \overset{\text{GJE}}{\Rightarrow} [R|\vec{d}]$

EXAMPLE #1. To illustrate Gauss elimination we consider an example:

$$\begin{aligned} 2x + y + z &= 1 && (1) \\ 4x + y &= -2 && (2) \\ -2x + 2y + z &= 7 && (3) \end{aligned}$$

It does not illustrate the procedure completely, but is a good starting point. The **solution set** S for (1), (2) and (3) is the set of all ordered triples, [x,y,z] which satisfy all three equations. That is,

$$S = \{ \vec{x} = [x, y, z] \in \mathbf{R}^3 : \text{Equations (1), (2), and (3) are true} \}$$

The coefficient and augmented matrices are

$$
A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \end{array}
\qquad
[A|\vec{b}\,] = \begin{array}{c} \begin{array}{cccc} x & y & z & \text{RHS} \end{array} \\ \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & 1 & 0 & -2 \\ -2 & 2 & 1 & 7 \end{array}\right] \end{array}
\begin{array}{l} \text{represents equation 1} \\ \text{represents equation 2} \\ \text{represents equation 3} \end{array}
$$

Note that matrix multiplication is not necessary for the process of solving a system of linear algebraic equations. However, you may be aware that we can use matrix multiplication to write the system (1), (2), and (3) as $A\ \vec{x} = \vec{b}$ where now it is mandatory that $\vec{x}$ be a column vector instead of a row vector. Also $\vec{b} = [1, -2, 7]^T$ is a column vector. (We use the transpose notation to save space and paper.)

<u>GAUSS ELIMINATION</u>. We now use the elementary row operations (ERO's) on the example in a systematic way known as (naive) **<u>Gauss</u> (Jordan) Elimination** to solve the system. The process can be divided into three (3) steps.

<u>Step 1</u>. **Forward Elimination** (obtain zeros below the pivots).This step is also called the (forward) sweep phase.

$$
\begin{array}{r} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}
\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & 1 & 0 & -2 \\ -2 & 2 & 1 & 7 \end{array}\right]
\Rightarrow
\begin{array}{r} \\ \\ R_3 + 3R_2 \end{array}
\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 3 & 2 & 8 \end{array}\right]
\Rightarrow
\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{array}\right]
$$

This completes the forward elimination step. The **pivots** are the diagonal elements 2, -1, and -4. Note that in getting zeros below the pivot 2, we can do 2 ERO's and only rewrite the matrix once. The last augmented matrix represents the system.

$2x + y + z = 1$	We could now use	$-4z = -4 \Rightarrow z = 1$
$-y - 2z = -4$	**<u>back</u> substitution**	$-y = -4 + 2z = -4 + 2(1) = -2 \Rightarrow y = 2$
$-4z = -4$.	to obtain	$2x = 1 - y - z = 1 - (2) - (1) = -2 \Rightarrow x = -1.$

The unique solution is sometimes written in the scalar form as $x = -1$, $y = 2$, and $z = 1$, but is more correctly written in the vector form as the column vector $[-1,2,1]^T$. Instead of back substituting with the equations, we can use the following two additional steps using the augmented matrix. When these steps are used, we refer to the process as **Gauss-Jordan elimination**.

<u>Step 2</u>. **Gauss-Jordan Normalization** (Obtain ones along the diagonal).

$$
\begin{array}{r}
1/2R_1 \\ -R_2 \\ -1/4R_3
\end{array}
\left[\begin{array}{ccc|c}
2 & 1 & 1 & 1 \\
0 & -1 & -2 & -4 \\
0 & 0 & -4 & -4
\end{array}\right]
\Rightarrow
\left[\begin{array}{ccc|c}
1 & 1/2 & 1/2 & 1/2 \\
0 & 1 & 2 & 4 \\
0 & 0 & 1 & 1
\end{array}\right]
$$

We could now use back substitution but instead we can proceed with the augmented matrix.

