An important concept in abstract linear algebra is that of a subspace. After we have established a number of important examples of vector spaces, we may have reason to examine subsets of these vector spaces. Some subsets are subspaces, some are not. It is important to be able to determine if a given subset is in fact a subspace.

In general in mathematics, a set with structure is called a **space**. A subset of a space that has the same structure as the space itself is called a **subspace**. Sometimes, all subsets of a space are subspaces with the induced mathematical structure. However, all subsets of a vector (or linear) space need not be subspaces. This is because not all subsets have a **closed algebraic structure**. The sum of two vectors in a subset might not be in the subset. A scalar times a vector in the subset might not be in the subset.

DEFINITION #1. Let W be a nonempty subset of a vector space V. If for any vectors $\vec{x}$, $\vec{y} \in$ W and scalars $\alpha,\beta \in$ **K** (recall that normally the set of scalars **K** is either **R** or **C**), we have that $\alpha\,\vec{x} + \beta\,\vec{y} \in$ W, then W is a **subspace** of V.

THEOREM #1. A nonempty subset W of a vector space V is a subspace of V if and only if for $\vec{x}$, $\vec{y} \in$ V and $\alpha \in$ **K** (i.e. α is a scalar) we have.

i) $\vec{x}$, $\vec{y} \in$ W implies $\vec{x} + \vec{y} \in$ W, and

ii) $\vec{x} \in$ W implies $\alpha\,\vec{x} \in$ W.

TEST FOR A SUBSPACE. Theorem #1 gives a good test to determine if a given subset of a vector space is a subspace since we can test the closure properties separately. Thus if $W \subset V$ where V is a vector space, to determine if W is a subspace,we check the following three points.

1) Check to be sure that W is nonempty. (We usually look for the zero vector since if there is $\vec{x} \in$W, then $0\,\vec{x} = \vec{0}$ must be in W. Every vector space and every subspace must contain the zero vector.)
2) Let $\vec{x}$ and $\vec{y}$ be arbitrary elements of W and check to see if $\vec{x} + \vec{y}$ is in W. (Closure of vector addition)
3) Let $\vec{x}$ be an arbitrary element in W and check to see if $\alpha\,\vec{x}$ is in W. (Closure of scalar multiplication).

The checks can be done in different ways. For example, by using an English description of the subspaces, by visualizing the subspaces geometrically, and by using mathematical notation (which you may have to develop). Although, in some sense any clear argument to validate the three properties constitutes a proof, mathematicians prefer a proof using mathematical notation since this does not limit one to three dimensions, avoids the potential pitfalls of vagueness inherent in English prose, and usually is more terse. Hence we will refer to the use of English prose and the visualization of subspaces geometrically as checks, and the use of mathematical notation as a proof. Use English prose and geometry to check in your mind. Learn to write algebraic proofs using mathematical notation to improve your understanding of the concept.

EXAMPLE #1. Let $\mathbf{R}^3 = \{(x,y,z): x,y,z \in \mathbf{R}\}$ and W be the subset of V which consists of vectors whose first and second components are the same and whose third component is zero. In mathematically (more concise)notation we can define W as $W = \{(\alpha,\alpha,0): \alpha \in \mathbf{R}\}$. (Since matrix multiplication is not needed, we use row vectors rather than column vector to save space.) Note that the scalars are the reals.

English Check. We first check to see if W is a subspace using English.

1) "Clearly" W is nonempty since, for example if $\alpha = 0$, the vector (0,0,0) is in W. Recall that the zero vector must always be in a subspace.
2) If we add two vectors in W (i.e. two vectors whose first two components are equal and whose third component is zero), "clearly" we will get a vector in W (i.e. a vector whose first two components are equal and whose third component is zero). Hence W is closed under vector addition.
3) If we multiply a vector in W (i.e. a vector whose first two components are equal and whose third component is zero) by a scalar, "clearly" we will get a vector in W (i.e. a vector whose first two components are equal and whose third component is zero). Hence W is closed under scalar multiplication.

Note that if the parenthetical expressions are removed, the above "check" does not explain or prove anything. Hence, after a geometrical check, we give a more mathematical proof using the definition of W as $W = \{(\alpha,\alpha,0): \alpha \in \mathbf{R}\}$.