Step 3. **Gauss-Jordan Completion Phase** (Obtain zeros above the pivots)
Variation #1 Right to Left.

$$
\begin{array}{r}
nR_1 - 1/2nR_3 \\ R_2 - 2R_3 \\
\end{array}
\left[\begin{array}{ccc|c}
1 & 1/2 & 1/2 & 1/2 \\
0 & 1 & 2 & 4 \\
0 & 0 & 1 & 1
\end{array}\right]
\Rightarrow
\begin{array}{r}
nR_1 - 1/2nR_2 \\ \\
\end{array}
\left[\begin{array}{ccc|c}
1 & 1/2 & 0 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1
\end{array}\right]
\Rightarrow
\left[\begin{array}{ccc|c}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1
\end{array}\right]
\Rightarrow
\begin{array}{l}
x = -1 \\ y = 2 \\ z = 1
\end{array}
$$

Variation #2 (Left to Right)

$$
\begin{array}{r}
nR_1 - 1/2\,R_2 \\ \\
\end{array}
\left[\begin{array}{ccc|c}
1 & 1/2 & 1/2 & 1/2 \\
0 & 1 & 2 & 4 \\
0 & 0 & 1 & 1
\end{array}\right]
\Rightarrow
\begin{array}{r}
nR_1 + 1/2nR_3 \\ R_2 - 2R_3 \\
\end{array}
\left[\begin{array}{ccc|c}
1 & 0 & -1/2 & 3/2 \\
0 & 1 & 2 & 4 \\
0 & 0 & 1 & 1
\end{array}\right]
\Rightarrow
\left[\begin{array}{ccc|c}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1
\end{array}\right]
\Rightarrow
\begin{array}{l}
x = -1 \\ y = 2 \\ z = 1
\end{array}
$$

Note that both variations as well as back substitution result in the same solution, $\vec{x} = [-1, 2, 1]^T$. It should also be reasonably clear that this algorithm can be programmed on a computer. It should also be clear that the above procedure will give a unique solution for n equations in n unknowns in a finite number of steps provided **<u>zeros never appear on the diagonal</u>**. The case when zeros appear on the diagonal and we still have a unique solution is illustrated below. The more general case of m equations in n unknowns where three possibilities exist is discussed later.

<u>EXAMPLE #2</u>. To illustrate the case of zeros on the diagonal that are eliminated by row exchanges consider:

$$
\begin{aligned}
-y - 2z &= -4 \qquad (4)\\
-4z &= -4 \qquad (5)\\
2x + y + z &= 1 \qquad (6)
\end{aligned}
$$

The augmented matrix is

$$
[A|\vec{b}] =
\begin{array}{c}
\begin{array}{cccc} x & y & z & \text{RHS} \end{array} \\
\left[\begin{array}{ccc|c}
0 & -1 & -2 & -4 \\
0 & 0 & -4 & -4 \\
2 & 1 & 1 & 1
\end{array}\right]
\end{array}
\begin{array}{l}
\\ \text{representsnequationn1} \\ \text{representsnequationn2} \\ \text{representsnequationn3}
\end{array}
$$

Note that there is a zero in the first row first column so that Gauss Elimination temporarily breaks down. However, note that the augmented matrix is row equivalent to those in the previous problem since it is just the matrix at the end of the forward step of that problem with the rows in a different order To establish a standard convention for fixing the breakdown, we go down the first column until we find a nonzero number. We then switch the first row with the first row with a nonzero entry in the first column. (If all of the entries in the first column are zero, then x can be anything and is not really involved in the problem.) Switching Rows one and three we obtain

$$\begin{array}{c} \to \\ \\ \to \end{array}\left[\begin{array}{ccc|c} 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \\ 2 & 1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 0 & -4 & -4 \\ 0 & -1 & -2 & -4 \end{array}\right]$$

The 2 in the first row, first column is now our first pivot. We go to the second row, second column. Unfortunately, it is also a zero. But the third row, second column is not so we switch rows.

$$\begin{array}{c} \\ \to \\ \to \end{array}\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 0 & -4 & -4 \\ 0 & -1 & -2 & -4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{array}\right]$$

We now have the same augmented matrix as given at the end of the forward step for the previous problem. Hence the solution is $x = -1$, $y = 2$, and $z = 1$. This can be written as the column vector $\mathbf{x} = [-1, 2, 1\,]^T$.

EXERCISES on Introduction to Gauss Elimination

EXERCISE #1. True or False.

_____ 1. An elementary equation operation (EEO) that can be used on a set of linear algebraic equations which does not change the solution set is “Exchange two equations”.

_____ 2. An elementary equation operation (EEO) that can be used on a set of linear algebraic equations which does not change the solution set is “Multiply an equation by a nonzero constant”.

_____ 3. An elementary equation operation (EEO) that can be used on a set of linear algebraic equations which does not change the solution set is “Replace an equations by itself plus a scalar multiple of another equation”.

_____ 4. An elementary row operation (ERO) of type one that can be used on a matrix is “Exchange two rows”.

_____ 5. An elementary row operation (ERO) of type two that can be used on a matrix is “Multiply a row by a nonzero constant”

_____ 6. An elementary row operation (ERO) of type three that can be used on a matrix is “Replace a row by itself plus a scalar multiple of another row”.

_____ 7. There is a clear one-to-one correspondence between systems of linear algebraic equations and augmented matrices including a correspondence between EEO’s and ERO’s so that we say that the two structures are isomorphic.

_____ 8. Every ERO has an inverse ERO of the same type.

_____ 9. If A and B are $m\times n$ (coefficient or augmented) matrices over the field **K**, we say that B is row-equivalent to A if B can be obtained from A by finite sequence of ERO's.

_____ 10. Row-equivalence is an equivalence relation

_____ 11. If A and B are $m\times n$ augmented matrices which are row-equivalent, then the systems they represent are equivalent (i.e., have the same solution set)

_____ 12. If A is a $m\times n$ (coefficient or augmented) matrix over the field **K**, we call the first nonzero entry in each row its leading entry.

EXERCISE #2. Solve
$$\begin{aligned} 4x + y \quad &= -2 \\ -2x + 2y + z &= 7 \\ 2x + y + z &= 1 \end{aligned}$$

EXERCISE #3. Solve
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + 2x_2 + x_3 \quad &= 0 \\ x_3 + x_4 &= 1 \\ x_2 + 2x_3 + x_4 &= 1 \end{aligned}$$

EXERCISE #4. Solve
$$\begin{aligned} 4x + y \quad &= -2 \\ -2x + y + z &= 7 \\ 2x + y + z &= 1 \end{aligned}$$

EXERCISE #5. Solve
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + 2x_2 + x_3 \quad &= 0 \\ x_3 + x_4 &= 1 \\ x_2 + 2x_3 + 2x_4 &= 3 \end{aligned}$$

EXERCISE #6. Solve
$$\begin{aligned} 4x + y \quad &= -2 \\ -2x + 2y + z &= 5 \\ 2x + y + z &= 1 \end{aligned}$$

EXERCISE #7. Solve
$$\begin{aligned} 4x + y \quad &= -2 \\ -2x + 2y + z &= 1 \\ 2x + y + z &= 1 \end{aligned}$$

By using the definition of **matrix equality** we can think of the system of **scalar equations**

$$\begin{aligned} a_{11}\ x_1 + a_{12}\ x_2 + \cdots + a_{1n}\ x_n &= b_1 \\ a_{21}\ x_1 + a_{22}\ x_2 + \cdots + a_{2n}\ x_n &= b_2 \\ \vdots \qquad \vdots \qquad & \quad \vdots \\ a_{m1}\ x_1 + a_{m2}\ x_2 + \cdots + a_{mn}\ x_n &= b_m \end{aligned} \qquad (1)$$

as one **"vector" equation** where "vector" means an n-tuple or column vector. By using matrix multiplication (1) can be written as

$$\underset{m\times n}{A}\ \underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}} \qquad (2)$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = [x_i] \in \mathbf{K}^n, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix} = [b_i] \in \mathbf{K}^m, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdot\ \cdot\ \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot\ \cdot\ \cdot & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m21} & \cdot\ \cdot\ \cdot & a_{mn} \end{bmatrix} = [a_{ij}] \in \mathbf{K}^{mxn}.$$