Geometrical Check. Since we can sketch $\mathbf{R}^3$ in the plane $\mathbf{R}^2$ we can give a geometrical interpretation of W and can in fact check to see if W is a subspace geometrically. Recall that geometrically we associate the algebraic vector $\vec{x} = (x,y,z) \in \mathbf{R}^3$ with the geometric **position vector** (directed line segment) whose tail is the origin and whose head is at the point (x,y,z). Hence W is the set of position vectors whose heads fall on the line through the origin and the point (1,1,0). Although technically incorrect, we often shorten this and say that W is the line through the origin and point (1,1,0).

1) The line W, (i.e. the line through (0,0,0) and (1,1,0)) "clearly" contains the origin (i.e. the vector $\vec{0} = (0,0,0)$).
2) If we add two vectors whose heads lie on the line W (i.e. the line through (0,0,0) and (1,1,0)), "clearly" we will get a vector whose head lies on the line W (i.e. the line through (0,0,0) and (1,1,0)). (Draw a picture.) Hence W is closed under vector addition.
3) If we multiply a vector whose head lies on the line through W (i.e. the line through (0,0,0) and (1,1,0)) by a scalar, "clearly" we will get a vector whose head lies on the line through (0,0,0) and (1,1,0). (Again, draw a picture.) Hence W is closed under scalar multiplication.

THEOREM #2. The set $W = \{(\alpha,\alpha,0): \alpha \in \mathbf{R}\}$ is a subspace of the vector space $\mathbf{R}^3$.

Proof. Let $W = \{(\alpha,\alpha,0): \alpha \in \mathbf{R}\}$.

1) Letting $\alpha = 0$ we see that $(0,0,0) \in W$ so that $W \neq \emptyset$.

2) Now let $\vec{x} = (\alpha,\alpha,0)$ and $\vec{y} = (\beta,\beta,0)$ so that $\vec{x}$, $\vec{y}$ y $\in$ W. Then

STATEMENT	REASON
$\vec{x} + \vec{y} = (\alpha,\alpha,0) + (\beta,\beta,0)$	Notation
$= (\alpha+\beta,\alpha+\beta,0)$.	Definition of Vector Addition in $\mathbb{R}^3$

"Clearly" $(\alpha+\beta,\alpha+\beta,0) \in$ W since it is of the correct form. Hence W is closed under vector addition.

3) Let $\vec{x} = (\alpha,\alpha,0)$ and a $\in$ **R** be a scalar. Then

STATEMENT	REASON
$a\,\vec{x} = a(\alpha,\alpha,0)$	Notation
$= (a\alpha,a\alpha,0)$.	Definition of Scalar Multiplication in $\mathbb{R}^3$

"Clearly" $(a\alpha,a\alpha,0) \in$ W since it is of the correct form. Hence W is closed under scalar multiplication.

Q.E.D.

Note that the proof is in some sense more convincing (i.e. more rigorous) than the English check or a geometric check. However we did have to develop some notation to do the proof.

<u>EXAMPLE #2</u>. Let V = $\mathscr{F}$(I) be the set of real valued functions on I=(a,b) where a<b. Now let W=C(I) be the set of continuous functions on I. Clearly W=C(I) is a subset of V =$\mathscr{F}$(I). Note that the scalars are the reals.

<u>English Check</u>. We first check to see if W is a subspace using English.

1) "Clearly" W is nonempty since, for example f(x) =x^2 is a continuous function in C(I). Also, f(x) =0 is continuous as is required for W to be a subspace.
2) If we add two continuous functions, we get a continuous function so that sums of functions in W are in W. Hence W is closed under vector addition. (Recall that this is a theorem from calculus.)
3) If we multiply a continuous function by a scalar, we get a continuous function. Hence W is closed under scalar multiplication. (Recall that this is anther theorem from calculus.)

“Clearly” there is no “geometrical” check that W is a subspace. The reason is that C(I) is infinite dimensional and hence a geometrical picture is not possible. The concept of a basis for a vector space or subspace and that of dimension will be discussed later.)

The informal English check can be upgraded to a mathematical proof by a careful rewording and by citing references for the appropriate theorems from calculus.

We close by noting that a subspace is in fact a vector space in its on right using the vector addition and scalar multiplication of the larger vector space.

THEOREM #3. Suppose W⊆V where V is a vector space over the scalars **K**. If W is a subspace of V, then all of the eight properties hold for all elements in W and all scalars in **K** (without relying on any of the elements in V that are not in W). Thus W with the vector addition and scalar multiplication of V (but without the elements in V that are not in W) is a vector space.

EXERCISES on Subspace of a Vector Space

EXERCISE #1. True or False.
For all questions, assume V is a vector space over a field **K**, $\vec{x}$, $\vec{y}$, $\vec{z}$ ∈V are any vectors and α,ß ∈**K** are any scalars.

_____ 1. If W⊆V is a nonempty subset of a vector space V and for any vectors $\vec{x}$, $\vec{y}$ ∈ W and scalars α,β ∈ **K**, we have that $\alpha\vec{x} + \beta\vec{y}$ ∈ W, then W is a subspace of V.
_____ 2. A nonempty subset W of a vector space V is a subspace of V if and only if for $\vec{x}$, $\vec{y}$ ∈ V and α ∈ **K** (i.e. α is a scalar) we have i) $\vec{x}$, $\vec{y}$ ∈ W implies $\vec{x} + \vec{y}$ ∈ W, and ii) $\vec{x}$ ∈ W implies $\alpha\vec{x}$ ∈ W.
_____ 3. If W is a subspace of V, then W is nonempty.
_____ 4. If W is a subspace of V, then for any arbitrary vectors $\vec{x}$ and $\vec{y}$ in W, the sum $\vec{x} + \vec{y}$ is in W.
_____ 5. If W is a subspace of V, then for any arbitrary vector $\vec{x}$ in W and scalar α∈**K**, $\alpha\vec{x}$ is in W.
_____ 6. If W be the subset of V= $\mathbf{R}^3$ which consists of vectors whose first and second components are the same and whose third component is zero, then W is a subspace of V.
_____ 7. W = {(α,α,0): α ∈ **R**} is a subspace of $\mathbf{R}^3$.
_____ 8. W = {(α,0,0): α ∈ **R**} is a subspace of $\mathbf{R}^3$.
_____ 9. If I = (a,b), then C(I,**R**) is a subspace of *F* (I,**R**).
_____ 10. If I = (a,b), then *A* (I,**R**) is a subspace of *F* (I,**R**).
_____ 11. If I = (a,b), then *A* (I,**R**) is a subspace of C(I,**R**).

EXERCISE #2. Let $\mathbf{R}^3$ = {(x,y,z): x,y,z∈**R**} and W be the subset of V which consists of vectors where all three components are the same. In mathematically (more concise) notation we can define W as W = {(α,α,α): α ∈ **R**}. (Since matrix multiplication is not needed, we use row vectors rather than column vector to save space.) Note that the scalars are the reals. If possible, do an English check, a geometric check, and a proof that W is a subspace.

EXERCISE #3. Let $\mathbf{R}^3$ = {(x,y,z): x,y,z∈**R**} and W be the subset of V which consists of vectors where second ands third components are zero. In mathematically (more concise) notation we can define W as W = {(α,0,0): α ∈ **R**}. (Since matrix multiplication is not needed, we use row vectors rather than column vector to save space.) Note that the scalars are the reals. If possible, do an Englishcheck, a geometric check, and a proof that W is a subspace.

EXERCISE #4. Let $\mathbf{R}^3 = \{(x,y,z): x,y,z\in\mathbf{R}\}$, $\mathbf{R}^4 = \{(x_1,x_2,x_3, x_4): x_1,x_2,x_3, x_4\in\mathbf{R}\}$. Consider the following subsets W of these vector sapaces. Determine which are subspaces. (Since matrix multiplication is not needed, we use row vectors rather than column vector to save space.) Note that the scalars are the reals. For each subset, first write an English description of the subset. If possible, do an English check, a geometric check, and a proof that W is a subspace. If W is not a subspace explain clearly why it is not. In mathematically (more concise) notation we define W as

1. $W = \{(\alpha,1,0): \alpha \in \mathbf{R}\} \subseteq\mathbf{R}^3$.
2. $W = \{(x,y,z): x^2 + y^2 + z^2 =1 \text{ and } x,y,z\in \mathbf{R}\} \subseteq\mathbf{R}^3$
3. $W = \{(\alpha,\beta,0): \alpha,\beta \in \mathbf{R}\}\subseteq\mathbf{R}^3$.
4. $W = \{(\alpha,\beta,0,0): \alpha,\beta \in \mathbf{R}\}\subseteq\mathbf{R}^4$.
5. $W = \{(\alpha,\alpha,\alpha,\alpha): \alpha \in \mathbf{R}\}\subseteq\mathbf{R}^4$
6. $W = \{(\alpha,\alpha,\beta,\beta): \alpha,\beta \in \mathbf{R}\}\subseteq\mathbf{R}^4$