If we think of $\mathbf{K}^n$ and $\mathbf{K}^m$ as vector spaces, we can define

$$T(\vec{x}) = \underset{m\times n}{A}\ \underset{n\times 1}{\vec{x}} \qquad (3)$$

so that T is a mapping from $\mathbf{K}^n$ to $\mathbf{K}^m$. We write T: $\mathbf{K}^n \rightarrow \mathbf{K}^m$. A mapping from a vector space to another vector space is called an **operator**. We may now view the system of scalar equations as the operator equation:

$$T(\vec{x}) = \vec{b} \qquad (4)$$

A column vector $\vec{x} \in \mathbf{K}^n$ is a solution to (4) if and only if it is a solution to (1) or (2). Solutions to (4) are just those vectors $\vec{x}$ in $\mathbf{K}^n$ that get mapped to $\vec{b}$ in $\mathbf{K}^m$ by the operator T. The equation

$$T(\vec{x}) = \vec{0} \qquad (5)$$

is called **homogeneous** and is the **complementary equation** to (4). Note that it always has

$\vec{x} = \vec{0} = [0, 0, \ldots, 0]^T \in \mathbf{K}^n$ as a solution. $\vec{x} = \vec{0}$ is called the **trivial solution** to (5) since it is always a solution. The question is "Are there other solutions?" and if so "How many?". We now have a connection between solving linear algebraic equations, matrix algebra, and abstract linear algebra. Also we now think of solving linear algebraic equations as an example of a **mapping problem** where the operator $T:\mathbf{R}^n \to \mathbf{R}^m$ is defined by (3) above. We wish to find all solutions to the **mapping problem** (4). To introduce this topic, we define what we mean by a **linear operator** from one vector space to another.

DEFINITION #1. An operator $T:V \to W$ is said to be **linear** if $\forall\ \vec{x}, \vec{y} \in V$ and scalars α,β, it is true that

$$T(\alpha\vec{x} + \beta\vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y}). \tag{6}$$

$T: \mathbf{K}^n \to \mathbf{K}^m$ defined by $T(\vec{x}) = \underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}}$ is one example of a linear operator. We will give others later. To connect the mapping problem to matrix algebra we reveal that if m = n, then (1) (2) and (4) have a unique solution if and only if the matrix A is invertible.

THEOREM #1. Suppose m = n so that (1) has the same number of equations as unknowns. Then (1), (2), and (4) have a unique solution if and only if the matrix A is invertible.

EXERCISES on Connection with Matrix Algebra and Abstract Linear Algebra

EXERCISE #1. True or False.

_____ 1. Since $\mathbf{K}^n$ and $\mathbf{K}^m$ are vector spaces, we can define $T(\vec{x}) = \underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}}$ so that T is a mapping from $\mathbf{K}^n$ to $\mathbf{K}^m$.

_____ 2. A mapping from a vector space to another vector space is called an operator.

_____ 3. Solutions to $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$ are just those vectors $\vec{x}$ in $\mathbf{K}^n$ that get mapped to $\vec{b}$ in $\mathbf{K}^m$ by the operator $T(\vec{x}) = \underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}}$.

_____ 4. The equation $T(\vec{x}) = \vec{0}$ is called homogeneous and is the complementary equation to $\underset{m\times n}{T}\left(\underset{n\times 1}{\vec{x}}\right) = \underset{m\times 1}{\vec{b}}$ where $T(\vec{x}) = \underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}}$.

_____ 5. $\vec{x} = \vec{0}$ is called the trivial solution to the complementary equation to $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{b}}$.

_____ 6. An operator $T:V \to W$ is said to be linear if $\forall\ \vec{x}, \vec{y} \in V$ and scalars α,β, it is true that $T(\alpha\vec{x} + \beta\vec{y}) = \alpha T(\vec{x}) + \beta T(\vec{y})$.

_____ 7. The operator T: $\mathbf{K}^n$ to $\mathbf{K}^m$ defined by $T(\vec{x}) = \underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}}$ is a linear operator.

_____ 8. $\underset{n\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{n\times 1}{\vec{b}}$ has a unique solution if and only if the matrix A is invertible.